Key exchange

Introduction to cryptology

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https://membres-ljk.imag.fr/Bruno.Grenet/IntroCrypto.html
https://membres-ljk.imag.fr/Pierre.Karpman/tea.html

Introduction

Up to now: Symmetric cryptography

- Symmetric encryption *confidentiality*
- Message authentication codes *authenticity/integrity*

→ Both require parties to share a common secret

> How do two parties agree on a common secret?

Bad solution

- Any pair of parties agree on a common key
  - If N parties, it requires $\Theta(N^2)$ keys!
  - To share a key, the parties must meet

One possible solution: key distribution centers (KDCs)

Idea

- Each party shares a (secret) key with the KDC
- If Alice wants to talk to Bob:
  - Alice gets an encrypted *session key* k: $\mathrm{Enc}_{k_a}(k)$
  - Bob gets the same encrypted temporary key: $\mathrm{Enc}_{k_b}(k)$
  - Alice and Bob decrypt k and use it to communicate

Advantages

- Each party retains (in the long run) only one key
- Each party only needs to meet the KDC, once

Disadvantages

- The KDC is the central security point:
  - If it is attacked, all security falls
  - If it fails, no communication is possible
- Does not work in *open system* like Internet

Public-key cryptography

Key-exchange protocols

- Two parties discuss publicly
- At the end, both parties know a same secret k
- External observers do not learn the secret, even after reading all exchanged messages

Public-key encryption and signatures

- Direct protocols to ensure confidentiality, authenticity and/or integrity
- Based on a pair (public key, private key) $\rightarrow$ no common secret

In this course

- This lecture: Key-exchange protocols
- Next week: Public-key encryption
- In two weeks: Signatures *public-key equivalent to MACs*

Contents

1. Key exchange protocols

2. Cyclic groups and discrete logarithm

3. Diffie-Hellman protocol

The goal of a key exchange

Allow two parties to agree on a key, remotely

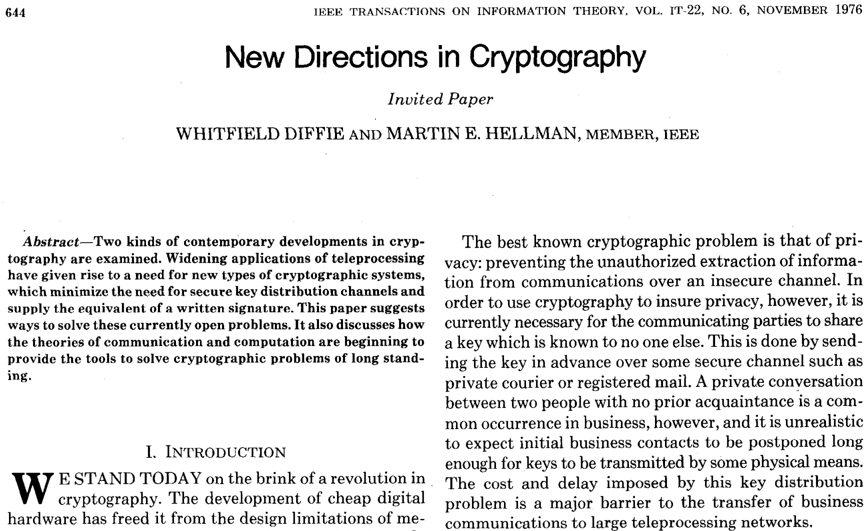
Objective

- Alice and Bob exchange messages
- At the end of the exchange, they both know the same key k
- An attacker who sees all the messages has no information about k

Is this possible?

- The attacker sees as much as Alice and Bob ‽
- No information → computational security

New Directions in Cryptography



644 IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. IT-22, NO. 6, NOVEMBER 1976

New Directions in Cryptography

Invited Paper

WHITFIELD DIFFIE AND MARTIN E. HELLMAN, MEMBER, IEEE

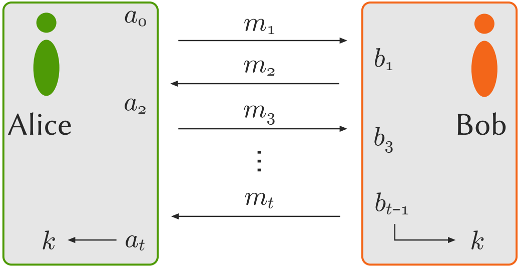
***Abstract*—Two kinds of contemporary developments in cryptography are examined. Widening applications of teleprocessing have given rise to a need for new types of cryptographic systems, which minimize the need for secure key distribution channels and supply the equivalent of a written signature. This paper suggests ways to solve these currently open problems. It also discusses how the theories of communication and computation are beginning to provide the tools to solve cryptographic problems of long standing.**

I. Introduction

WE STAND TODAY on the brink of a revolution in cryptography. The development of cheap digital hardware has freed it from the design limitations of me-

The best known cryptographic problem is that of privacy: preventing the unauthorized extraction of information from communications over an insecure channel. In order to use cryptography to insure privacy, however, it is currently necessary for the communicating parties to share a key which is known to no one else. This is done by sending the key in advance over some secure channel such as private courier or registered mail. A private conversation between two people with no prior acquaintance is a common occurrence in business, however, and it is unrealistic to expect initial business contacts to be postponed long enough for keys to be transmitted by some physical means. The cost and delay imposed by this key distribution problem is a major barrier to the transfer of business communications to large teleprocessing networks.

Definition of a protocol



Key exchange protocol

- Public : messages $m_1, \ldots, m_t$; key space $\mathcal{K}$
- Private : the a_i known only to Alice, the b_i only to Bob
- Correct protocol if Alice and Bob compute the same key $k \in \mathcal{K}$

Vocabulary

- $m_1, \ldots, m_t$: the *transcript*

Freely inspired from: J. Katz, Y. Lindell. Introduction to modern cryptography. 3rd ed, CRC Press, 2021.

Security of a protocol

Secure key exchange protocol: given $m_1, \dots, m_t$, it is difficult to compute k

Game: key exchange indistinguishability

Challenger	simulates the protocol $\rightsquigarrow$ transcript $m_1, \dots, m_t$ and key $k \in \mathcal{K}$ draws $b \twoheadleftarrow \{0,1\}$ and returns $\hat{k} = k$ if $b = 1$ and $\hat{k} \twoheadleftarrow \mathcal{K}$ otherwise
Adversary	sees the transcript and $\hat{k}$, and returns a bit b'

Advantages

- For a specific adversary A:

$$\mathrm{Adv}_{\mathrm{KE}}^{\mathrm{IND-EAV}}(A) = \left|\Pr\left[b' = 1 \middle| b = 1\right] - \Pr\left[b' = 1 \middle| b = 0\right]\right| = 2\Pr\left[\textsc{success}\right] - 1$$

- For the protocol:

$$\mathrm{Adv}_{\mathrm{KE}}^{\mathrm{IND-EAV}}(t) = \max_{A_t} \mathrm{Adv}_{\mathrm{KE}}^{\mathrm{IND-EAV}}(A_t)$$

where A_t denotes an algorithm with running time $\leq t$

Eavesdropper security and *person-in-the-middle* attack

Indistinguishability in the presence of an eavesdropper

- Security definition assumes an authenticated channel between Alice and Bob
- The adversary is only *passive*

Person-in-the-middle attack

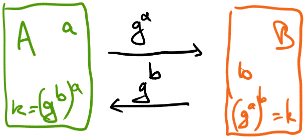
- Eve intercepts messages between Alice and Bob
- She impersonates both Alice and Bob
- She creates a common secret with Alice, and another one with Bob
- Alice and Bob incorrectly think they share a common secret

Key exchange is not enough

Combine with authentication signatures

- *Authenticated* key exchange

A glimpse of Diffie-Hellman protocol



Protocol sketch

Public: a *number* g

Alice chooses a *random* a, computes g^a and sends g^a to Bob

Bob chooses a *random* b, computes g^b and sends g^b to Alice

Alice computes $k = (g^b)^a = g^{ab}$

Bob computes $k = (g^a)^b = g^{ab}$

Outstanding issues

- How to choose g?
  - If it is an integer, g^a, g^b and g^{ab} are *huge* integers
- Why is this scheme secure?
  - Eve sees g^a and g^b

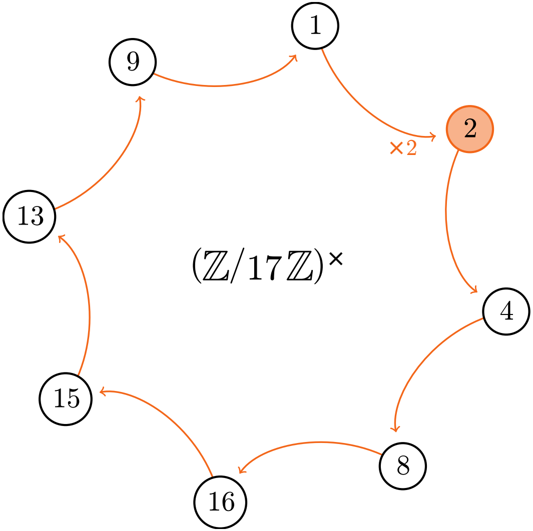
Contents

1. Key exchange protocols

2. Cyclic groups and discrete logarithm

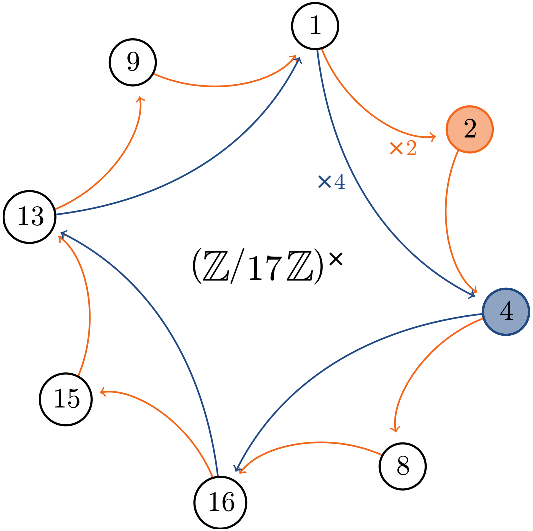
3. Diffie-Hellman protocol

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



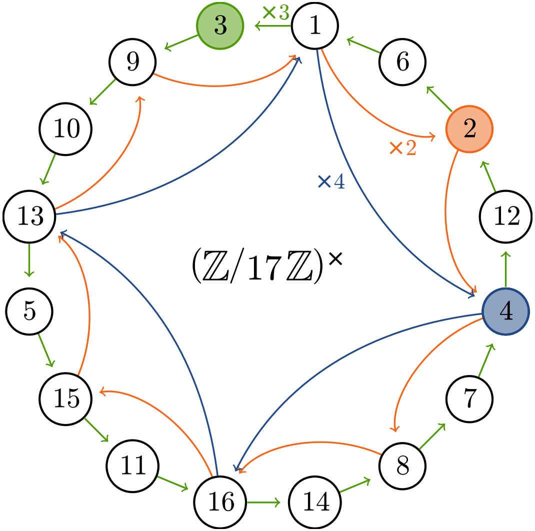
- 2 has order 8

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



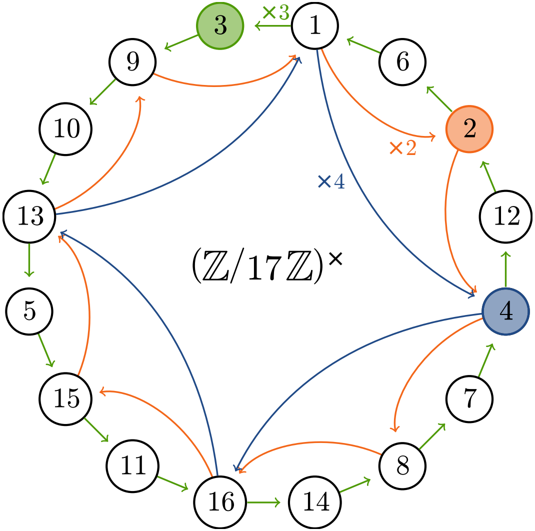
- 2 has order 8
- 4 has order 4

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



- 2 has order 8
- 4 has order 4
- 3 has order 16 → generator

The multiplicative group of $\mathbb{Z}/p\mathbb{Z}$



- ▶ 2 has order 8
- ▶ 4 has order 4
- ▶ 3 has order 16 → generator

Theorem

For every p, $(\mathbb{Z}/p\mathbb{Z})^\times$ is a cyclic group: there exists a generator $g \in (\mathbb{Z}/p\mathbb{Z})^\times$ such that

$$\begin{aligned}(\mathbb{Z}/p\mathbb{Z})^\times &= \{g^n : n \in \mathbb{Z}\} \\ &= \{g^n : 0 \leq n < p-1\}\end{aligned}$$

Remark

The generator is *not* unique (ex.: 3, 5, 6, 7, 10, 11, 12, 14)

Cyclic groups

Definition

A multiplicative cyclic group with generator g is a set G such that $G = \{g^t : t \in \mathbb{Z}\}$

Remarks

- The generator is not unique
- If $|G| = n$ is finite with generator g, $G = \{g^t : 0 \leq t < n\}$ *n: order*
  - for all $x \in G$, there exists a *unique* $t \in \{0, \ldots, n-1\}$ s.t. $x = g^t$
- Each element $x \in G$ *generates a subgroup* $G_x = \{x^t : t \in \mathbb{Z}\} \subset G$
  - order of x = order of G_x
  - if x has order s, x^r has order $s/\textsc{gcd}(s, r)$

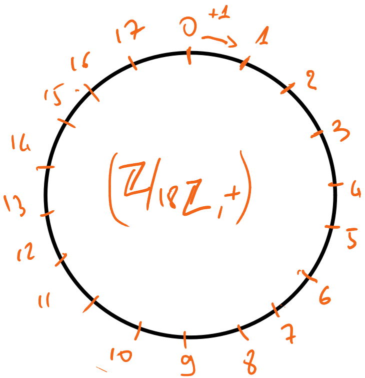
More general definitions

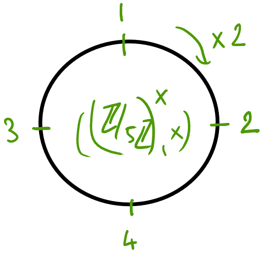
- Cyclic group $(G, \star)$ with any binary operation $\star$
  - Additive cyclic group with generator g: $G = \{t \cdot g : t \in \mathbb{Z}\}$
  - Notation: $(G, \times)$, $(G, +)$ or $(G, \star)$ to specify the type of cyclic group
- General (non-cyclic) groups *cf.* CM 7

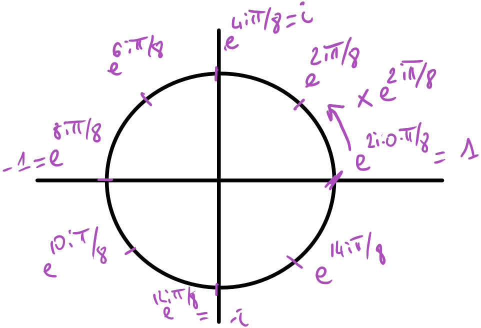
Examples

	Additive	Multiplicative
Infinite	$(\mathbb{Z},+)$ with gen 1, -1 $\mathbb{Z} = \{t \cdot 1 : t \in \mathbb{Z}\}$	$\{2^t : t \in \mathbb{Z}\}$ (gen 2)
Finite	$(\mathbb{Z}/n\mathbb{Z},+)$ with gen 1 $\mathbb{Z}/n\mathbb{Z} = \{t \cdot 1 \bmod n : t \in \mathbb{Z}\}$ (any k s.t. $\gcd(k,n)=1$ is a generator)	$(\mathbb{Z}/p\mathbb{Z}^{\times}, \times)$ $(\{-1,1\}, \times)$ with gen -1 $\{e^{2ik\pi/n} : k \in \mathbb{Z}\}$

Graphical representation







Discrete logarithm problem

Definitions

Given a cyclic group G with generator g,

- the discrete logarithm of x in base g is the unique $0 \leq t < |G|$ such that $x = g^t$
- the discrete logarithm problem is, given x, to compute t

The naive algorithm

- Compute $g^0, g^1, g^2, \ldots$ until we get $g^t = x$
- Complexity $O(t) = O(|G|)$ operations in G

Easy case: $(\mathbb{Z}/n\mathbb{Z}, +)$

- Generators: 1 or any g such that $\textsc{gcd}(g, n) = 1$
- Discrete logarithm of x: t s.t. $x = t \cdot g \bmod n$
- Case $g = 1$: nothing to do!
- General case:
  1. Compute u, v s.t. $u \cdot g + v \cdot n = 1$
  2. Return $t = u \cdot x \bmod n$

$n = 12$
$g = 5$
$x = 7$

I need t s.t. $t . 5 \bmod 12 = 7$

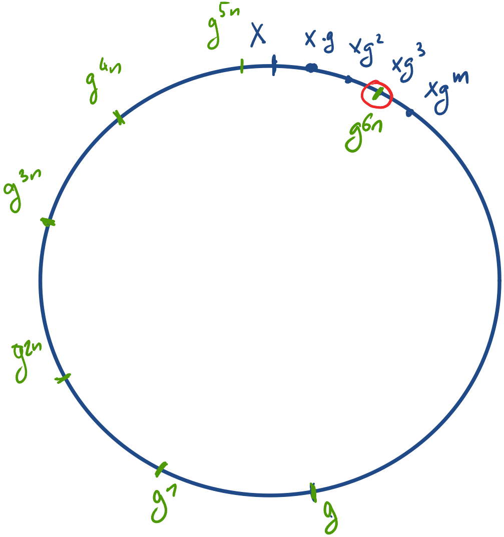
$\Rightarrow t = 11$

Extended Euclidean Algorithm

$t \cdot g = u \cdot x \cdot g = x \bmod n$

Baby step-giant step: A picture is worth a thousand words

(Shanks, 1971)



$(\mathbb{Z}/17\mathbb{Z}^{\times}, \times)$ $\quad g = 3$

$x = 4$

$x = g^{t}$

$xg^{3} = g^{6n}$

$g^{t+3} = g^{6n}$

$t = 6n - 3 \pmod{|G|}$

Baby step-giant step: the algorithm

(Shanks, 1971)

Input: a cyclic group G of order n, with generator g, and $x \in G$

Output: the discrete logarithm t of x in base g

```
$m \leftarrow \lceil \sqrt{n} \rceil$
$B \leftarrow [1, g, g^2, \ldots, g^{m-1}]$                    Baby steps
$(h, y, j) \leftarrow (g^m, x, 0)$
while $y \notin B$: $(y, j) \leftarrow (y \cdot h, j + 1)$        Giant steps: $y = x \cdot g^{m \cdot j}$
$i \leftarrow$ index such that $y = g^i$                  Collision found: $x \cdot g^{m \cdot j} = g^i$
return $(i - m \cdot j) \bmod n$
```

Analysis

Correction: by Euclidean division, there exist $i, j < m$ such that $t = i - mj \bmod n$

Complexity: Baby steps & giant steps: $O(\sqrt{n})$

Collision search: $O(\sqrt{n})$ (naive), $O(\log n)$ (dichotomy), $O(1)$ (hash tables)

$\Rightarrow O(\sqrt{n})$ (same in space)

DLP hardness

Theorem (baby-step giant-step)

In any cyclic group G, the discrete logarithm problem can be computed in time $O(\sqrt{|G|})$

- Easily parallelizable
- Variants with better space complexity: Pollard's ρ or kangaroos (*a.k.a.* λ) algorithms
- Pohlig-Hellman (1978): $O(\sqrt{p})$ *largest prime divisor of* $|G|$

Choice of G

- $(\mathbb{Z}/n\mathbb{Z}, +)$ or $|G|$ small: easy DLP!
- $(\mathbb{Z}/p\mathbb{Z}^\times, \times)$: usually hard, though not *maximally* hard $\ll \sqrt{p}$
  - record: p of 795 bits, 3100 core-year Boudot *et al.*, 2019
- Points of an elliptic curve over a finite field: maximally hard $O(\sqrt{n})$
  - record: group of 114 bits, 13 days on GPU Zieniewicz & Pons, 2020

Additional remarks

- One should use prime order groups
- Algorithm polynomial in log n on a quantum computer Shor, 1997

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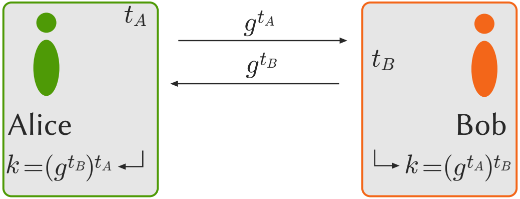
The protocol



Diffie-Hellman protocol

Input : Group G of order n and generator $g \in G$

Alice draws $t_A \twoheadleftarrow \{0, \ldots, n-1\}$, computes $h_A = g^{t_A}$ and sends h_A to Bob

Bob draws $t_B \twoheadleftarrow \{0, \ldots, n-1\}$, computes $h_B = g^{t_B}$ and sends h_B to Alice

Alice computes $k_A = h_B^{t_A}$

Bob computes $k_B = h_A^{t_B}$.

Correctness

The protocol is correct: $k_A = (g^{t_B})^{t_A} = g^{t_A t_B} = (g^{t_A})^{t_B} = k_B$

Freely inspired from: J. Katz, Y. Lindell. Introduction to modern cryptography. 3rd ed, CRC Press, 2021.

Use in practice

Where do the secret lives?

- Shared secret $k \in G$, while usually one needs it in $\{0,1\}^*$
- *Key derivation function* $\mathsf{KDF} : G \to \{0,1\}^*$ $\simeq$ hash function

Person-in-the-middle attack *a.k.a Man-in-the-middle*

- Requires an authentication between Alice and Bob
- Out-of-scope of this lecture → *c.f.* signatures CM6

Cost of the protocol

- Requires two exponentiations in G
- $O(\log |G|)$ operations in G binary powering
- $O(\log^2 p \log\log p)$ bit-operations for $\mathbb{Z}/p\mathbb{Z}$

Binary powering

$$g^t = \begin{cases} g^{\lfloor t/2 \rfloor} \cdot g^{\lfloor t/2 \rfloor} & \text{if } t \text{ is even} \\ g \cdot g^{\lfloor t/2 \rfloor} \cdot g^{\lfloor t/2 \rfloor} & \text{if } t \text{ is odd} \end{cases}$$

```
 Input: $g \in G$, $t \in \mathbb{Z}_{\geq 0}$
Output: $g^t$
$h \leftarrow 1$
while $t \neq 0$:
    if $t$ is odd: $h \leftarrow h \cdot g$
    $g \leftarrow g \cdot g$
    $t \leftarrow \lfloor t/2 \rfloor$
return $h$
```

Complexity

- $O(\log t)$ multiplications in G

Correctness

- Invariant: $h \cdot g^t = g^{t_{init}}$

The Computational Diffie-Hellman (CDH) hypothesis

CDH Game

Challenger simulates the DH protocol → transcript $g, x_1, x_2 \in G$

Adversary is given the transcript and outputs $y \in G$

Success of the adversary if $y = g^{t_1 t_2}$ where $x_1 = g^{t_1}$ and $x_2 = g^{t_2}$

Advantage

- $\mathrm{Adv}_G^{\mathrm{CDH}}(t) = \max_{A_t} \Pr\left[\textsc{success}(A_t)\right]$ where A_t is an algorithm that runs in time $\leq t$

CDH hypothesis: $\mathrm{Adv}_G^{\mathrm{CDH}}(t)$ is *negligible* for *reasonable* t in particular $\ll \sqrt{|G|}$

Remarks

- CDH for $G \Rightarrow$ the discrete log. is *hard* in G *cf contrapositive*
- $g^{t_1 t_2} = (g^{t_1})^{t_2} = (g^{t_2})^{t_1} \neq g^{t_1} g^{t_2}$

The *decisional* Diffie-Hellman (DDH) hypothesis

DDH Game

Challenger simulates the DH protocol $\rightarrow (x_1, x_2, k) \leftarrow (g^{t_1}, g^{t_2}, g^{t_1 t_2})$
draws $b \twoheadleftarrow \{0, 1\}$ and sets $\hat{k} \leftarrow k$ if $b = 1$, $\hat{k} \twoheadleftarrow G$ if $b = 0$

Adversary is given the transcript (g, x_1, x_2) and $\hat{k}$, and outputs b'

Advantages

- $\mathrm{Adv}_G^{\mathrm{DDH}}(A) = |\Pr[b' = 1|b = 1] - \Pr[b' = 0|b = 1]|$
- $\mathrm{Adv}_G^{\mathrm{DDH}}(t) = \max_{A_t} \mathrm{Adv}_G^{\mathrm{DDH}}(A_t)$ where A_t runs in time $\leq t$

DDH hypothesis: $\mathrm{Adv}_G^{\mathrm{DDH}}(t)$ is *negligible* for *reasonable* t in particular $\ll \sqrt{|G|}$

Relation with other hypotheses

DDH for $G \Rightarrow$ CDH for $G \Rightarrow$ hardness of DLP

Security of the Diffie-Hellman protocol

Theorem

If the DDH hypothesis holds for G, then the Diffie-Hellman protocol with group G is IND-EAV secure

Proof. $\mathsf{Adv}^{\mathsf{IND-EAV}}_{\mathsf{DH}(G)}(t) = \mathsf{Adv}^{\mathsf{DDH}}_{G}(t)$ by definition!

Conclusion

50 shades of Diffie-Hellman

- The DH protocol is essentially the only key exchange protocol
- But many choices of cyclic group $G : (\mathbb{Z}/p\mathbb{Z})^\times$, elliptic curve, isogenies, …
- Key derivation function to go from G to $\{0,1\}^*$

Security of the protocol

- Three hypotheses: DLP hardness, CDH, DDH
- DDH $\iff$ IND-EAV security
- DDH $\Rightarrow$ CDH $\Rightarrow$ DLP hardness

Inherent vulnerability: *person-in-the-middle*

- Charlie stands between Alice and Bob, and intercepts, modifies, etc. all messages between Alice and Bob
- Requires *authentication* between Alice and Bob *signatures*