Symmetric Determinantal Representations of Polynomials

Bruno Grenet*†
Joint work with Erich L. Kaltofen‡, Pascal Koiran*† and Natacha Portier*†

*MC2 – LIP, ÉNS Lyon
†Theory Group – DCS, U. of Toronto
‡Dept. of Mathematics – North Carolina State U.

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Motivation from Convex Geometry

- **Linear Matrix Expression (LME):** for $A_i$ symmetric in $\mathbb{R}^{t \times t}$

$$A_0 + x_1 A_1 + \cdots + x_n A_n$$
Motivation from Convex Geometry

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- Lax conjecture: express a real zero polynomial $f$ as

\[ f = \det A \]

with $A$ LME and $A_0 \succeq 0$. 
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- Drop condition $A_0 \succeq 0 \sim\sim$ exponential size matrices
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- What about *polynomial size matrices*?
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- **Drop condition** $A_0 \succeq 0 \sim \text{ exponential size matrices}$

- **What about polynomial size matrices?**

- **Applications to Semi-Definite Programming**
Valiant (1979)

- Arithmetic formula $\leadsto$ Determinant

\[
\begin{vmatrix}
0 & x & 1 \\
x & 0 & 1 \\
1 & 0 & 0
\end{vmatrix}
\begin{vmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1
\end{vmatrix}
= 2x_1 \cdot (x_2 + y) + z \cdot (x_2 + y)
\]
Valiant (1979)

- Arithmetic formula $\Leftrightarrow$ Determinant

$$\begin{bmatrix}
0 & x_1 & 1 & y & x_2 & 0 & 0 & z & 0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix} = 2x_1 \cdot (x_2 + y) + z \cdot (x_2 + y)$$
Valiant (1979)

- Arithmetic formula $\leadsto$ Determinant

\[
\begin{pmatrix}
0 & x_1 & x_1 & 0 & 0 & z & 0 \\
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0 & 0 & 1 & x_2 & y & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 \\
y & 0 & 0 & 0 & 0 & 1 & x_2 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
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Valiant (1979)

- **Arithmetic formula** $\leadsto$ **Determinant**

\[
\begin{pmatrix}
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-1 & 0 & 0 & 0 & 1 & 0 & 0 \\
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-1 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

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\]

- Weakly-skew circuit $\leadsto$ Determinant

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\[
\begin{bmatrix}
0 & y & x_2 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & -1 & x_1 & z & 0 & 0 \\
x_1 & 0 & 0 & 0 & -1 & 0 & 2 \\
x & 0 & 0 & 0 & 0 & -1 & 0 \\
\end{bmatrix}
= 2x_1 \cdot (x_2 + y) + z \cdot (x_2 + y)
\]

- Weakly-skew circuit $\leadsto$ Determinant

\[
\begin{pmatrix}
0 & y & x_2 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & x_1 & z & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

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0 & 0 & 0 & -1 & x_1 & z & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 2 \\
1 & 0 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & -1
\end{pmatrix}
\]

\[
= 2x_1 \cdot (x_2 + y) + z \cdot (x_2 + y)
\]
Strategy

- Valiant’s, Toda’s and Malod’s contructions $\leadsto$ polynomial size matrices

Remark: valid for any field
Strategy

- Valiant’s, Toda’s and Malod’s constructions $\leadsto$ polynomial size matrices
- But nonsymmetric matrices
Valiant’s, Toda’s and Malod’s contructions $\rightsquigarrow$ polynomial size matrices

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- Is is possible to symmetrize their constructions?
Strategy

- Valiant’s, Toda’s and Malod’s contructions $\leadsto$ polynomial size matrices
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(Improved) Valiant’s and Malod’s constructions
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Symmetrization for fields of characteristic $\neq 2$
Contents

- (Improved) Valiant’s and Malod’s constructions
- Symmetrization for fields of characteristic $\neq 2$
- Case of characteristic 2
Valiant’s and Malod’s constructions

Symmetric determinantal representations

Characteristic 2
Graph-theoretic interpretation of determinants

Let $G$ be a graph, $A$ its adjacency matrix
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$$
\det A = \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}
$$
Graph-theoretic interpretation of determinants

- Let $G$ be a graph, $A$ its adjacency matrix.

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\det A = \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}
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- permutation in $A = \text{cycle cover in } G$
Let $G$ be a graph, $A$ its adjacency matrix

\[
\det A = \sum_{\sigma} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}
\]

- permutation in $A =$ cycle cover in $G$
- Up to signs, $\det A =$ sum of the weights of cycle covers in $G$
Valiant’s construction (1/3)

- Input: a formula representing a polynomial $\varphi \in K[X_1, \ldots, X_n]$ of size $e$
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  **Size of a formula**: number of computation gates
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- **Output:** a matrix \( A \) of dimension \((e + 1)\), with entries in 
  \( K \cup \{X_1, \ldots, X_n\} \), s.t. \( \det A = \varphi \)
Valiant’s and Malod’s constructions

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- In between: a graph \( G \) of size \((e + 1)\) whose adjacency matrix is \( A \)
Valiant’s construction (2/3)

Invariant \[ \phi = \pm \sum_{s-t \text{-paths } P} (-1)^{|P|} w(P) \]
Valiant's and Malod's constructions

Valiant's construction (2/3)

$$\varphi = \pm \sum s-t \text{-paths } P (-1)^{|P| w(P)}$$

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Valiant's construction (2/3)

\[ \phi = \pm \sum_{s \rightarrow t} -1^{|P|} w(P) \]

\[ G_1 \]

\[ G_2 \]

\[ s \]

\[ t \]

\[ t_1 \]

\[ t_2 \]

\[ \pm 1 \]
Valiant’s and Malod’s constructions

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Valiant’s construction (3/3)

- $G$ s.t. $\varphi = \pm \sum_{s-t\text{-paths } P} (-1)^{|P|} w(P)$, with $s$, $t$ distinguished.
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$\leadsto G'$: merge $s$ and $t$ + add weight-1 loops on vertices $\neq s$. 

Theorem

For a size-$e$ formula, this construction yields a size-$(e+1)$ graph. Let $A$ be the adjacency matrix of $G$. Then

$$\det(A) = \varphi.$$
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For a size-e formula, this construction yields a size-$(e + 1)$ graph. Let $A$ be the adjacency matrix of $G$. Then $\det(A) = \varphi$. 

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Malod’s construction (1/3)

- Input: a weakly-skew circuit of size $e$ with $i$ variable inputs representing $\varphi$
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$$e = 5 \text{ and } i = 4$$
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- Input: a weakly-skew circuit of size $e$ with $i$ variable inputs representing $\varphi$
- Output: a matrix $A$ of dimension $(e + i + 1)$ s.t. $\det A = \varphi$

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- $\varphi_\alpha$: polynomial computed by gate $\alpha$
- Reusable gate: not in a closed subcircuit

$e = 5$ and $i = 4$
Valiant’s and Malod’s constructions

Malod’s construction (2/3)

For each reusable gate $\alpha$, there exists $t_{\alpha}$ s.t. $w(s \rightarrow t_{\alpha}) = \phi_{\alpha}$.
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Invariant

For each reusable gate $\alpha$, there exists $t_\alpha$ s.t. $w(s \rightarrow t_\alpha) = \phi_\alpha$. 

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- As in Valiant’s, $G \sim G'$: same idea
Malod’s construction (3/3)

- As in Valiant’s, $G \rightsquigarrow G'$: same idea

**Theorem**

*For a ws circuit of size $e$ with $i$ variable inputs representing $\varphi$, this construction yields a size-$(e + i + 1)$. The determinant of its adjacency matrix equals $\varphi$.*
Outline

1. Valiant’s and Malod’s constructions

2. Symmetric determinantal representations

3. Characteristic 2
Introduction

- Symmetric matrices $\iff$ undirected graphs
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Difficulty: no DAG anymore!
Introduction

- Symmetric matrices ⇔ undirected graphs
- Difficulty: no DAG anymore!
- Solution: some changes in the construction, and new invariants
Symmetric determinantal representations

**Introduction**

- Symmetric matrices $\iff$ undirected graphs
- Difficulty: no DAG anymore!
- Solution: some changes in the construction, and new invariants
- N.B.: $\text{char}(\mathbb{K}) \neq 2$ in this section
Case of formulas

\[ \varphi = \sum_{s-t-\text{paths } P} \left| P \right| / 2 + 1 \cdot w(P) \]

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Case of formulas

\[ \varphi = \sum_{s - t \text{-paths } P} \left| P \right| / 2 + w(P) \]
Case of formulas

\[ \phi = \sum_s^{s-t-\text{paths}} P_{-1} |P|/2 + w(P) \]
Case of formulas

\[ \varphi = \sum_{s-t\text{-paths } P} (-1)^{|P|/2+1} w(P) \]

and...

Invariants

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Invariants for formula’s construction

\[ \varphi = \sum_{s-t\text{-paths } P} (-1)^{|P|/2 + 1} w(P) \]
\[ \varphi = \sum_{s-t\text{-paths } P} (-1)^{|P|/2+1} w(P) \]

- \( |G| \) is even, every cycle in \( G \) is even, and every \( s-t \)-path is even

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Invariants for formula’s construction

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- \(|G|\) is even, every cycle in \(G\) is even, and every \(s-t\)-path is even
- \(G \setminus \{s, t\}\) is either empty or has a unique cycle cover
Invariants for formula’s construction

\[ \varphi = \sum_{s-t\text{-paths } P} (-1)^{|P|/2+1} w(P) \]

- \(|G|\) is even, every cycle in \(G\) is even, and every \(s-t\)-path is even
- \(G \setminus \{s, t\}\) is either empty or has a unique cycle cover

\[ \Rightarrow \text{Perfect matching of weight 1} \]
Invariants for formula’s construction

- $\varphi = \sum_{s-t\text{-paths } P} (-1)^{|P|/2+1} w(P)$
- $|G|$ is even, every cycle in $G$ is even, and every $s-t$-path is even
- $G \setminus \{s, t\}$ is either empty or has a unique cycle cover

$\Rightarrow$ Perfect matching of weight 1

- For any $s-t$-path $P$, $G \setminus P$ is either empty or has a unique cycle cover
\[ \varphi = \sum_{s-t\text{-paths } P} (-1)^{|P|/2+1} w(P) \]

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- For any \(s-t\)-path \(P\), \(G \setminus P\) is either empty or has a unique cycle cover

\[ \Rightarrow \] Perfect matching of weight 1
From $G$ to $G'$

The determinant of its adjacency matrix equals $\phi$.

Theorem
For a formula $\phi$ of size $e$, this construction yields a graph of size $2^e + 3$.
From $G$ to $G'$

- $|G'|$ is odd. An odd cycle in $G'$ has to go through $c$
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- $|G'|$ is odd. An odd cycle in $G'$ has to go through $c$
- Cycle covers in $G'$ $\iff$ $s \rightarrow t$-paths in $G$

\[ (-1)^{|G'|/2 + 1} \]

\[ \frac{1}{2} \]
From $G$ to $G'$

- $|G'|$ is odd. An odd cycle in $G'$ has to go through $c$
- Cycle covers in $G'$ $\iff$ $s \to t$-paths in $G$ $\iff$ $t \to s$-paths in $G$

$$(-1)^{|G'|/2} + 1$$
$|G'|$ is odd. An odd cycle in $G'$ has to go through $c$

Cycle covers in $G' \iff s \to t$-paths in $G \iff t \to s$-paths in $G$

$(-1)^{|G/2|+1}$ ensures that the signs are OK.
From $G$ to $G'$

- $|G'|$ is odd. An odd cycle in $G'$ has to go through $c$.
- Cycle covers in $G' \iff s \to t$-paths in $G \iff t \to s$-paths in $G$.
- $(-1)^{|G|/2} + 1$ ensures that the signs are OK.
- $1/2$: to deal with $s \to t$ and $t \to s$-paths, implies char($\mathbb{K}$) $\neq 2$. 

Theorem

For a formula $\phi$ of size $e$, this construction yields a graph of size $2^e + 3$.

The determinant of its adjacency matrix equals $\phi$. 

From $G$ to $G'$

- $|G'|$ is odd. An **odd cycle** in $G'$ has to go through $c$.
- Cycle covers in $G'$ $\iff$ $s \to t$-paths in $G$ $\iff$ $t \to s$-paths in $G$.
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**Theorem**

*For a formula $\varphi$ of size $e$, this construction yields a graph of size $2e + 3$. The determinant of its adjacency matrix equals $\varphi$.***
Case of weakly-skew circuits

Main difficulty:
Case of weakly-skew circuits

- Main difficulty:

- Definition: an path $P$ is said **acceptable** if $G \setminus P$ admits a cycle cover
Constructions
Constructions

\[
\begin{align*}
\text{Symmetric determinantal representations} \\
\text{Constructions} \\
\end{align*}
\]
Constructions
For each reusable $\alpha$, there exists $t_\alpha$ s.t.
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$$\varphi_\alpha = \sum_{s \text{-t}_\alpha \text{-paths } P} (-1)^{|P|-1} w(P)$$

- Every $s$-$t_\alpha$-path is odd
- For a $s$-$t_\alpha$-path $P$, $G \setminus P$ is either empty or has a unique cycle cover $\Rightarrow$ Perfect matching of weight 1
- $|G|$ is odd, every cycle in $G$ is even
- $G\{s\}$ is either empty or has a unique cycle cover $\Rightarrow$ Perfect matching of weight 1
For each reusable $\alpha$, there exists $t_{\alpha}$ s.t.

\[
\varphi_{\alpha} = \sum_{\text{acceptable}\ s-t_{\alpha}-paths\ P} (-1)^{\frac{|P|-1}{2}} w(P)
\]

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$\Rightarrow$ Perfect matching of weight 1
Invariants in the case of weakly-skew circuits

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Invariants in the case of weakly-skew circuits

- For each reusable $\alpha$, there exists $t_\alpha$ s.t.
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Invariants in the case of weakly-skew circuits

- For each reusable $\alpha$, there exists $t_\alpha$ s.t.
  \[ \varphi_\alpha = \sum_{\text{acceptable } s-t_\alpha\text{-paths } P} (-1)^{|P|-1} w(P) \]
  ▶ Every $s-t_\alpha\text{-path}$ is odd
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    ~ Perfect matching of weight 1

- $|G|$ is odd, every cycle in $G$ is even
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  ~ Perfect matching of weight 1
From $G$ to $G'$

- Add an edge between $s$ and $t$, of weight $\frac{1}{2}(-1)^{\frac{|G|-1}{2}} \leadsto G'$.
From $G$ to $G'$

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- $|G' \setminus \{s, t\}|$ is odd, cycles are even: no cycle cover with $s \leftrightarrow t$. 
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- As for every path $P$, $G' \setminus P$ has an only cycle cover, of weight 1:
From $G$ to $G'$

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  Cycle covers of $G'$ $\iff$ $s \to t$-paths in $G$ $\iff$ $t \to s$-paths in $G$. 

With some sign considerations, we get:

Theorem

For a weakly skew circuit of size $e$, with $i$ input variables, computing a polynomial $\phi$, this construction yields a graph $G'$ with $2(e+i)+1$ vertices.

The adjacency matrix of $G'$ has its determinant equal to $\phi$. 

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From $G$ to $G'$

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- $|G' \setminus \{s, t\}|$ is odd, cycles are even: no cycle cover with $s \leftrightarrow t$.
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  Cycle covers of $G'$ $\iff$ $s \rightarrow t$-paths in $G$ $\iff$ $t \rightarrow s$-paths in $G$.
- With some sign considerations, we get:
From $G$ to $G'$

- Add an edge between $s$ and $t$, of weight $\frac{1}{2}(-1)^{\frac{|G|-1}{2}} \sim G'$.
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Theorem

For a weakly skew circuit of size $e$, with $i$ input variables, computing a polynomial $\varphi$, this construction yields a graph $G'$ with $2(e + i) + 1$ vertices. The adjacency matrix of $G'$ has its determinant equal to $\varphi$. 
Outline

1. Valiant’s and Malod’s constructions
2. Symmetric determinantal representations
3. Characteristic 2
Introduction

- Scalar $1/2$ in the constructions $\Rightarrow$ not valid for characteristic 2
Introduction

- Scalar 1/2 in the constructions $\implies$ not valid for characteristic 2
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Which polynomials can be represented as determinant of symmetric matrices in characteristic 2?
A positive result

**Theorem**

Let $p$ be a polynomial, represented by a weakly-skew circuit of size $e$ with $i$ input variables. Then there exists a symmetric matrix $A$ of size $2(e + i) + 2$ such that $p^2 = \det A$. 
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- Cycle Covers in \( G \) ⇐⇒ Perfect Matching in \( G' \)

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\det M = \sum_{\mu} w(\mu) \quad (\mu \text{ ranges over the Perfect Matchings})
\]

- As there is no loop in \( G' \), \( \det A = \sum_{\mu} w(\mu)^2 = \left( \sum_{\mu} w(\mu) \right)^2 \)
This result raises the question:
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If $p^2$ has a small weakly-skew circuit, what about $p$?

In technical terms:

If $f$ is a family of polynomials s.t. $f^2 \in \text{VP}$, does $f$ belong to $\text{VP}$?

It appears to be related to an open problem of Bürgisser:

Is the partial permanent VNP-complete in characteristic 2?
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Valiant’s classes

- Complexity of a polynomial: size of the smallest circuit computing it.
Valiant’s classes

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**Definition**

A family \((f_n)\) of polynomials is in VP if for all \(n\), the number of variables, the degree, and the complexity of \(f_n\) are polynomially bounded in \(n\).
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A family \((f_n)\) of polynomials is in VNP if there exists a family \((g_n(y_1, \ldots, y_{v(n)})) \in VP\) s.t.

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f_n(x_1, \ldots, x_{u(n)}) = \sum_{\bar{c} \in \{0,1\}^{v(n)-u(n)}} g_n(x_1, \ldots, x_{u(n)}, \bar{c}).
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- \((\text{DET}_n) \in VP, (\text{PER}_n) \in VNP, \ldots\)
VNP-completeness

**Definition**

A family $(g_n)$ is a $p$-projection of a family $(f_n)$ is there exists a polynomial $t$ s.t. for all $n$, $g_n(\bar{x}) = f_{t(n)}(a_1, \ldots, a_n)$, with $a_1, \ldots, a_n \in \mathbb{K} \cup \{x_1, \ldots, x_n\}$. 
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- \((PER_n)\) is VNP-complete in characteristic \(\neq 2\)
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- \((\text{PER}_n)\) is VNP-complete in characteristic \(\neq 2\)
- \((\text{HC}_n)\) is VNP-complete (in any characteristic)
Partial Permanent

\[ \text{per}^* M = \sum_{\pi} \prod_{i \in \text{def}(\pi)} M_{i, \pi(i)} \]

where \( \pi \) ranges over the injective partial maps from \([n]\) to \([n]\).
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Lemma

Let \( G = K_{n,n} \). Let \( A \) and \( B \) be the respective adjacency and biadjacency matrices of \( G \). Then in characteristic 2,

\[ \det(A + I_{2n}) = (\text{per}^* B)^2 \]

where \( I_{2n} \) is the identity matrix.
Characteristic 2

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Same kind of ideas as the previous proof.
Partial permanents as family of polynomials

\( (\text{PER}_n^*) \): family of polynomials defined as partial permanents of \( n \times n \) matrices of indeterminates.
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\((\text{PER}^*_n)\): family of polynomials defined as partial permanents of \(n \times n\) matrices of indeterminates.

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\(((\text{PER}^*)_n^2)\): family of polynomials defined as \textit{square of partial permanents} of \(n \times n\) matrices of indeterminates.

**Theorem**

\(((\text{PER}^*)_n^2) \in \text{VP} \text{ in characteristic } 2.\)
Partial permanents as family of polynomials

$(\text{PER}_n^*)$: family of polynomials defined as partial permanents of $n \times n$ matrices of indeterminates.

$((\text{PER}_n^*)^2)$: family of polynomials defined as square of partial permanents of $n \times n$ matrices of indeterminates.

**Theorem**

$((\text{PER}_n^*)^2) \in \text{VP}$ in characteristic 2.

**Proof.** $((\text{PER}_n^*)^2)$ is a $p$-projection of $(\text{DET}_n)$.
Answer to Bürgisser’s problem

Problem

Is the partial permanent VNP-complete in characteristic 2?
Answer to Bürgisser’s problem

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Is the partial permanent VNP-complete in characteristic 2?

Theorem
If it is the case, $\oplus \mathbb{P}/\text{poly} = \text{NC}^2/\text{poly}$, and $\text{PH} = \Sigma_2$. 

Proof sketch.
If the case arises, $\text{VNP}^2 \subseteq \text{VP}$. This translates into boolean complexity result via Bürgisser’s boolean parts of Valiant’s classes.
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A negative result?

Question
Which polynomials can be represented as determinant of symmetric matrices in characteristic 2?
A negative result?

**Question**
Which polynomials can be represented as determinant of symmetric matrices in characteristic 2?

**Conjecture**
The polynomial $xy + z$ has no such representation

**Two-day-old Proof.** To do on a board!
We obtained **Symmetric Determinantal Representations** for Formulas and Weakly-Skew Circuits of **linear size**
Conclusion

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**Theorem**

Let $M$ be an $n \times n$ matrix. Then there exists a symmetric matrix $M'$ of size $O(n^5)$ s.t. $\det M = \det M'$.

For characteristic 2:

- Answer to Bürgisser’s Open Problem
- Proof (?) of a negative result (to be verified...)

Bruno Grenet (LIP – ÉNS Lyon)
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Future work

- In Convex Geometry: $\mathbb{K} = \mathbb{R}$ and polynomials are *real zero polynomials*. 
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$\Rightarrow$ what can be done in that precise case?

- Characterize polynomials with a symmetric determinantal representation in characteristic 2.

- Symmetric matrices in Valiant’s theory?
Thank you!