Aggregation of Uncertain Qualitative Preferences for a Group of Agents

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Abstract. We consider aggregation of partially known qualitative preferences for a group of agents, considering necessary and potentially optimal choices with respect to different notions of optimality (consensus, extreme choices, Pareto optimality) and provide a theoretical characterization. We report statistics (obtained with simulations with synthetic data) about the cardinality of the sets of possible and necessarily optimal choices for the different cases. Finally we introduce preliminary ideas on a qualitative notion of \textit{fairness} and on interactive elicitation.

1 Introduction

In this paper we consider aggregation of partially known qualitative preferences of different agents. By qualitative we mean that preferences are explicitly and directly encoded by binary relations \textsuperscript{[6,11]}. We are in a setting of strict uncertainty, meaning that no probabilistic assumption is made. Preference information is incomplete, meaning that we only know a fraction of the binary preferences. We assume that any consistent extension of the currently known preference relations is considered possible. In particular, we consider possible and necessary Pareto optimality and study the relations with other notions of optimality.

This work is an effort towards the direction of effective, practical methods for preference assessment (preference elicitation) with purely qualitative statements. Our work is similar to \textsuperscript{[4]}; however our setting differs since we assume qualitative preferences expressed as orders (while they assume numeric utility functions).

2 Model

General Assumptions. We assume a set $\mathcal{C}$ of $m$ items or elements (choices) and a set $\mathcal{G} = \{1, \ldots, n\}$ of $n$ agents; preferences are explicitly modeled by binary relations. For each agent the preference relation $\succeq_i$ (for each agent $i$) models the fact that, if $o \succeq_i o'$ holds, option $o$ is at least as preferred as option $o'$ for agent $i$. As usual \textsuperscript{[5]} a preference relation $\succeq_i$ induces a preference structure $(\succ_i, \approx_i, \sim_i)$ for each agent, where $\succ_i$ is the strict preference relation, $\approx_i$ is the indifference relation and $\sim_i$ is the incomparability relation: $o \succ_i o'$ if $o \succeq_i o'$ and $o' \not\succeq_i o$, $o \approx_i o'$ if $o \succeq_i o'$ and $o' \succeq_i o$ and $o \sim_i o'$ if $o \not\succeq_i o'$ and $o' \not\succeq_i o$. 

For a given relation $\succeq$, we use the operator top to denote the maximum element (if it exists):
\[
\text{top}(\succeq) = \{ o \in C | \forall o' \in C, o' \neq o : o \succ o' \}.
\] (2.1)

Notice that the cardinality of $\text{top}(\succeq)$ is at most 1. The operator max denotes maximal elements:
\[
\text{max}(\succeq) = \{ o \in C | \nexists o' \in C : o' \succ o \}.
\] (2.2)

Obviously it holds $\text{top}(\succeq) \subseteq \text{max}(\succeq)$. If the preferences of an agent over the available items constitute a chain (i.e. $\succeq_i$ is a total order), then his optimal option is $\text{top}(\succeq_i) = \text{max}(\succeq_i)$.

As usual in multi-agent systems and multi-objective decision analysis, an option $o$ dominates another option $o'$ if it holds $o \succeq_i o'$ for all $i$, and for at least one $j$ it holds $o \succ_j o'$ (strict preference); the Pareto optimal choices for the group of agents are the those choices that are not dominated by any other choice. We can conveniently express Pareto Optimality using the notion of aggregate group dominance relation $\succeq_G$, defined as $o \succeq_G o'$ iff $\forall i \in G, o \succeq_i o'$. A consensus choice, if it exists, is an option that is the best preferred item for all agents.

\[
\text{Cons}(\succeq_1, ..., \succeq_n) = \text{top}(\succeq_G) = \text{top}(\bigcap_{i \in G} \succeq_i).
\] (2.3)

By representing relations in their extensive form as subset of the Cartesian product $C \times C$, the group dominance relation can be conveniently written as $\succeq_G = \bigcap_{i} \succeq_i$. Then, Pareto Optimal choices with respect to $\succeq_G$ are the maximal elements of the aggregated group preference order:

\[
\text{ParetoOpt}(\succeq_1, ..., \succeq_n) = \text{max}(\succeq_G) = \text{max}(\bigcap_{i \in G} \succeq_i).
\] (2.4)

A particular kind of solutions (that are also Pareto Optimal) are those that are extreme in the sense that they are the best choice for one (or more) agents but might not be good choices for the other agents. The set of these extreme choices is

\[
\text{Ext}_G(\succeq_1, ..., \succeq_n) = \bigcup_{i \in G} \text{top}(\succeq_i \cap G).
\] (2.5)

**Partial Knowledge of Preference Orders.** We suppose that only partial information about the agents’ preferences (the order relation) is known. We consider that only a subset of each preference order is known, meaning that we are aware of some pairwise preferences but not others. Since we assume a context where preferences are transitive, we assume that also the known preference relation is transitive, i.e. it is a partial order$^1$. Let $\succeq_i^*$ be the true preference order for

$^1$ If preferences are given in terms of pairwise comparison statements, we consider the transitive closure of the binary relation containing all such pairwise comparisons.
agent $i$ (unknown to the system); the knowledge about the agent’s preferences is encoded by a partial order $\succeq_i \subseteq \succeq_i^*$ for each agent $i$. While the true maximal elements of each agent $i$ are $\max(\succeq_i^*)$, we can only deduce a set $\max(\succeq_i)$ of “current” maximal items. Of course, current maximal items might not be maximal in the “true” complete preference model. It holds $\max(\succeq_i^*) \subseteq \max(\succeq_i)$, since a maximal element according to the $\succeq_i^*$ must be also maximal for the “sparser” relation $\succeq_i$.

From these partial orders, we consider the “known” aggregate relation of dominance for group $G$, $\succeq_G = \bigcap_{i \in G} \succeq_i$. Notice that $\succeq_G$ is also a subset of $\succeq_G^*$, as it is the intersection of partial orders that are included in the underlying true orders: $\succeq_G \subseteq \succeq_G^*$ as $\succeq_G = \bigcap_{i \in G} \succeq_i \subseteq \bigcap_{i \in G} \succeq_i^* = \succeq_G^*$.

The true Pareto optimal choices are $\text{ParetoOpt}(\succeq_1^*, ..., \succeq_n^*)$ and the extrema $\text{Ext}_G(\succeq_1, ..., \succeq_n)$ (one simply substitute $\succeq_i$ with $\succeq_i^*$ in Equation 2.4 and in Equation 2.5); however we only know the “current” Pareto Optimal choices are $\text{ParetoOpt}(\succeq_1, ..., \succeq_n)$ and the current extrema choices are $\text{Ext}_G(\succeq_1, ..., \succeq_n)$.

When considering aggregation, the ideal case (but usually rare in practice) is that all agents agree on the best option (consensus); the agents share a common maximum element. If we have incomplete knowledge about the individual preferences, we might wonder if, given the current information, a consensus might exist. Hence, we define the notion of possible and necessary consensus for a group of agents, and similarly for Pareto Optimal and Extreme choice.

### 3 Consensus, Extreme and Pareto Optimal Items: The Possible and the Necessary

**Definition 1.** Given $(\succeq_1, ..., \succeq_n)$, the possible consensus choices $\text{PossCons}_G$ for group $G$ are those for which there exists a set of total orders $(\succeq_1', ..., \succeq_n')$, with each $\succeq_i'$ extending $\succeq_i$, such that they are a maximum element of the derived aggregate dominance relation $\succeq_G' = \bigcap_i \succeq_i'$.

$$\text{PossCons}_G(\succeq_1, ..., \succeq_n) = \{ o \in C | \exists (\succeq_1', ..., \succeq_n'): (\succeq_1' \supseteq \succeq_1), ..., (\succeq_n' \supseteq \succeq_n) \land o \in \text{Cons}(\succeq_1', ..., \succeq_n') \} = \bigcup_{\succeq_i' \supseteq \succeq_i, \forall i \in G} \text{top}(\bigcap_i \succeq_i') = \bigcup_{\succeq_i' \supseteq \succeq_i, \forall i \in G} \text{top}(\succeq_G') \text{.} \quad (3.1)$$

The necessary consensus choices $\text{NecCons}_G$ for group $G$ are those that are a maximum element for all complete orders extending $\succeq_G$.

$$\text{NecCons}_G(\succeq_1, ..., \succeq_n) = \{ o \in C | \forall (\succeq_1', ..., \succeq_n'): (\succeq_1' \supseteq \succeq_1), ..., (\succeq_n' \supseteq \succeq_n) \land o \in \text{top}(\succeq_G') \} = \bigcap_{\succeq_i' \supseteq \succeq_i, \forall i \in G} \text{top}(\succeq_G') \text{.} \quad (3.4)$$
Proposition 1. Necessary and possible consensus for $\mathcal{G}$ can be formulated as:

1. NecCons$_{\mathcal{G}} = \bigcap_{i \in \mathcal{G}} \text{top}(\succeq_i)$
2. PossCons$_{\mathcal{G}} = \bigcap_{i \in \mathcal{G}} \text{max}(\succeq_i)$

Proof. 1) NecCons$_{\mathcal{G}}$: If a choice belongs to $\bigcap_i \text{top}(\succeq_i)$ then (by definition) it means that it dominates all other choices for each agent $i$; it will still be the maximum in any extension of the preference relations. Moreover, if an option is dominated in a preference relation $\succeq_i$, it will still be dominated in any extension. Therefore NecCons$_{\mathcal{G}}$ is exactly the intersection of all $\text{top}(\succeq_i)$.

2) PossCons$_{\mathcal{G}}$: If an option $o_1$ is dominated wrt agent $i$ (e.g. there is a choice $o_2$ such that $o_2 \succ_i o_1$), it will also be dominated in any extension of $\succ_i$, and cannot be a possible consensus. Therefore only options that are maximal elements for all agents can potentially be a consensus. Consider an option $o \in \bigcap_i \text{max}(\succeq_i)$; we construct an extension of the preference order by breaking all incomparabilities in favor of $o$. Since $o$ is a consensus in the extension, the argument follows.

While theoretically interesting, the concept of possible/necessary consensus can be of limited interest, as frequently agents will have conflicting preferences. We therefore consider weaker notions of optimality.

Definition 2. The possible extreme choices PossExt$_{\mathcal{G}}$ for group $\mathcal{G}$ are those for which there is a set of total orders $(\succeq'_1, \ldots, \succeq'_n)$, with each $\succeq'_i$ extending $\succeq_i$, for which they are an extreme choice.

$$\text{PossExt}_\mathcal{G} = \{ o \mid \exists (\succeq'_1, \ldots, \succeq'_n) : (\succeq'_1 \succeq \succeq_1), \ldots, (\succeq'_n \succeq \succeq_n) \land o \in \text{Ext}(\succeq'_1, \ldots, \succeq'_n) \} \quad (3.7)$$

The necessary extreme choices NecExt$_{\mathcal{G}}$ for group $\mathcal{G}$ are those that are extreme for all total orders $(\succeq'_1, \ldots, \succeq'_n)$, with each $\succeq'_i$ extending $\succeq_i$.

$$\text{NecExt}_\mathcal{G} = \{ o \mid \forall (\succeq'_1, \ldots, \succeq'_n) : (\succeq'_1 \succeq \succeq_1), \ldots, (\succeq'_n \succeq \succeq_n), o \in \text{Ext}(\succeq'_1, \ldots, \succeq'_n) \} \quad (3.8)$$

Proposition 2. Necessary extreme and possible extreme choices for a group $\mathcal{G}$ can be rewritten as follows:

- NecExt$_{\mathcal{G}} = \bigcup_{i \in \mathcal{G}} \text{top}(\succeq_i)$
- PossExt$_{\mathcal{G}} = \bigcup_{i \in \mathcal{G}} \text{max}(\succeq_i)$

We now consider potential and necessary Pareto optimal choices.

Definition 3. A choice is a possible Pareto optimal for group $\mathcal{G}$ if there exists a set of total orders $(\succeq'_1, \ldots, \succeq'_n)$, with each $\succeq'_i$ extending $\succeq_i$, for which they are a Pareto optimal choice.

$$\text{PossParetoOpt}_{\mathcal{G}} = \{ o \mid \exists (\succeq'_1, \ldots, \succeq'_n) : (\succeq'_1 \succeq \succeq_1), \ldots, (\succeq'_n \succeq \succeq_n) \land o \in \max(\bigcap_{i \in \mathcal{G}} \succeq'_i) \}$$

A choice is a necessary Pareto optimal for group $\mathcal{G}$ if it is a maximal element of the aggregate preference relation with respect to all extensions of the current preference orders.

$$\text{NecParetoOpt}_{\mathcal{G}} = \{ o \mid \forall (\succeq'_1, \ldots, \succeq'_n) : (\succeq'_1 \succeq \succeq_1), \ldots, (\succeq'_n \succeq \succeq_n), o \in \max(\bigcap_{i \in \mathcal{G}} \succeq'_i) \}$$
An equivalent statement is the following: a choice is necessary Pareto Optimal if there is no option that possibly dominates it. An option $o_1$ is a possible dominator for $o_2$ if there is a consistent extension of the known preference orders that make $o_1$ dominate $o_2$. Possible dominance can be formalized accordingly:

**Definition 4.** The relation $\succeq_{GD}^P$ of possible dominance for a group $G$ is such that

$$o_1 \succeq_{GD}^P o_2 \iff \exists (\succeq'_1, \ldots, \succeq'_n) : o_1 \succ_G o_2$$

with each $\succeq'_i$ extending $\succeq_i$ and $\succeq'_G = \bigcap_i \succeq'_i$ being the associated aggregate relation.

Given this definition, an item $o$ is a necessary Pareto Optimal if $o$ is a maximal element of $\succ^P_G$ (the induced strict relation). The relation $\succeq^P_G$ can be characterized in terms of the currently known preferences in the following way: an option $o_1$ is a potential dominator of $o_2$ if, for every agent $i$, $o_2$ is not strictly preferred to $o_1$, and $o_1$ and $o_2$ are not equally preferred for all agents.

**Proposition 3.** The relation of possible dominance ($\succeq_{GD}^P$) can be written as:

$$\succeq_{GD}^P = \left( \bigcap_{i \in G} \succeq_i \right) - \bigcap_{i \in G} \approx_i = \left( \bigcap_{i \in G} \succeq_i \cup \sim_i \right) - \bigcap_{i \in G} \approx_i$$

In the case of underlying linear orders, two options are never equally preferred, and the expression simplifies to

$$\succeq_{GD}^P = \left( \bigcap_{i \in G} \succeq_i \cup \sim_i \right)$$

One could make a similar reasoning and define a relation of necessary dominance $\succ_{GD}^N$. Notice that, if $o_1 \succeq_G o_2$ then $o_1 \succeq_{GD}^N o_2$, therefore $\succeq_G \subseteq \succ_{GD}^N$. Moreover it can be shown that it holds exactly that $\succeq_{GD}^N = \succeq_G$.

We can now characterize the sets PossParetoOpt and NecParetoOpt of possible and necessary Pareto Optimal choices in the following way.

**Proposition 4.** The set of Possible Pareto Optimal choices coincides with the set of the current undominated (Pareto Optimal) options given the known preference orders $\succeq_i$.

$$\text{PossParetoOpt}(\succeq_1, \ldots, \succeq_n) = \max(\succeq_G) = \max \left( \bigcap_{i \in G} \succeq_i \right)$$

The set of Necessary Pareto Optimal choices coincides with the maximal choices with respect to the strict relation of possible dominance.

$$\text{NecParetoOpt}(\succeq_1, \ldots, \succeq_n) = \max(\succ_{GD}^P)$$

**Proof.** 1) PossParetoOpt$_G$: We show the argument by constructions. Let choice $o_1$ belong to the Pareto set of the currently known preference orders; $o_1 \in \max(\succeq_G)$. Then for all preference orders $\succeq_i$ construct a complete (linear) order $\succeq'_i$ extending $\succeq_i$ such that, for any item $o_2$, if the preference between $o_1$
Proposition 5. We derive the following taxonomy: PossCons, NecExt, PossExt, NecParetoOpt and PossParetoOpt. 

Consider the following case with \( 1 \). NecCons as the pair \( o \) Pareto optimal choices wrt maximum. There is one necessary extreme, is no necessary optimal choice as only the preference order of agent 1 has a 

\( 1 \). NecCons \( 2 \). PossCons \( 3 \). NecCons \( 2 \) and \( 3 \). NecCons \( 2 \), NecExt \( 2 \) and \( \text{PossExt} \) \( \text{PossParetoOpt} \) and \( \text{PossParetoOpt} \).

Proposition 5. We derive the following taxonomy:

1. NecCons \( G \) \( \subseteq \) NecExt \( G \) \( \subseteq \) NecParetoOpt \( G \)
2. PossCons \( G \) \( \subseteq \) PossExt \( G \) \( \subseteq \) PossParetoOpt \( G \)
3. NecCons \( G \) \( \subseteq \) PossCons \( G \), NecExt \( G \) \( \subseteq \) PossExt \( G \) and 

\( \text{PossParetoOpt} \subseteq \) PossParetoOpt \( G \).

Proof. 1. NecCons \( G = \bigcap_{i \in G} \text{top}(\succeq_i) \subseteq \bigcup_{i \in G} \text{top}(\succeq_i) = \text{NecExt} \subseteq \max(G) = \text{PossParetoOpt}G; \) the last inclusion inequality holding as an element \( o \) of \( \text{NecExt} \) belongs to \( \text{top}(\succeq_j) \) for a given \( j \in G \) (by definition); \( o \) cannot be possibly dominated (wrt \( \succ_P = \bigcap_{i \in G} \succeq_i \cup \sim_i \) since it is the maximum in \( \succeq_j \), therefore it has to be a maximal element of \( G \).

2. PossCons \( G = \bigcap_{i \in G} \max(\succeq_i) \subseteq \bigcup_{i \in G} \max(\succeq_i) = \text{PossExt} \subseteq \max(\succeq) = \text{PossParetoOpt} \).

3. Straightforward from definition of possible and necessary: if an element is, respectively, a consensus, extreme, or Pareto optimal for all extensions of the preference orders, then in particular there exists an extension for which it is, respectively, a consensus, extreme or Pareto optimal.

Example 1. Consider the following case with \( C = \{o_1, o_2, o_3\} \). It is known that agent 1 prefers option \( o_1 \) to \( o_2 \), and option \( o_2 \) to \( o_3 \) (that is a linear order, or chain, \( o_1 \succ o_2 \succ o_3 \)). We also know that agent 2 prefers \( o_1 \) to \( o_3 \) and also \( o_2 \) to \( o_3 \) (\( o_1 \succ_2 o_3 \) and \( o_3 \succ_2 o_3 \)), but nothing is known about his preference between \( o_1 \) and \( o_2 \). For agent 3 we only know \( o_1 \succ o_3 \), meaning that he prefers \( o_1 \) to \( o_3 \).

There is only one maximal element for agent 1 \( (o_1, \text{that is also the maximum element for this agent}) \), while \( o_1 \) and \( o_2 \) are maximal for agents 2 and 3. The intersection is \( \{o_1\} \) and therefore \( o_1 \) is the only possible consensus choice. There is no necessary optimal choice as only the preference order of agent 1 has a maximum. There is one necessary extreme, \( o_1 \), and the possible extreme options are \( o_1, o_2 \) (the union of maximal elements). Then the group relation \( \succ_G \) is such that \( o_1 \succ_G o_3 \) (the pair \( o_1 \) and \( o_2 \) is incomparable with respect to \( \succ_G \), as well as the pair \( o_2 \) and \( o_3 \)). Then \( o_1 \) and \( o_2 \) are the maximal elements for \( \succ_G \) and the possible Pareto optimal choices for the agents. The relation of potential dominance is \( \succeq^{PD} \) that in this case is a linear order \( o_1 \succ^{PD} o_2 \succ^{PD} o_3 \); option \( o_1 \) is the only necessary Pareto optimal choice.
The taxonomy that we obtained (Figure 1) is perhaps not very surprising: the stricter the notion of optimality (consensus, extreme, Pareto) the smaller the sets. Sets of possible optimal items are included in the sets of necessary optimal items. However, our theoretical characterisation is useful as it allows to compute the possible and necessary optimal items reasoning only with respect to the current available preference information (the $\succeq_i$) without the need to individually consider all possible extensions of the currently available preference information. In particular, the set of possible Pareto optimal choices $\text{PossParetoOpt}$ coincides with the current Pareto optimal set, the set of maximal items with respect to the group dominance relation computed with the currently available preferences. The set of necessary Pareto optimal choices are those items that are non dominated with respect to the strict relation of possible dominance ($\succ^{PD}$), that can be expressed in a convenient way thanks to Equation 3.13. Furthermore the intersection of the maximal elements of the $\succeq_i$ coincides $\text{PossCons}$ and the union of the maximal elements of the $\succeq_i$ coincides with $\text{PossExt}$.

\begin{align*}
\text{PossParetoOpt} & \quad \text{PossExt} & \quad \text{PossCons} \\
\text{NecParetoOpt} & \quad \text{NecExt} & \quad \text{NecCons}
\end{align*}

Fig. 1. Inclusion membership between the different classes

4 Cardinality

The previous section provided a theoretical characterization of different kinds of “optimality” (consensus, extreme choices and Pareto Optimal choices) when dealing with partial binary preference information, providing a mathematical formulation of possible and necessary optimal choices under the different semantics. Here we perform a number of simulations in order to assess the cardinality of the sets $\text{NecCons}$, $\text{PossCons}$, $\text{NecExt}$, $\text{PossExt}$, $\text{NecParetoOpt}$ and $\text{PossParetoOpt}$ in practical circumstances.

- We randomly generate (uniformly) a permutation of the $n$ elements for each agent of the $m$ agents, this is assumed to be their true preference ranking.
- For each agent, from the incidence matrix representing the preference relation, we randomly cancel a fraction $f$ of pairs in relation. We compute the transitive closure of the relation.
- From the obtained partial orders, we compute the sets $\text{NecCons}$, $\text{PossCons}$, $\text{NecExt}$, $\text{PossExt}$, $\text{NecParetoOpt}$ and $\text{PossParetoOpt}$. 
In the table below we report the cardinality of these sets averaged over 100 runs, for some values of $n$, $m$ and $f$. Further studies might consider cardinality under different ranking probability models (for instance considering the models in [3]).

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5 Current Works

This section discusses current work dealing with elicitation of qualitative preferences and with the identification of “fair” choices.

_Elicitation_. We are interested in interactive settings, where the agents provide new information to the system (statements of the type $o_1 \succeq o_2$) at each step; this results in an update of the preference order $\succeq_i = \succeq_i \cup (o_1 \succeq o_2)$ for the agent $i$ who entered new information. The challenge is to define effective strategies for elicitation, that pick the items to compare in some smart way. Of course, we want...
to ask agents to compare two items that are currently incomparable given the available information. One heuristic strategy would be to consider queries about maximal items in the currently known preference relation. When choosing to which agent ask queries, one heuristic could consist in targeting the agent whose current preference relation is sparser (the lowest number of pairwise comparison is known).

We believe that the development of efficient query strategies, as well as practical evaluation and comparison of different techniques, is an important next step involving a substantial research effort. One challenge would be the definition of suitable measures of the value of information [7] of a candidate query in this intrinsically qualitative setting.

**Fairness.** We aim to provide a characterisation of fairness in a qualitative way (in a way similar to [2]). Intuitively, a choice is more fair than another if it is less “extreme” in the satisfaction of the agents. Fairness is a well developed concept in numerical approaches, where one can consider the least satisfied agent, or more refined aggregators such as OWA. However, typical fairness measures are not meaningful in this context, as we deal with qualitative preferences. One could of course map choices according to their position in each agent’s ranking, and then consider fairness using numerical methods, but then the advantages of a qualitative approach would disappear.

We want to work on partial preference orders without the need of a distance-from-equality measure. In order to define what equity means in this qualitative setting, we propose to use the notion of reference point. The reference point $e$ expresses an intermediate level of preference satisfaction, encoding a somewhat intermediate level of satisfaction. Note that the option at which the level $e$ is assessed, can be different for each agent (for instance, the $e$ might corresponds to the second best choice for agent 1, while to the third choice for agent 2). This point is such that, if each agent could achieve $e$, this would correspond to a maximally fair situation, as all agents will be equality satisfied.

**Definition 5.** A choice $o_1$ is more fair than choice $o_2$ with respect to agent $i$ and $j$, written as $o_1 \preceq_{F}(i,j) o_2$, if $o_1 \succeq_i o_2 \succeq_i e$ and if $e \succeq_j o_2 \succeq_j o_2$.

From $\succeq_{F}(i,j)$ a single fairness relation $\succeq^F$ for all agents can be obtained by intersection: $\succeq^F = \bigcap_{(i,j) \in G} \succeq_{F}(i,j)$. In order to obtain a practical way to rank items accounting for both preferences and fairness, we now combine the two preference relations $\succeq_G$ and $\succeq^F$ into a single preference order. A choice $o_1$ is preferred to another choice $o_2$ if it is either that $o_1$ dominates $o_2$ or if $o_1$ is more fair than $o_2$.

**Definition 6.** The combined dominance-fairness preference relation $\succeq^C$ is defined as $\succeq^C = \succeq_G \cup \succeq^F$.

We propose to consider the Pareto optimal choices wrt the obtained order $\succeq^C$: these solutions might be considered candidate choices for the groupwise decision making. These solutions can be considered the qualitative counterpart of
the concept of well balanced solutions in multi-objective optimization (Lorentz dominance). Future works consist in an investigation of the mathematical properties of the relation $\succeq^C$ and in practical evaluation (including simulations).

6 Conclusions

In this paper we considered the case of partially known preferences of different agents, considering purely qualitative preferences (partial preference relations). We derived a mathematical characterisation and a taxonomy of the possible and necessary optimal choices, according to three different notions of optimality: shared optimality, extrema and Pareto optimality. In simulations we reported the cardinality measures of these sets in a number of different settings.

Notice that we have considered a setting with multiple agents, but this work can be very easily adapted to a multi-criteria setting: in this other setting each $i$ would refer to a different criteria and $G$ to the group of criteria. This work is related to works in computational social choice theory, interested in establishing how hard is to compute necessary and possible winners given common election rules [8]. Researchers in Operations Research [4] have also considered necessary and possibly optimal items with several feasible utility functions.

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