Simultaneous Conversions with the Residue Number System using Linear Algebra

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Context

How to perform arithmetic operations on ... ?

1. Multi-precision integer matrices $\mathcal{M}_{n\times n}(\mathbb{Z})$ or $\mathcal{M}_{n\times n}(\mathbb{Z}/_{N\mathbb{Z}})$
2. Multi-precision integer polynomials $\mathbb{Z}[X]$ or $\mathbb{Z}/_{N\mathbb{Z}}$

Different approaches:

1. **Direct algorithm:** matrix arithmetic then multi-precision integer arithmetic

   $\rightsquigarrow$ Best for matrix/polynomial size $\ll$ integer bitsize

2. **Modular approach:** Split the big integer matrix into many small integer matrices

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Conversion to/from Residue Number System is frequently a bottleneck

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Outline

Euclidean division

Problem: If $a, m \in \mathbb{N}$, compute $q = a \operatorname{quo} m$ and $r = a \operatorname{rem} m$.

Note: $(n := \mathrm{bitsize}(a)) \geqslant (t := \mathrm{bitsize}(m))$

Algorithms for Euclidean division:

1. $n = 2t$ in time $\mathcal{O}(\mathsf{I}(t))$ – **Balanced case** [Cook '66], [Barrett '86]

   Idea: Newton iteration to approximate $1/m$ then $q = \lfloor a/m \rfloor$, finally $r = a - qm$

   Note: Only integer operations, need to control numerical newton iteration

2. $n \geqslant 2t$ in time $\mathcal{O}\left(n \frac{\mathsf{I}(t)}{t}\right)$,

   Idea: Iterate base case reduction

3. $n \leqslant 2t$ in time $\mathcal{O}\left(t \frac{\mathsf{I}(n-t)}{n-t}\right)$ [Giorgi et al, '13]

   Idea: Compute q only at precision $n - t$

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Chinese remainder theorem:

$$\mathbb{Z}/(m_1 \cdots m_s)\mathbb{Z} \quad \underset{\text{Reconstruction}}{\overset{\text{Reduction}}{\rightleftarrows}} \quad \prod_{i=1}^{s} \mathbb{Z}/m_i\mathbb{Z}$$

From now on, assume that m_i are bounded (e.g. 64 bits machine-word)

Let's recall the algorithms for reduction and reconstruction

RNS reduction

Problem: Compute $(a \operatorname{rem} m_i)_{i=1\dots s}$ where $a < M$ and $M = m_1 \cdots m_s$.

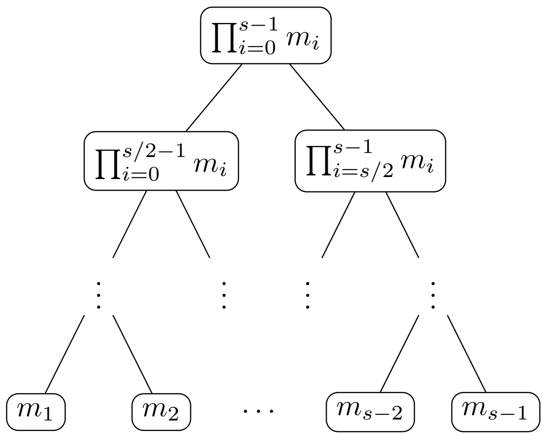
Naive Algorithm

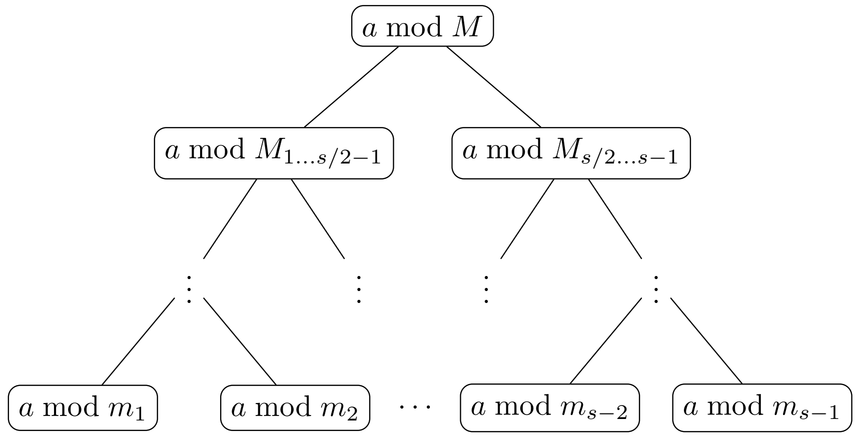
Apply $a \operatorname{rem} m_i$ in the case $(t=1) \ll (n=s)$ $\rightsquigarrow$ Quadratic time $\mathcal{O}(s^2)$

Note: Small constant factor in the big $\mathcal{O}$, faster for small s

Divide-and-conquer algorithm

Sub-product tree followed by Divide-and-conquer reductions





Quasi-linear complexity: $\mathcal{O}(\mathsf{I}(s) \log s)$

Note: Similar to polynomial case of multi-point evaluation

RNS reconstruction – Polynomial variant

Problem: Given the evaluations $a_i = a(m_i) = a(X) \operatorname{rem} (X - m_i)$, reconstruct $a(X)$ using

$$a(X) = \sum_i a_i \prod_{j \neq i} \frac{(X - m_j)}{(m_i - m_j)} = \sum_i \underbrace{{}^{a_i}\!/_{M_i(m_i)}}_{b_i} M_i(X) \quad \text{where} \quad M_i = M \,/\, (X - m_i)$$

Algorithm

1. Compute $M_i(m_i)$'s using $M_i(m_i) = M'(m_i)$ so multi-point evaluation of M'
2. Let $b_i = {}^{a_i}\!/_{M_i(m_i)}$ and recursively compute $P = \sum_i b_i M_i$:

Idea: If $M^0 = \prod_{j=1}^{s/2} (X - m_j)$ and $M^1 = \prod_{j=s/2+1}^{s} (X - m_j)$, note that

$$\sum_{i=1}^{s} b_i \frac{M}{(X - m_i)} = \left(\sum_{i=1}^{s/2} b_i \frac{M^0}{(X - m_i)} \right) M^1 + \left(\sum_{i=s/2+1}^{s} b_i \frac{M^1}{(X - m_i)} \right) M^0$$

Complexity: $\mathcal{O}(\mathsf{M}(s) \log s)$

RNS reconstruction – Maths

Problem: Given the reductions $a_i = a \operatorname{rem} m_i$, reconstruct a.

Integer equivalent of Lagrange interpolation

$$\mathbb{Z}/M\,\mathbb{Z} \longrightarrow (\mathbb{Z}/m_1\,\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_i\,\mathbb{Z}) \times \cdots \times (\mathbb{Z}/m_s\,\mathbb{Z})$$

$$M_i \longmapsto (0, \ldots, M_i \operatorname{rem} m_i, \ldots, 0) \qquad M_i = m_1 \cdots \widehat{m_i} \cdots m_s$$

$$u_i\, M_i \longmapsto (0, \ldots, 1, \ldots, 0) \qquad u_i = 1/M_i \bmod m_i$$

$$u_i\, a_i\, M_i \longmapsto (0, \ldots, a_i, \ldots, 0)$$

$$\sum_{i=1}^{s} u_i\, a_i\, M_i \longmapsto (a_1, \ldots, a_i, \ldots, a_s)$$

So a given by

$$a := \left(\sum_{i=1}^{s} u_i\, a_i\, M_i\right) \operatorname{rem} M = \left(\sum_{i=1}^{s} ((u_i\, a_i) \operatorname{rem} m_i)\, M_i\right) \operatorname{rem} M$$

is the reconstruction of $(a_1, \ldots, a_s)$.

Remark: $((u_i\, a_i) \operatorname{rem} m_i)\, M_i$ corresponds to $\frac{a_i \prod_{j \neq i} (X - m_j)}{\prod_{j \neq i} (m_i - m_j)}$ in Lagrange formula

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RNS reconstruction – Algorithm

Algorithm

1. Compute $M_i \operatorname{rem} m_i$ $\rightsquigarrow M_i(m_i) = M'(m_i)$? No derivative M' !
2. Compute $u_i = 1 / M_i \bmod m_i$
3. Compute $b_i := (u_i\, a_i) \operatorname{rem} m_i$
4. Compute $P = \sum_i b_i\, M_i$ $\rightsquigarrow$ Same recursion $P = P^0\, M^1 + P^1\, M^0$

Trick for $M_i \operatorname{rem} m_i$

Reduce M modulo $m_1^2, \ldots, m_s^2$

Indeed if $M = q\, m_i^2 + r$ then

$$m_i \,|\, r \quad \text{and} \quad M_i = M / m_i = q\, m_i + r / m_i$$

Total complexity: $\mathcal{O}(\mathsf{M}(s) \log s)$

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Our contribution

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Motivation: Integer matrix multiplication needs many simultaneous RNS conversions

Problem: Given $a_1, \ldots, a_r < (M = m_1 \cdots m_s)$, compute $(a_i \operatorname{rem} m_j)_{i=1\ldots r, j=1\ldots s}$

State-of-the-art complexities for simultaneous RNS conversions

1. Naive algorithms: $\mathcal{O}(r\, s^2)$
2. DAC algorithms: $\mathcal{O}(r\, \mathsf{I}(s) \log s)$

Our contribution [Doliskani, Giorgi, Lebreton, Schost '17]

Simultaneous conversions from/to RNS in time $\mathcal{O}(r\, s^{\omega - 1})$

(using precomputation of time $\mathcal{O}(s^2)$)

Outline

RNS reduction using linear algebra - Idea

Problem: Given $\boldsymbol{a} = (a_1, \ldots, a_r) \in [\![0; M[\![^r$, compute $(a_i \operatorname{rem} m_j)_{i=1\ldots r, j=1\ldots s}$

Note: $[\_]_\ell = \_ \bmod m_\ell$ and t such that $m_i \leqslant 2^t$

Idea:

1. Decompose a_i in base 2^t: $a_i = \sum_{j=0}^{s-1} a_{i,j}\, 2^{jt}$

2. If $d_{i,\ell} := (\sum_{j=0}^{s-1} a_{i,j}\, [2^{jt}]_\ell)$ then

   $d_{i,\ell}$ is a *pseudo-reduction* of a_i modulo m_ℓ, i.e. $a_i = d_{i,\ell} \bmod m_\ell$ and $d_{i,\ell} \leqslant 2^{4t}$

Algorithm

1. Precompute $([2^{jt}]_i)_{1 \leqslant i,j \leqslant s}$ $\mathcal{O}(s^2)$
2. Matrix multiplication with small integer entries $\mathcal{O}(\mathsf{MM}(s,s,r))$

$$\begin{bmatrix} d_{1,1} & \dots & d_{r,1} \\ \vdots & & \vdots \\ d_{1,s} & \dots & d_{r,s} \end{bmatrix} = \begin{bmatrix} 1 & [2^t]_1 & [2^{2t}]_1 & \dots & [2^{(s-1)t}]_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & [2^t]_s & [2^{2t}]_s & \dots & [2^{(s-1)t}]_s \end{bmatrix} \times \begin{bmatrix} a_{1,0} & \dots & a_{r,0} \\ \vdots & & \vdots \\ a_{1,s-1} & \dots & a_{r,s-1} \end{bmatrix}$$

3. Final reduction $a_i = d_{i,\ell} \operatorname{rem} m_\ell$ with $d_{i,\ell}$ small $\mathcal{O}(r\,s)$

Complexity: $\mathcal{O}(r\,s^{\omega-1})$ when $r \geqslant s$.

Speed-up: $s^{3-\omega}$ compared to naive algorithm

Problem: Given residues $a_{i,j} = (a_i \,\mathrm{rem}\, m_j)_{\substack{1 \leqslant i \leqslant r \\ 1 \leqslant j \leqslant s}}$, reconstruct $(a_1, \ldots, a_r) \in [\![0; M[\![^r$

Formula: $a_i = \left(\sum_{j=1}^{s} \underbrace{((u_j\, a_{i,j}) \,\mathrm{rem}\, m_i)}_{\gamma_{i,j}} M_i \right) \mathrm{rem}\, M$

Idea:

1. If $\ell_i := \sum_{j=1}^{s} \gamma_{i,j}\, M_j$, then

   ℓ_i is a *pseudo-reconstruction* of $(a_{i,j})_{1 \leqslant j \leqslant s}$, i.e. $\ell_i = a_{i,j} \bmod m_j$ and $\ell_i < s\, M$.

2. Decompose in linear operation with small entries :

   a. Write $M_j = \sum_{k=0}^{s-1} \mu_{j,k}\, 2^{kt}$ in base 2^t

   b. Compute $\ell_i = \sum_{k=0}^{s-1} \left(\sum_{j=1}^{s} \gamma_{i,j}\ \mu_{j,k} \right) 2^{kt}$ $\simeq$ decomposition of ℓ_i in base 2^t

RNS reconstruction using linear algebra - Complexity

Algorithm

1. Precompute $M_j = M / m_j$ and $u_j = (1 / M_j \operatorname{rem} m_j)$ — $\mathcal{O}(s^2)$
2. Compute all $\gamma_{i,j} = (u_j \, a_{i,j}) \operatorname{rem} m_i$ — $\mathcal{O}(r\,s)$
3. Compute pseudo base 2^t $(d_{i,j})$ decomposition of ℓ_i : — $\mathcal{O}(\mathsf{MM}(r,s,s))$

$$\begin{bmatrix} d_{1,0} & \cdots & d_{1,s-1} \\ \vdots & & \vdots \\ d_{r,0} & \cdots & d_{r,s-1} \end{bmatrix} = \begin{bmatrix} \gamma_{1,1} & \cdots & \gamma_{1,s} \\ \vdots & & \vdots \\ \gamma_{r,1} & \cdots & \gamma_{r,s} \end{bmatrix} \begin{bmatrix} \mu_{1,0} & \cdots & \mu_{1,s-1} \\ \vdots & & \vdots \\ \mu_{s,0} & \cdots & \mu_{s,s-1} \end{bmatrix}$$

4. Recover exact base 2^t decomposition of $\ell_i = \sum d_{i,k} \, 2^{kt}$ — $\mathcal{O}(r\,s)$
5. Final reconstruction: $a_i = \ell_i \operatorname{rem} M$ — $\mathcal{O}(r\,s)$

Complexity: $\mathcal{O}(r\,s^{\omega-1})$ when $r \geqslant s$.

Speed-up: $s^{3-\omega}$ compared to naive algorithm

Implementation details

Question: How to choose modulus bitsize t ?

Constraints:

1. Matrix entries bitsize:

   Use BLAS so all matrices integer entries should fit in double, i.e.

$$s\, m_i\, 2^t < 2^{53} \tag{1}$$

2. Limited number of primes of at most t bits

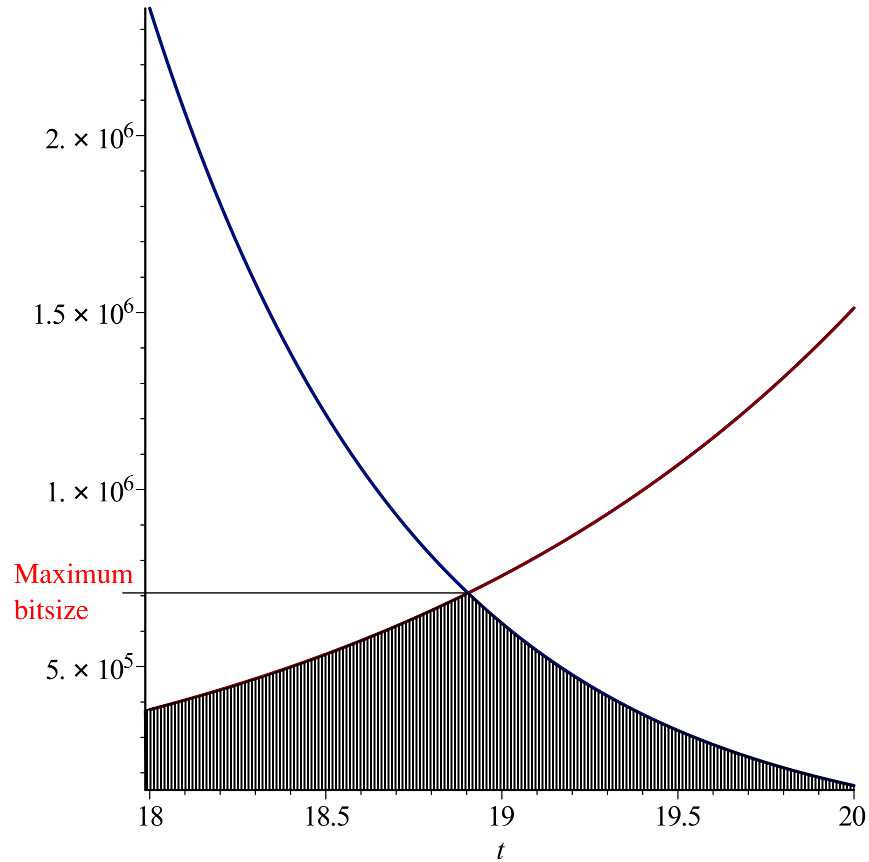
$$s \leqslant 2^t / (t \ln(2)) \tag{2}$$

Implementation details – Maximum bitsize

Goal 1: Maximize reachable bitsize of M

Since $\log_2(M) \simeq s\,t$, constraints give

$$\log_2(M) \leqslant \min\left(2^{53-2t}\,t,\, 2^t/\ln(2)\right)$$



So take $t = 19$ and maximum M has 76 KBytes (2^{15} moduli of bitsize $\leqslant 19$)

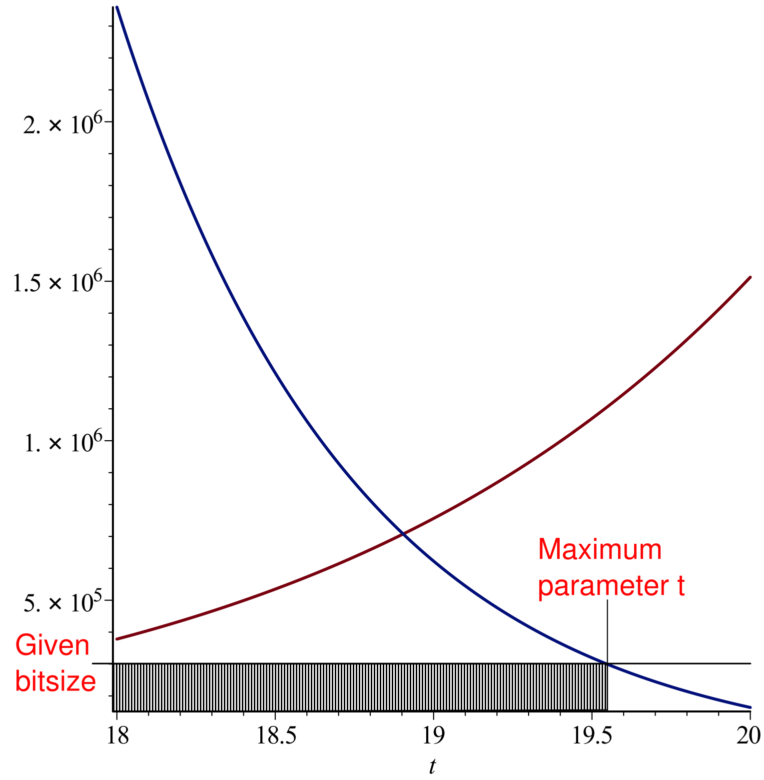
Implementation details – Optimize t

Now if M is less than 76 KBytes,

Goal 2: Maximize moduli bitsize t ($=$ Reduce number s of moduli)

Constraints give

$$\log_2(M) \leqslant \min\left(2^{53-2t}\,t, 2^t/\ln(2)\right)$$



Example: If M is 128 Bytes, take 45 moduli of bitsize 23

(instead of 54 moduli of bitsize $19 \rightsquigarrow$ speedup $(54/45)^{\omega-1} \simeq 1.44$)

Timings – RNS reduction

	RNS bitsize	Naive - MMX	DAC - FLINT	LinAlg - FFLAS [speedup]
	2^8	0.34	0.17	0.06 [x 2.8]
	2^9	0.75	0.35	0.13 [x 2.7]
	2^{10}	1.77	0.84	0.27 [x 3.1]
	2^{11}	4.26	2.73	0.75 [x 3.6]
	2^{12}	11.01	7.03	1.92 [x 3.7]
	2^{13}	29.86	17.75	5.94 [x 3.0]
	2^{14}	88.95	50.90	21.09 [x 2.4]
	2^{15}	301.69	165.80	80.82 [x 2.0]
	2^{16}	1055.84	506.91	298.86 [x 1.7]
	2^{17}	3973.46	1530.05	1107.23 [x 1.4]
	2^{18}	15376.40	4820.63	4114.98 [x 1.2]
limit	2^{19}	59693.64	13326.13	15491.90 [none]

Figure. Simultaneous RNS reductions (time per integer in $\mu\, s$)

Timings – RNS reconstruction

	RNS bitsize	Naive - MMX	DAC - FLINT	LinAlg - FFLAS [speedup]
	2^8	0.74	0.63	0.34 [x 1.8]
	2^9	1.04	1.34	0.39 [x 3.4]
	2^{10}	1.86	3.12	0.72 [x 4.3]
	2^{11}	4.29	6.92	1.57 [x 4.4]
	2^{12}	12.18	16.79	3.94 [x 4.3]
	2^{13}	43.89	40.73	12.77 [x 3.2]
	2^{14}	144.57	113.19	43.13 [x 2.6]
	2^{15}	502.18	316.61	161.44 [x 2.0]
	2^{16}	2187.65	855.48	609.22 [x 1.4]
	2^{17}	10356.08	2337.96	2259.84 [x 1.1]
	2^{18}	39965.23	7295.26	8283.64 [none]
limit	2^{19}	156155.06	18529.38	31382.81 [none]

Figure. Simultaneous RNS reconstruction (time per integer in $\mu\, s$)

Note: Our precomputations are more costly: we need $\simeq 1000$ a_i's to amortize them.

Extension for larger moduli

Application to integer polynomial multiplication

Problem: For multiplication in $\mathbb{Z}[x]$, we prefer Fourier primes ($\exists\ 2^k$-root for k large)

But there are not so many Fourier primes $\leqslant 2^{19}$!

$\rightsquigarrow$ **How can we extend our moduli bitsize limit ?**

Recall: When $m_i \leqslant 2^t$ and $a_i = \sum_{j=0}^{s-1} a_{i,j}\, 2^{jt}$, *pseudo-reduction* $d_{i,\ell} := (\sum_{j=0}^{s-1} a_{i,j}\, [2^{jt}]_\ell)$

$$\underbrace{\begin{bmatrix} d_{1,1} & \dots & d_{r,1} \\ \vdots & & \vdots \\ d_{1,s} & \dots & d_{r,s} \end{bmatrix}}_{D} = \underbrace{\begin{bmatrix} 1 & [2^t]_1 & [2^{2t}]_1 & \dots & [2^{(s-1)t}]_1 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & [2^t]_s & [2^{2t}]_s & \dots & [2^{(s-1)t}]_s \end{bmatrix}}_{B} \times \underbrace{\begin{bmatrix} a_{1,0} & \dots & a_{r,0} \\ \vdots & & \vdots \\ a_{1,s-1} & \dots & a_{r,s-1} \end{bmatrix}}_{A}$$

Idea: If $m_i \leqslant (2^t)^\kappa$, cut B into $B = B_0 + B_1\, 2^t + \cdots + B_{\kappa-1} 2^{t(\kappa-1)}$ and compute $B\,A$ as

$$D = (B_0\, A) + \ldots + 2^{(\kappa-1)t}\, (B_{\kappa-1}\, A)$$

Complexity and timings of extended moduli variant

Cost of the extension is the same:

1. s moduli of bitsize t: One multiplication of matrices $(s \times s) \times (s \times r)$
2. s/κ moduli of bitsize $\kappa\, t$: κ multiplications of matrices $(s/\kappa \times s) \times (s \times r)$

	RNS reduction			RNS reconstruction		
RNS bitsize	**FLINT**	**FFLAS** ($\kappa = 1$)	**FFLAS** ($\kappa = 2$)	**FLINT**	**FFLAS** ($\kappa = 1$)	**FFLAS** ($\kappa = 2$)
m_i	$<2^{59}$	$<2^{19}$	$<2^{38}$	$<2^{59}$	$<2^{19}$	$<2^{38}$
2^9	0.35	0.13	0.24	1.34	0.39	0.70
2^{10}	0.84	0.27	0.53	3.12	0.72	1.39
2^{11}	2.73	0.75	1.20	6.92	1.57	2.46
2^{12}	7.03	1.92	2.92	16.79	3.94	5.15
2^{13}	17.75	5.94	8.01	40.73	12.77	14.98
2^{14}	50.90	21.09	25.05	113.19	43.13	47.54
2^{15}	165.80	80.82	85.38	316.61	161.44	167.93
2^{16}	506.91	298.86	299.11	855.48	609.22	629.69
2^{17}	1530.05	1107.23	1099.52	2337.96	2259.84	2375.98
2^{18}	4820.63	4114.98	4043.68	7295.26	8283.64	8550.81
2^{19}	13326.13	15491.90	15092.94	18529.38	31382.81	33967.42

Conclusion

Conclusions:

1. Our approach is complementary with asymptotically fast algorithms

   We improves run-times for small and medium size

2. We exploit the available optimized implementations of matrix multiplication (BLAS)

   Reach peak performance of processors, gain a significant constant

3. If our gain is only constant, its impact is substantial to many important applications

   multiplication in $\mathcal{M}_{u,v}(\mathbb{Z}), \mathbb{Z}[x]$, polynomial factorization...

4. When prime bitsize limitation is a problem, we are still able to reduce the computation to matrix multiplication with small entries

Perspectives:

1. Implement hybrid version of linear algebra and divide-and-conquer strategies
2. Use different cutting for large moduli to provide further improvement