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On star and caterpillar arboricity

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Abstract

We give new bounds on the star arboricity and the caterpillar arboricity of planar graphs with given girth. One of them answers an open problem of Gyárfás and West: there exist planar graphs with track number 4. We also provide new NP-complete problems.

Keywords: NP-completeness; Partitioning problems; Edge coloring

1. Introduction

Many graph parameters in the literature are defined as the minimum size of a partition of the edges of the graph such that each part induces a graph of a given class C. The most common is the chromatic index $\chi'(G)$, in this case C is the class of graphs with maximum degree one. Vizing [18] proved that $\chi'(G)$ either equals $\Delta(G)$ or $\Delta(G)+1$, where $\Delta(G)$ denotes the maximum degree of G. Deciding whether $\chi'(G) = 3$ is shown to be NP-complete for general graphs in [13]. The arboricity $a(G)$ is another well studied parameter, for which C is the class of forests. In [15], Nash-Williams proved that:

$$a(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{|V(H)|-1} \right\rceil \tag{1}$$

with the maximum being over all the subgraphs $H = (E(H), V(H))$ of G. Even with this nice formula, the polynomial algorithm computing the arboricity of a graph is not trivial [12]. Other similar parameters have been studied. A *star* is a tree of diameter at most two. A *caterpillar* is a tree whose non-leaf vertices form a path. For the *star arboricity* $sa(G)$ and the *caterpillar arboricity* $ca(G)$, the corresponding class C is respectively the class of star forests and the class of caterpillar forests. Since stars are caterpillars which are trees, and since trees are easily partitionable into two star forests, we have the following two inequalities for any graph G.

$$sa(G) \geq ca(G) \geq a(G) \tag{2}$$

$$2a(G) \geq sa(G). \tag{3}$$

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A proper vertex coloring of a graph is *acyclic* if there is no bicolored cycle. Let $\chi_a(G)$ denote the acyclic chromatic number of a graph G. Hakimi et al. [11] showed the following relation between $\chi_a(G)$ and $sa(G)$.

Theorem 1 (*Hakimi et al.*). *For any graph G, we have $\chi_a(G) \geq sa(G)$.*

Other interesting graph parameters include the track number $t(G)$ [10,14] and the subchromatic index $\chi'_{\text{sub}}(G)$ [6]. For the track number, C is the class of interval graphs. For the subchromatic index, C is the class of graphs whose connected components are stars or triangles. Notice that the class of triangle-free interval graphs is equivalent to the class of caterpillar forests. Thus, if G is triangle-free, then $t(G) = ca(G)$ and $\chi'_{\text{sub}}(G) = sa(G)$.

Given a tree T, a *T-free forest* is a forest without subgraphs isomorphic to T. For example, the P_n-free forests and the $K_{1,n}$-free forests correspond to, respectively, the forests with diameter at most $n-2$ and to the forests with degree at most $n-1$. Given a tree T, the T-free arboricity $T\text{-}fa(G)$ of a graph G is the minimum number of T-free forests needed to cover the edges of G. In this case C is the class of T-free forests. Using this terminology, we can redefine some of the parameters we introduced. For $n \geq 2$, let S_n be the tree arised from $K_{1,n}$ by subdividing each edge once. The chromatic index, the star arboricity, and the caterpillar arboricity correspond to, respectively, the P_3-free arboricity, the P_4-free arboricity, and the S_3-free arboricity.

If a tree T_1 is a subtree of a tree T_2, then $T_1\text{-}fa(G) \geq T_2\text{-}fa(G)$. So, the poset of trees produces a poset of arboricities. For example, since $P_4 \subset S_2 \subset S_3 \subset \cdots \subset S_n$, we have $P_4\text{-}fa(G) \geq S_2\text{-}fa(G) \geq S_3\text{-}fa(G) \geq \cdots \geq S_n\text{-}fa(G)$, for any graph G.

In [11], it is proved that deciding whether a graph G satisfies $sa(G) \leq 2$ is NP-complete. We obtain the same result for a very restricted graph class.

Theorem 2. *For any $g \geq 3$, deciding whether a bipartite planar graph G with girth at least g and maximum degree 3 satisfies $sa(G) \leq 2$ is NP-complete.*

This implies that there exist planar graphs of arbitrarily large girth with star arboricity at least 3. This lower bound is tight for $g \geq 7$. Since planar graphs of girth $g \geq 7$ are acyclically 3-colorable [1], their star arboricity is at most 3 by Theorem 1.

As we already mentioned, if G is triangle-free, then $sa(G) = \chi'_{\text{sub}}(G)$. So, this theorem answers a question of Fiala and Le [6].

Corollary 3. *Deciding whether a planar graph G satisfies $\chi'_{\text{sub}}(G) \leq 2$ is NP-complete.*

Let us denote by $L(G)$ the line graph of G and by $\mathcal{L}$ the class of line graphs of "planar bipartite graphs with maximum degree three and girth at least six". Notice that graphs in $\mathcal{L}$ are planar with maximum degree four and line graphs of bipartite graphs, thus perfect [3]. This class of graph is very restricted, it corresponds to planar ($K_{1,3}$, K_4, K_4^-, C_4, odd-hole)-free graphs. The complexity of determining the subchromatic number of a graph is an interesting question. Deciding whether a graph G satisfies $\chi_{\text{sub}}(G) \leq 2$ is NP-complete if G is planar [8] or if G is perfect [4]. Theorem 2 shows that it is also the case for perfect planar graphs since $\chi'_{\text{sub}}(G) = \chi_{\text{sub}}(L(G))$.

Corollary 4. *Deciding whether a graph $G \in \mathcal{L}$ satisfies $\chi_{\text{sub}}(G) \leq 2$ is NP-complete.*

A graph is 2-*degenerate* if all of its subgraphs contain a vertex of degree at most 2.

Theorem 5. *Deciding whether a 2-degenerate bipartite planar graph G satisfies $sa(G) \leq 3$ is NP-complete.*

Shermer [17] proved that it is NP-complete to decide whether a graph G has caterpillar arboricity 2. We generalize here his result to S_n-free arboricity and consider more restricted graph classes.

Theorem 6. *The following problems are NP-complete:*

(1) *For every $n \geq 2$, deciding whether a 2-degenerate bipartite planar graph G satisfies $S_n\text{-}fa(G) \leq 3$.*
(2) *For every $n \geq 3$, deciding whether a 2-degenerate bipartite planar graph G of girth $g \geq 6$ satisfies $S_n\text{-}fa(G) \leq 2$.*

Theorem 6(1) implies the existence of bipartite planar graphs with caterpillar arboricity four and, as we already mentioned, the track number of a triangle-free graph equals its caterpillar arboricity. It is proved in [10] that deciding whether a graph G has track number $t(G) \leq k$ is NP-complete for $k = 2$ and conjectured that it is also the case for higher k. Here we proved that it is the case for a restricted family of graphs and for $k = 2$ or 3.

Corollary 7. *It is NP-complete to decide whether a* 2*-degenerate bipartite planar graph* G *satisfy* $t(G) \le 2$ *(resp.* $t(G) \le 3$*).*

The *interval number* $i(G)$ is the smallest k such that every vertex of the graph G can be represented as a set of at most k intervals of a line and there is an edge uv iff the segments of u and v intersect. Scheinerman and West [16] proved that the interval number of planar graphs is at most 3 and the first author [9] proved that the caterpillar arboricity of planar graphs is at most 4. This implies that the maximum track number of planar graphs is either 3 or 4. In [14], Kostochka and West proved that the maximum track number of outerplanar graphs equals their maximum interval number, 2. We can deduce from Theorem 6(1) that this is not the case for planar graphs, which answers an open question of Gyárfás and West [10].

Corollary 8. *There exist bipartite planar graphs with track number four.*

Theorem 6(2) implies that there exist planar graphs of girth $g \ge 6$ with caterpillar arboricity at least three. The next theorem shows that this lower bound is tight.

Theorem 9. *For any planar graph* G *with girth* $g \ge 6$*,* $ca(G) \le 3$*.*

Contrarily to the star arboricity, the caterpillar arboricity of planar graphs with sufficiently large girth is two.

Theorem 10. *For any planar graph* G *with girth* $g \ge 10$*,* $ca(G) \le 2$*.*

In the next section, we define the notion of T-fa coloring, which is needed in the proofs. Sections 3 and 4 are devoted to the proofs of the upper bounds and of the NP-completeness results, respectively.

2. T-free arboricity and T-fa coloring

We define the T-fa colorings for $T = P_4$ and $T = S_n$ with $n \ge 2$.

Definition 11. For $T = P_4$ (resp. $T = S_n$ with $n \ge 2$), a k-T-fa *coloring* of a graph G is a k-edge-coloring of G and a partial orientation of its edges such that:

- The graph induced by a color class is a T-free forest.
- If the edge uv is colored i and is oriented toward v, then v is a leaf in the ith forest F_i (the T-free forest induced by the edges colored i).
- The graph induced by the unoriented edges has maximum degree 0 (resp. $n-1$).

It is clear that if a graph G has a k-T-fa coloring, then T-$fa(G) \le k$ (by the first point of the definition). The reverse also holds. Indeed, given an edge partition of G into k T-free forests we can construct a k-T-fa coloring of G. For this, color each edge set with a given color and then orient to a leaf of F_i each edge of F_i incident to leaf. With this construction, any vertex v incident to an unoriented edge colored i is incident to another edge colored i. So, if there was a vertex incident to an (resp. to $n-1$) unoriented edge(s) colored i, this would contradict the "T-freeness" of the partition.

If the graph G is k-T-fa colored, for each of its k forests, we distinguish two types of vertices. The *ends*, which have an incident arc in this forest oriented toward them, and the *inner vertices*. A k-T-fa coloring of G is *suitable* if every vertex of G is an end in at most one forest (*i.e.* is an inner vertex in $k-1$ or k forests).

3. Upper bounds on the caterpillar arboricity

The maximum average degree $\mathrm{mad}(G)$ of a graph G is defined by $\mathrm{mad}(G) = \max\{2|E(H)|/|V(H)|,\ H \subseteq G\}$. Since we consider planar graphs, we will use the following well known observation based on Euler's formula:

Lemma 12. *If* G *is a planar graph with girth at least* g*, then* $\mathrm{mad}(G) < \frac{2g}{g-2}$.

Theorems 9 and 10 will be deduced, using this lemma, from a proposition of the form "every graph G of girth at least g with $\text{mad}(G) < q = \frac{2g}{g-2}$ has a suitable k-P-fa coloring". The proof of these propositions is based on the *discharging method*, as used in [2]. We consider a graph H of girth at least g that has no suitable coloring and is minimal for the subgraph partial order. This means that every proper subgraph H' of H has a suitable coloring.

First, we provide a set S of configurations that H cannot contain due to its minimality property. To show that a configuration $C \in S$ is forbidden, we suppose that H contains C and then argue that any suitable coloring of some proper subgraph of H can be extended in a suitable coloring of the whole graph H, which is a contradiction. Then we have to prove that any graph K avoiding every configuration in S satisfies $\text{mad}(K) \geq q$. To do that, we assume that every vertex v is assigned an initial charge equal to its degree $d(v)$ and we define a *discharging procedure* that preserves the total charge of the graph. We then show that if the discharging procedure is applied to a graph K avoiding S, then the final charge $d^*(v)$ of every vertex $v \in V(K)$ satisfies $d^*(v) \geq q$. We thus have

$$\text{mad}(K) \geq \frac{2|E(K)|}{|V(K)|} = \frac{\sum\limits_{v \in V(K)} d(v)}{|V(K)|} = \frac{\sum\limits_{v \in V(K)} d^*(v)}{|V(K)|} \geq \frac{q|V(K)|}{|V(K)|} = q.$$

In every figure depicting forbidden configurations, every neighbor of a "white" vertex is drawn, whereas a "black" vertex may have other neighbors in the graph. Two or more black vertices may coincide in a single vertex, provided they do not share a common white neighbor.

A simple example of forbidden configuration we use for Theorem 9 and Theorem 10 is the following. If H is a minimal graph having no suitable k-S_3-fa coloring, then its minimum degree $\delta(H) \geq 2$. If there was a vertex $v \in V(H)$ of degree 0 or 1 it would be easy to extend any suitable k-S_3-fa coloring of $H \setminus \{v\}$ to H. Before proving the theorems let us define the k-vertices (resp. $^{\leq}k$-vertices and $^{\geq}k$-vertices) as the vertices of degree $d = k$ (resp. $d \leq k$ and $d \geq k$). Similarly a k-neighbor of v (resp. a $^{\leq}k$-neighbor and a $^{\geq}k$-neighbor) is a neighbor of v that is a k-vertex (resp. a $^{\leq}k$-vertex and a $^{\geq}k$-vertex).

3.1. Proof of Theorem 9

Lemma 13. *Let H be a minimal graph of girth at least 6 having no suitable 3-S_3-fa coloring. Then $\delta(H) \geq 2$ and H does not contain a 2-vertex adjacent to a $^{\leq}5$-vertex.*

Proof. Suppose that H contains a 2-vertex u adjacent to both a $^{\leq}5$-vertex v and a vertex w. Consider a suitable coloring of $H \setminus \{u\}$ into three forests F_1, F_2, F_3. In this coloring, the vertex v is an inner vertex in at least two forests and has degree at most 4 in $H \setminus \{u\}$. This implies that there is a forest, say F_1, in which v is an inner vertex and such that v is incident to at most 1 unoriented edges of F_1. The vertex w is an inner vertex in at least two forests. Let F_i be one of these forests with $i \neq 1$. Now we can extend the coloring to H by coloring the edges uv and uw respectively 1 and i, letting uv unoriented, and orienting uw toward u. This S_3-fa coloring is suitable since u is just an end in F_i. □

We apply the following discharging rule to the graph H considered in Lemma 13: each $^{\geq}6$-vertex gives $\frac{1}{2}$ to each of its 2-neighbors. Let us check that for every $v \in V(H)$, $d^*(v) \geq 3$:

- $d(v) = 2$: v has two $^{\geq}6$-neighbors by Lemma 13, so $d^*(v) = 2 + 2\frac{1}{2} = 3$.
- $d(v) = k$, $3 \leq k \leq 5$: the charge of v is unchanged, so $d(v) = d^*(v) = k \geq 3$.
- $d(v) = k \geq 6$: $d^*(v) \geq k - k\frac{1}{2} = \frac{k}{2} \geq 3$.

This shows that the maximum average degree of a minimal graph of girth at least 6 having no suitable 3-S_3-fa coloring is at least 3. By Lemma 12, we thus have that every planar with girth at least 6 has a suitable 3-S_3-fa coloring, which proves Theorem 9.

3.2. Proof of Theorem 10

Lemma 14. *Let H be a minimal graph of girth at least 10 having no suitable 2-S_3-fa coloring. Then $\delta(H) \geq 2$ and H does not contain any of the configurations depicted in Fig. 1.*

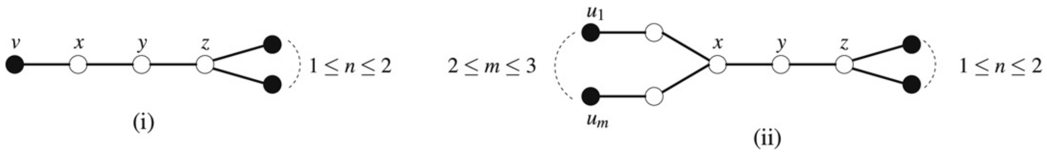

Fig. 1. Forbidden configurations for Lemma 14.

Proof. We consider each of the configurations:

(i) Suppose H contains the configuration (i) depicted in Fig. 1. Consider a suitable 2-S_3-fa coloring of $H \setminus \{y\}$. In every case, z is an inner vertex in some forest F_i such that z is incident to at most one non-oriented edge colored i. We can extend this coloring to H such that xy and yz are non-oriented, vx and xy get different colors, and yz gets color i.

(ii) Suppose H contains the configuration (ii) depicted in Fig. 1. Consider a suitable 2-S_3-fa coloring of the graph H' obtained from H by deleting the edge yz. We can always modify this coloring into a suitable 2-S_3-fa coloring of H' such that xy is non-oriented and there exists no monochromatic path connecting y to any u_i. In every case, z is an inner vertex in some forest F_i such that z is incident to at most one non-oriented edge colored i. We can extend this coloring to H such that yz is non-oriented and gets color i. □

A 3-vertex is *weak* if it has three 2-neighbors. A 2-vertex is *weak* if is adjacent to a 2-vertex or a weak 3-vertex. We apply the following discharging rules to the graph H considered in Lemma 14: each $^{\geq}4$-vertex gives $\frac{1}{2}$ to its weak 2-neighbors and $\frac{1}{4}$ to its non-weak 2-neighbors, each non-weak 3-vertex gives $\frac{1}{4}$ to its 2-neighbors. Let us check that for every $v \in V(H)$, $d^*(v) \geq \frac{5}{2}$:

- $d(v) = 2$: if v is weak, then v has a $^{\geq}4$-neighbor (see Fig. 1(i) and (ii) with $m = 2$), so $d^*(v) = 2 + \frac{1}{2} = \frac{5}{2}$. Otherwise v receives $\frac{1}{4}$ from each neighbor, so $d^*(v) \geq 2 + 2\frac{1}{4} = \frac{5}{2}$.
- $d(v) = 3$: if v is weak, then $d^*(v) = d(v) = 3 > \frac{5}{2}$. Otherwise v has at most two 2-neighbors, so $d^*(v) \geq 3 - 2\frac{1}{4} = \frac{5}{2}$.
- $d(v) = 4$: if v has four 2-neighbors, then its 2-neighbors are not weak (see Fig. 1(ii) with $m = 3$), so $d^*(v) \geq 4 - 4\frac{1}{4} = 3 > \frac{5}{2}$. Otherwise v has at most three 2-neighbors, so $d^*(v) \geq 4 - 3\frac{1}{2} = \frac{5}{2}$.
- $d(v) = k \geq 5$: $d^*(v) \geq k - k\frac{1}{2} = \frac{k}{2} \geq \frac{5}{2}$.

This shows that the maximum average degree of a minimal graph of girth at least 10 having no suitable 2-S_3-fa coloring is at least $\frac{5}{2}$. By Lemma 12, we thus have that every planar with girth at least 10 has a suitable 2-S_3-fa coloring, which proves Theorem 10.

4. NP-completeness results

Theorems 5 and 6(1) are each obtained by a polynomial reduction from the problem 3-COLORABILITY which is NP-complete on planar graphs with maximum degree 4 [7]. A *subcoloring* of a graph is a partition of its vertex set such that each part induces a disjoint union of cliques. Theorems 2 and 6(2) are each obtained by a polynomial reduction from the problem 2-SUBCOLORABILITY which is NP-complete on triangle-free planar graphs with maximum degree 4 [5,8]. Notice that on triangle-free graphs, a 2-subcoloring corresponds to a vertex partition into two graphs with maximum degree 1. Let us now describe the reductions for Theorems 5 and 6(1) (resp. Theorems 2 and 6(2). Given a planar graph (resp. a triangle-free planar graph) G, we construct a graph G' that belong to the class specified in the theorem as follows: we add a "vertex gadget" to every vertex v of G and replace every edge uv of G by an "edge gadget". The vertex gadget forces the v to be an inner vertex in at most one forest F_i for any k-T-fa coloring (with k and T as mentioned in the theorem). The edge gadget is such that G' is k-T-fa colorable if and only if G is 3-colorable (resp. 2-subcolorable). More precisely if G has a vertex coloring c, then G' has k-T-fa coloring such that every original vertex v is an inner vertex in $F_{c(v)}$, and conversely, if G' has a k-T-fa coloring such that every original vertex v is an inner vertex of F_i, then taking $c(v) = i$ gives a vertex coloring of G.

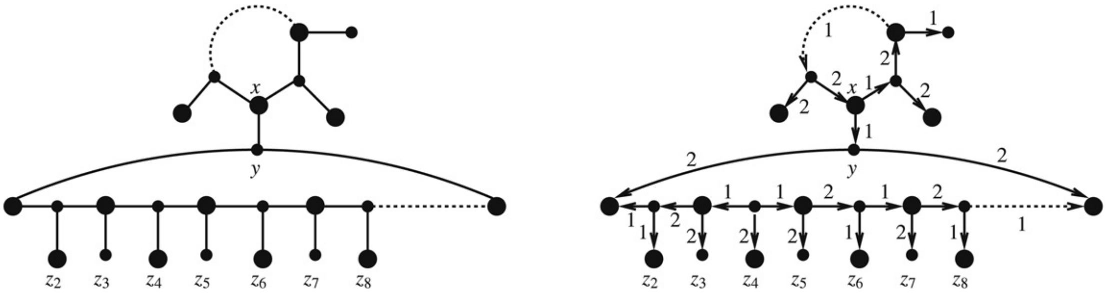

Fig. 2. The vertex gadget and its 2-P_4-fa colorings.

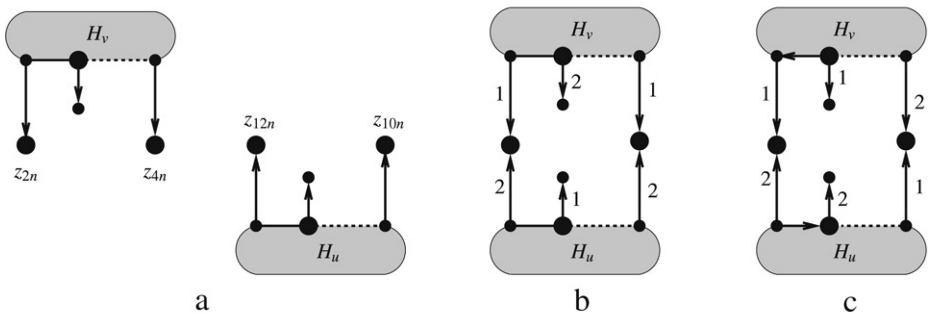

Fig. 3. The connection between two vertex gadgets.

4.1. Proof of Theorem 2

Let C'_n (resp. P'_n) be the graph obtained from the cycle C_n (resp. the path P_n) by adding, for each vertex $v \in V(C_n)$ (resp. $v \in V(P_n)$), a new vertex v' and an edge vv'. Note that in a 2-P_4-fa coloring, if a vertex v of degree at least three has an incident edge oriented to v and colored 1, then all the remaining edges incident to v are colored 2 and are oriented from v to the other end. This implies that in any 2-P_4-fa coloring of C'_{2n} each vertex of degree one is incident to an edge oriented toward it. The vertex gadget is the graph depicted in Fig. 2 obtained from C'_{2n} and P'_{18n-1}. This graph is bipartite and the size of the vertices in the figure indicate in which set of the bipartition they are. In any 2-P_4-fa coloring of the vertex gadget, the edge xy is oriented toward y. This imply that in any of these 2-P_4-fa colorings there is a $i \in \{1, 2\}$ such that at most one of the vertices $z_{2n}, z_{4n}, z_{6n}, \ldots, z_{16n}$ (z_4 in Fig. 2) is not an end of F_i. Furthermore, for any j with $1 \le j \le 8$, there is a 2-P_4-fa coloring in which the vertex z_{2jn} is an end of F_{3-i}. There are also 2-P_4-fa colorings of the vertex gadget in which all the vertices z_{2jn}, for $1 \le j \le 8$, are ends of F_i.

Given a triangle-free planar graph G with maximum degree 4, we construct G' by replacing every vertex v of G by a copy of the vertex gadget, denoted H_v. Every vertex v of G numbers its incident edges from 1 to $\deg(v) \le 4$ going around v in the clockwise sense. For every edge uv of G we connect H_u and H_v in the following way. Let i_u (resp. i_v) be the number of uv with respect to u (resp. v). Identify the vertices $z_{(2i_u-1)2n}$ and $z_{(2i_u)2n}$ of H_u respectively with the vertices $z_{(2i_v)2n}$ and $z_{(2i_v-1)2n}$ of H_v. In Fig. 3(a) we have $i_u = 3$ and $i_v = 1$ and the connection of H_u and H_v is depicted in Fig. 3(b) or (c). The graph G' is planar, bipartite, with maximum degree three and may have arbitrary girth (its girth is $2n$). We now have to show that $\chi_{\text{sub}}(G) \le 2$ if and only if $sa(G') \le 2$.

Given a 2-subcoloring c of G we obtain a 2-P_4-fa coloring of G' by coloring each vertex gadget H_u in such way that most of the vertices z_{2in} are ends in $F_{c(u)}$. If u has no neighbor v such that $c(u) = c(v)$, then all its vertices z_{2in} are ends in $F_{c(u)}$. If u has a neighbor v such that $c(u) = c(v)$, let i_u be the number of the edge uv with respect to u. By definition of a 2-subcoloring u has at most one such neighbor. In this case let the vertex $z_{(2i_u)2n}$ be an end in $F_{3-c(u)}$. This coloring of G' is a 2-P_4-fa coloring. Indeed, if an edge uv of G is such that $c(u) \ne c(v)$, then we see in Fig. 3(b) that the 2-P_4-fa colorings of H_u and H_v fit. If an edge uv of G is such that $c(u) = c(v)$, then we see in Fig. 3(c) that the 2-P_4-fa colorings of H_u and H_v also fit.

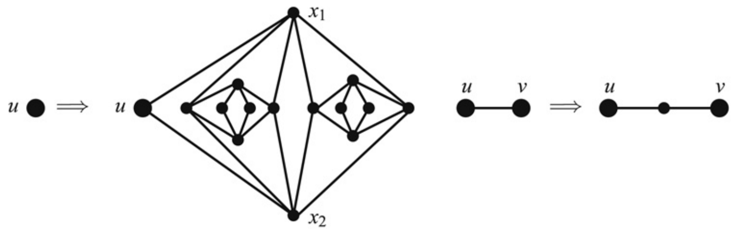

Fig. 4. The vertex gadget and the edge gadget for the reduction of Theorem 5.

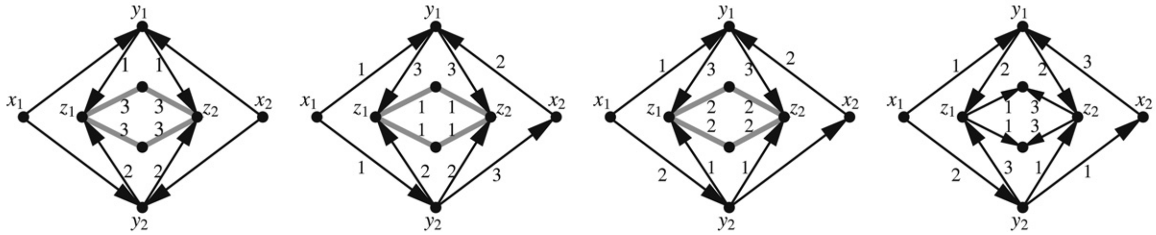

Fig. 5. 3-P_4-fa coloring of $\mathcal{A}$ with conditions on x_1 and x_2.

Conversely, given a 2-P_4-fa coloring of G' we obtain a 2-subcoloring of G by coloring each vertex u of G with the color $i \in \{1, 2\}$ that verify: most of the vertices z_{2jn} of H_u are ends of F_i. Since there is at most one vertex z_{2jn} that is not an end of F_i, the vertex u has at most one neighbor in G with the same color.

4.2. *Proof of Theorem* 5

Given a planar graph G, we construct G' by adding to every vertex of G the vertex gadget depicted in Fig. 4 and by subdividing every edge of G. The graph G' is clearly planar, bipartite and 2-degenerated.

First, we comment on how to 3-P_4-fa color the graph $\mathcal{A}$ depicted in Fig. 5. Notice that if a vertex has two incoming edges colored 1 and 2, all its remaining incident edges have to be colored 3. In the first drawing, we impose that all the edges $x_i y_j$ are oriented toward y_j. This implies that all the edges $y_i z_j$ are oriented toward z_j, and that we just used two colors for these edges. This finally implies that all the remaining edges incident to the z_i's have the same color, which is not allowed since each color induces a forest. In the second drawing, we impose that just one edge $x_i y_j$ is oriented toward x_2 and that the edges incident to x_1 have the same color, 1. The edges $x_2 y_1$ and $y_2 x_2$ have to be respectively colored 2 and 3. This implies that the edges $y_1 z_i$ are oriented toward z_i and colored 3. This implies that the edges $y_2 z_i$ are oriented toward z_i and colored 2. This finally implies that all the remaining edges incident to the z_i's have the same color, which is not allowed. In the third drawing, we impose that just one edge $x_i y_j$ is oriented toward x_2, that the edges incident to x_1 have distinct colors, 1 and 2, and that the edges $x_1 y_2$ and $x_2 y_1$ have the same color, 1. This implies that the edges $y_1 z_i$ are oriented toward z_i and colored 3. This implies that the edges $y_2 z_i$ are oriented toward z_i and colored 1. This finally implies that all the remaining edges incident to the z_i's have the same color, which is not allowed. In the last drawing, we see a 3-P_4-fa coloring of $\mathcal{A}$ in which only one edge is oriented toward x_2.

This implies that there is not much flexibility for coloring the vertex gadget in Fig. 4. Actually, in any 3-P_4-fa coloring of the vertex gadget, the two edges incident to u have to be oriented toward u and so u is an inner vertex in exactly one forest. Indeed, if ux_1 is oriented toward x_1 en colored 3 then one copie of $\mathcal{A}$, say $\mathcal{A}_1$, has both edges $x_1 y_1$ and $x_1 y_2$ oriented from x_1 to the other end. According to the possible 3-P_4-fa colorings of $\mathcal{A}_1$, this implies that either x_1 is an inner vertex in F_1 and F_2 and that x_2 is an end in F_1, either that in $\mathcal{A}_1$ both $x_2 y_1$ and $x_2 y_2$ are oriented toward x_2. In the first case the possible 3-P_4-fa colorings of $\mathcal{A}_2$ (the second copie of $\mathcal{A}$) are such that x_2 is an end in F_2. This implies that ux_2 is colored 3 and oriented toward u, which is impossible since ux_1 is also colored 3. In the second case $\mathcal{A}_2$ should have only one edge oriented toward x_1 and both edges incident to x_2 oriented from x_2 to the other end. Such 3-P_4-fa coloring of $\mathcal{A}_2$ would imply that x_2 is an inner vertex in two forests which is impossible.
In any 3-P_4-fa coloring of G', since any vertex $u \in V(G)$ is an inner vertex in exactly one forest, say F_1, the edges incident to u that belong to an edge gadget must be colored 1 and must be oriented toward the subdivision vertex.

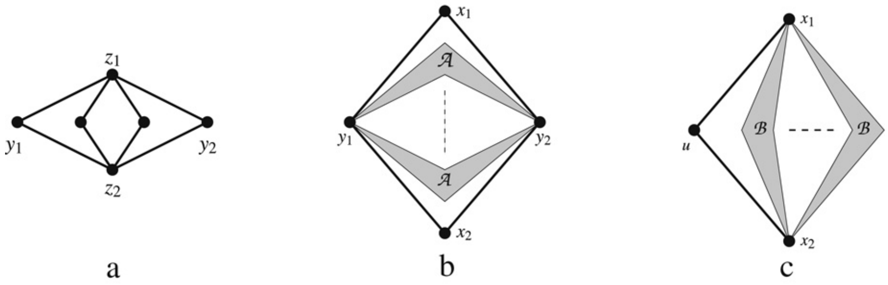

Fig. 6. (a) The graph $\mathcal{A}$, (b) the graph $\mathcal{B}$ and (c) the vertex gadget of u.

Thus the edge gadget forces two vertices u and $v \in V(G)$ to be inner vertices in distinct forests. Now we show that G is 3-colorable iff G' is 3-P_4-fa colorable.

Assume that G has a 3-coloring c, then for any vertex $u \in V(G)$ we color its vertex gadget in G' so that u is an inner vertex in $F_{c(u)}$. We then extend this 3-P_4-fa coloring to G', which is possible since for any $uv \in E(G)$ we have $c(u) \neq c(v)$. Conversely, suppose G' has a 3-P_4-fa coloring, we color the vertices of G accordingly to the forest for which they are an inner vertex in the 3-P_4-fa coloring of G'. This produces a 3-coloring of G.

4.3. *Proof of Theorem 6(1)*

The graph $\mathcal{A}$ depicted in Fig. 6(a) is such that in any of its 3-S_n-fa colorings if none of the edges y_1z_1, y_1z_2, y_2z_1 and y_2z_2 uses a given color, say 3, then one of these edges is not oriented from y_i to z_j. Furthermore, there is such 3-S_n-fa coloring of $\mathcal{A}$ for which only one of these edges is unoriented.

The graph $\mathcal{B}$ depicted in Fig. 6(b) is obtained using $2n-1$ copies of $\mathcal{A}$. The restrictions in the possible 3-S_n-fa colorings of $\mathcal{A}$ imply that in any 3-S_n-fa coloring of $\mathcal{B}$ one of the edges x_1y_1, x_1y_2, x_2y_1 and x_2y_2 is not oriented from x_i to y_j. Furthermore, there is a 3-S_n-fa coloring of $\mathcal{B}$ where x_1y_1, x_1y_2 and x_2y_1 are oriented toward y_j and respectively colored 1, 2 and 3; and where the edge x_2y_2 is unoriented and colored 1.

The graph vertex gadget depicted in Fig. 6(c) is obtained using $6(n-1)$ copies of $\mathcal{B}$. The restrictions in the possible 3-S_n-fa colorings of $\mathcal{B}$ imply that in any of its 3-S_n-fa colorings, the vertex u is an inner vertex in exactly one forest (the edges ux_1 and ux_2 are both oriented toward u).

Given a planar graph G, we construct G' by adding to every vertex u of G the vertex gadget and by replacing every edge uv of G by a cycle (u, x_{uv}, v, y_{uv}) where x_{uv} and y_{uv} are new vertices. The graph G' is clearly 2-degenerated, bipartite and planar. Now we prove that G is 3-colorable iff G' is 3-S_n-fa colorable.

If G has a 3-coloring c, for each vertex $u \in V(G)$ we 3-S_n-fa color its gadget so that u is an inner vertex in $F_{c(v)}$. Then we orient the remaining edges incident to u from u to the other end and we color them $c(u)$. It is clear that for any edge $uv \in E(G)$, since $c(u) \neq c(v)$ the cycle (u, x_{uv}, v, y_{uv}) of G' is properly 3-S_n-fa colored. So the graph G' is 3-S_n-fa colorable.

Conversely, the restrictions in the possible 3-S_n-fa colorings of a vertex gadget imply that if G' is 3-S_n-fa colored any vertex $u \in V(G)$ is an inner vertex in exactly one forest in G'. We define a 3-coloring c of G so that in G' any vertex $u \in V(G)$ is an inner vertex in $F_{c(u)}$. Since for any cycle (u, x_{uv}, v, y_{uv}) of G' the edges incident to u (resp. v) are colored $c(u)$ (resp. $c(v)$) then for any edge $uv \in E(G)$ we have $c(u) \neq c(v)$. So c is a 3-coloring of G.

4.4. *Proof of Theorem 6(2)*

Note that there is no 2-S_n-fa coloring of the path (a, b, c, d) where the edges ab and cd are oriented toward b and c and have distinct colors. This implies that in $\mathcal{A}$, there is a forest, say F_1, such that both vertices a_i and b_j are inner vertices in F_1. This implies that in $\mathcal{B}$, the vertices a_i and a_j are respectively inner vertices in F_2 and F_1. This implies that in $\mathcal{C}$, one of the a_i's, say a_1 (resp. a_2), is an inner vertex in F_1 (resp. F_2) and that a_3 is an inner vertex in both F_1 and F_2. This implies that at least one of the edges $u'a_i$ is unoriented. The possible colorings of $\mathcal{B}$ also imply that in every 2-S_n-fa coloring of $\mathcal{D}$ where the edges ua_1 and ua_2 have the same color, one of these edges is unoriented. All this implies that in the vertex gadget (depicted in the right of Fig. 7), the edge uu' is colored $x \in \{1, 2\}$ and oriented toward u and that u is incident to $n-2$ unoriented edges colored $3-x$.

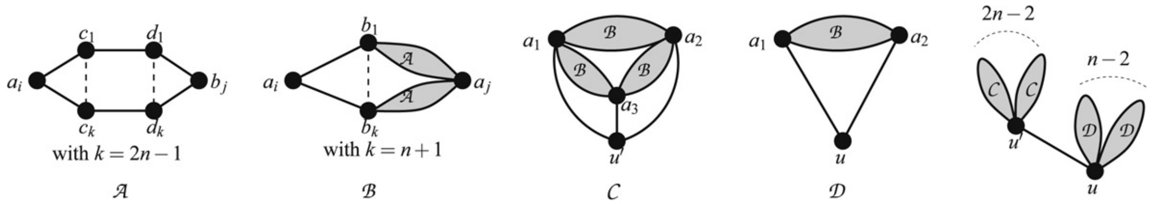

Fig. 7. The vertex gadget.

Given a triangle-free planar graph G, we construct G' by adding to every vertex $u \in V(G)$ the vertex gadget depicted in Fig. 7, right, and by subdividing every edge of G. A 2-S_n-fa coloring of the vertex gadget forces an original vertex of G to be an inner vertex in at most one forest, say F_1, and tobe incident to at least $n-2$ unoriented edges of F_1. We consider now 2-S_n-fa colorings of the edge gadget of an edge uv of G. If u and v are inner vertices in distinct forests, then we can 2-S_n-fa color the edges of the edge gadget and orient them toward the subdivision vertex. If u and v are inner vertices in the forest F_1, then both edges of the edge gadget have to be unoriented edges colored 1. Thus u and v are now incident to $n-1$ unoriented edges of F_1. This shows that G has a 2-subcoloring if and only if G' has a 2-S_n-fa coloring.

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