Symmetric Determinantal Representations of Weakly-Skew Circuits

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Introduction

The problem

\((x + y) + (y \times z) = \det\)

\[
\begin{vmatrix}
0 & x & y & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} \\
x & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & z & 0 & 0 & 0 \\
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- Formal polynomial
(x + y) + (y \times z) = \text{det}

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\end{vmatrix}
\]

- Formal polynomial
- Smallest possible dimension of the matrix
Introduction

Representations of polynomials

Arithmetic circuit:

Size $e = 5$
Inputs $i = 2$
Introduction

Representations of polynomials

Weakly-skew circuit:

Size $e = 5$

Inputs $i = 4$
Representations of polynomials

Formula:

Size $e = 5$
Inputs $i = 6$
**Introduction**

**Motivation**

L. G. Valiant, *Completeness classes in algebra*, STOC’79

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**Theorem (Universality of determinant and permanent)**

Let $P$ be a polynomial given by a formula of size $e$. There exist matrices $M$ and $N$ of size $(e + 2) \times (e + 2)$ such that

$$P = \det M = \text{per} N.$$
Subsequent works

- Improved bounds:
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  - $2e + 2$: J. von zur Gathen [1]

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- Extension to weakly-skew circuits, with bound

Introduction

Subsequent works

- **Improved bounds:**
  - $2e + 2$: J. von zur Gathen [1]

- Extension to *weakly-skew circuits*, with bound
  - $2e + 1$: S. Toda [3]

Subsequent works

- Improved bounds:
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- Extension to weakly-skew circuits, with bound
  - $2e + 1$: S. Toda [3]
  - $e + i + 1$: G. Malod & N. Portier [4]

Our results

- Extension to *symmetric matrices* (characteristic \( \neq 2 \))
Extension to symmetric matrices (characteristic $\neq 2$)

Char. 2: Partial permanent is (probably) not VNP-complete
Motivation from Convex Geometry

- **Linear Matrix Expression (LME):** for $A_i$ symmetric in $\mathbb{R}^{t \times t}$

\[ A_0 + x_1 A_1 + \cdots + x_n A_n \]
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- Lax conjecture: express a **real zero polynomial** $f$ as
  \[ f = \det A \]

with $A$ LME and $A_0 \succeq 0$. 
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- **Drop condition $A_0 \succeq 0$ $\supseteq$ exponential size matrices**
Motivation from Convex Geometry

- **Linear Matrix Expression (LME):** for $A_i$ symmetric in $\mathbb{R}^{t \times t}$

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  with $A$ LME and $A_0 \succeq 0$. $\leadsto$ disproved

- Drop condition $A_0 \succeq 0$ $\leadsto$ exponential size matrices

- What about polynomial size matrices?
Main construction

Overview

\[(x + y) + (y \times z)\]

Circuit: Weakly-skew circuit or formula
Main construction

Overview

Circuit: Weakly-skew circuit or formula
Main construction

Overview

Arithmetic Branching Program

Circuit $\Rightarrow$ ABP
Main construction

Overview

\[ \text{Circuit} \implies \text{ABP} \]
Main construction

Overview

\[
\begin{align*}
-1 & & -1 & & -1/2 \\
-1 & & -1 & & -1/2 \\
-1 & & -1 & & -1/2 \\
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\end{align*}
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Circuit \(\Rightarrow\) ABP \(\Rightarrow\) Graph
Main construction

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\[= (x + y) + (y \times z)\]

Circuit \[\Rightarrow\] ABP \[\Rightarrow\] Graph \[\Rightarrow\] Matrix
Main construction

Overview

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\end{vmatrix}
= (x + y) + (y \times z)
\]

Characteristic $\neq 2$

Circuit $\implies$ ABP $\implies$ Graph $\implies$ Matrix
Main construction

Main new difficulty

Symmetric matrices
Main construction

Main new difficulty

Symmetric matrices $\implies$ undirected graphs
Main construction

Main new difficulty

Symmetric matrices
\[ \Rightarrow \] undirected graphs
\[ \Rightarrow \] “undirected ABPs”
Main construction

Main new difficulty

Symmetric matrices
\[ \implies \text{undirected graphs} \]
\[ \implies \text{“undirected ABPs”} \]

Definition
A path $P$ is **acceptable** if $G \setminus P$ admits a cycle cover
Main construction

Weakly-Skew Circuit $\Rightarrow$ ABP

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Main construction

Weakly-Skew Circuit $\Rightarrow$ ABP

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Weakly-Skew Circuit $\implies$ ABP
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Weakly-Skew Circuit $\Rightarrow$ ABP
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Weakly-Skew Circuit $\implies$ ABP
Main construction

$\text{ABP} \iff \text{Graph}$

Add $s \xleftarrow{(1/2) \cdot (-1)^{\frac{|G|-1}{2}}} t$: new graph $G'$. 

**Diagram:**

- Graph with nodes $s$, $t$, $x$, $y$, $z$ with weights $-1$, $-1$, $-1$, $-1/2$.
- Edges with weights $-1$, $-1$, $-1$, $-1$.
Main construction

ABP $\implies$ Graph

- Add $s \xleftarrow{(1/2) \cdot (-1)^{\frac{|G|-1}{2}}} t$: new graph $G'$.
- Cycle covers of $G'$
  \[\iff s \rightarrow t\text{-paths in } G\]
Main construction

ABP $\implies$ Graph

- Add $s \leftarrow \frac{1}{2} \cdot \frac{|G| - 1}{2} \rightarrow t$: new graph $G'$.
- Cycle covers of $G'$
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Main construction

Graph $\implies$ Matrix

\[ s \]

\[ x, y \]

\[ z \]

\[ t \]

\[ -1 \]

\[ -1 \]

\[ -1 \]

\[ -1/2 \]

\[ -1 \]

\[ -1 \]

\[ -1 \]

\[ -1 \]

\[ \textbf{Determinant} \]

\[ S_n = \text{Permutation group of } \{1, \ldots, n\} \]

\[ \det A = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{n} A_{i, \sigma(i)} \]
Main construction

Graph $\implies$ Matrix

**Determinant**

$\mathfrak{S}_n = \text{Permutation group of } \{1, \ldots, n\}$

$$\det A = \sum_{\sigma \in \mathfrak{S}_n} (-1)^{\text{sgn}(\sigma)} \prod_{i=1}^{n} A_{i,\sigma(i)}$$

- permutation in $A \equiv$ cycle cover in $G'$
Main construction

Graph $\Longrightarrow$ Matrix

Determinant

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$$
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$$

- permutation in $A \equiv$ cycle cover in $G'$
- Up to signs, $\det A =$ sum of weights of cycle covers in $G'$
Main construction

Summary

\[ P(x_1, \ldots, x_n) \]  Weakly-Skew Circuit

\[ \text{Arithmetic Branching Program} = \sum_{\text{cycle cover}} C^{(j)} \cdot \text{sgn}(C) \cdot w(C) \]

\[ \text{Graph } G' = \det \text{Adj}(G') \]  Symmetric Matrix

Weakly-skew circuit

Non symmetric

\[ e + 1 + (e+i) + 1 \]

Symmetric

\[ 2e + 1 + 2(e+i) + 1 \]
Main construction

Summary

\[ P(x_1, \ldots, x_n) = \sum_{s-t \text{ path } P} (-1)^{|P| - 1} \frac{|P| - 1}{2} w(P) \]

Weakly-Skew Circuit

Arithmetic Branching Program
$P(x_1, \ldots, x_n)$

$= \sum_{s-t \text{ path } P} (-1)^{\frac{|P|-1}{2}} w(P)$

$= \sum_{\text{cycle cover } C} (-1)^{\text{sgn}(C)} w(C)$

Weakly-Skew Circuit

Arithmetic Branching Program

Graph $G'$
Summary

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Symmetric Matrix

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Symmetric Matrix

<table>
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<tr>
<th>Formula</th>
<th>Weakly-skew circuit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Non symmetric</td>
<td>( e + 1 )</td>
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Bruno Grenet – Symmetric Determinantal Representations of Weakly-Skew Circuits
Problem [Bürgisser 00]

Is the partial permanent VNP-complete in characteristic 2?
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Is the partial permanent VNP-complete in characteristic 2?

\[ \mathcal{P}_n = \text{Injective Partial Maps from } \{1, \ldots, n\} \text{ to itself} \]

\[ \text{per}^* M = \sum_{\pi \in \mathcal{P}_n} \prod_{i \in \text{def}(\pi)} M_{i,\pi(i)} \]
Characteristic 2

Problem

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- Injective Partial Maps \(\equiv\) Partial Matchings in a Bipartite Graph
Problem [Bürgisser 00]

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- Injective Partial Maps $\equiv$ Partial Matchings in a Bipartite Graph
- VP, VNP, VNP-complete $\equiv$ P, NP, NP-complete for polynomials
Is the partial permanent VNP-complete in characteristic 2?
Is the partial permanent VNP-complete in characteristic 2?

**Theorem**

No unless the *Polynomial Hierarchy collapses*. 

---

Bruno Grenet – Symmetric Determinantal Representations of Weakly-Skew Circuits
Is the partial permanent VNP-complete in characteristic 2?

**Theorem**

No unless the *Polynomial Hierarchy collapses.*

**Main lemma**

\[(\text{per}^* M)^2 \in \text{VP}\]
**Theorem**

Let $M$ be an $n \times n$ matrix. Then there exists a symmetric matrix $M'$ of size $O(n^3)$ s.t. $\det M = \det M'$. 
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Theorem (G., Monteil, Thomassé)

In characteristic 2, Symmetric Determinantal Representations do not always exist.
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Theorem (G., Monteil, Thomassé)

In characteristic 2, Symmetric Determinantal Representations do not always exist.

Theorem (Malod)

In characteristic 2, the partial permanent is in VP.
Conclusion

Summary & Future Work

- Symmetric Determinantal Representations of linear size
Conclusion

Summary & Future Work

- Symmetric Determinantal Representations of linear size
- Characteristic 2: Partial answer to Bürgisser’s Open Problem
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- Symmetric Determinantal Representations of linear size
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- Convex Geometry: $\mathbb{K} = \mathbb{R}$ and real zero polynomials
Summary & Future Work

- Symmetric Determinantal Representations of linear size
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- Convex Geometry: $\mathbb{K} = \mathbb{R}$ and real zero polynomials
  - what can be done in that precise case?
Summary & Future Work

- Symmetric Determinantal Representations of **linear size**
- Characteristic 2: Partial answer to Bürgisser’s Open Problem

**Convex Geometry:** $\mathbb{K} = \mathbb{R}$ and **real zero** polynomials

~~ what can be done in that precise case?~~

- Characteristic 2:
Symmetric Determinantal Representations of linear size

Characteristic 2: Partial answer to Bürgisser’s Open Problem

Convex Geometry: $\mathbb{K} = \mathbb{R}$ and real zero polynomials

what can be done in that precise case?

Characteristic 2:

- Characterize polynomials with a Symmetric Determinantal Representation
Summary & Future Work

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Convex Geometry: $\mathbb{K} = \mathbb{R}$ and real zero polynomials

What can be done in that precise case?

- Characteristic 2:
  - Characterize polynomials with a Symmetric Determinantal Representation
  - Explore other graph polynomials
Symmetric Determinantal Representations of \textit{linear size}

Characteristic 2: Partial answer to Bürgisser’s Open Problem

Convex Geometry: $\mathbb{K} = \mathbb{R}$ and real zero polynomials

\implies \text{what can be done in that precise case?}

Characteristic 2:

- Characterize polynomials with a Symmetric Determinantal Representation
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Symmetric matrices in Valiant’s theory?
Thank you!
1 Introduction
2 Main construction
3 Characteristic 2
4 Conclusion