Near-colorings: non-colorable graphs and NP-completeness

M. Montassier and P. Ochem*

LIRMM (Université de Montpellier, CNRS), Montpellier, France.

February 16, 2015

Abstract

A graph G is $(d_1, \ldots, d_l)$-*colorable* if the vertex set of G can be partitioned into subsets $V_1, \ldots, V_l$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leqslant i \leqslant l$. In this paper, we focus on complexity aspects of such colorings when $l = 2, 3$. More precisely, we prove that, for any fixed integers k, j, g with $(k, j) \neq (0, 0)$ and $g \geqslant 3$, either every planar graph with girth at least g is (k, j)-colorable or it is NP-complete to determine whether a planar graph with girth at least g is (k, j)-colorable. Also, for every fixed integer k, it is NP-complete to determine whether a planar graph that is either $(0, 0, 0)$-colorable or non-$(k, k, 1)$-colorable is $(0, 0, 0)$-colorable. Additionally, we exhibit non-$(3, 1)$-colorable planar graphs with girth 5 and non-$(2, 0)$-colorable planar graphs with girth 7.

1 Introduction

A graph G is $(d_1, \ldots, d_k)$-*colorable* if the vertex set of G can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leqslant i \leqslant k$. This notion generalizes those of proper k-coloring (when $d_1 = \cdots = d_k = 0$) and d-improper k-coloring (when $d_1 = \cdots = d_k = d \geqslant 1$).

Planar graphs are known to be $(0, 0, 0, 0)$-colorable (Appel and Haken [1, 2]) and $(2, 2, 2)$-colorable (Cowen, Cowen, and Woodall [13]). The $(2, 2, 2)$-colorability is optimal (for any integer k, there exist non-$(k, k, 1)$-colorable planar graphs) and holds in the choosability case (Eaton and Hull [15] or Škrekovski [23]). Improper colorings have then been considered for planar graphs with large girth or graphs with low maximum average degree. We recall that the girth of a graph G, denoted by $g(G)$, is the length of a shortest cycle in G, and the maximum average degree of a graph G, denoted by

*This work was partially supported by the ANR grant EGOS 12-JS02-002-01.

mad(G), is the maximum of the average degrees of all subgraphs of G, i.e. $\mathrm{mad}(G) = \max \{2|E(H)|/|V(H)|, H \subseteq G\}$.

$(1,0)$-coloring.
Glebov and Zambalaeva [20] proved that every planar graph with girth at least 16 is $(1,0)$-colorable. This was then strengthened by Borodin and Ivanova [3] who proved that every graph G with $\mathrm{mad}(G) < \frac{7}{3}$ is $(1,0)$-colorable. This implies that every planar graph G with girth at least 14 is $(1,0)$-colorable. Borodin and Kostochka [7] then proved that every graph G with $\mathrm{mad}(G) \leqslant \frac{12}{5}$ is $(1,0)$-colorable. In particular, it follows that every planar graph with girth at least 12 is $(1,0)$-colorable. On the other hand, they constructed graphs G with $\mathrm{mad}(G)$ arbitrarily close (from above) to $\frac{12}{5}$ that are not $(1,0)$-colorable; hence their upper bound on the maximum average degree is best possible. The last result was strengthened for triangle-free graphs: Kim, Kostochka, and Zhu [22] proved that triangle-free graphs G satisfying $11|V(G)| - 9|E(G)| \geqslant -4$ are $(1,0)$-colorable. This implies that planar graphs with girth at least 11 are $(1,0)$-colorable. On the other hand, Esperet, Montassier, Ochem, and Pinlou [16] proved that determining whether a planar graph with girth 9 is $(1,0)$-colorable is NP-complete. To our knowledge, the question whether all planar graphs with girth at least 10 are $(1,0)$-colorable is still open.

$(k,0)$-coloring with $k \geqslant 2$.
Borodin, Ivanova, Montassier, Ochem, and Raspaud [4] proved that every graph G with $\mathrm{mad}(G) < \frac{3k+4}{k+2}$ is $(k,0)$-colorable. The proof in [4] extends the one in [3] but does not work for $k = 1$. Moreover, they exhibited a non-$(k,0)$-colorable planar graph with girth 6. Finally, Borodin and Kostochka [8] proved that for $k \geqslant 2$, every graph G with $\mathrm{mad}(G) \leqslant \frac{3k+2}{k+1}$ is $(k,0)$-colorable. This result is tight in terms of maximum average degree.

$(k,1)$-coloring.
Recently, Borodin, Kostochka, and Yancey [9] proved that every graph with $\mathrm{mad}(G) \leqslant \frac{14}{5}$ is $(1,1)$-colorable, and the restriction on $\mathrm{mad}(G)$ is sharp. In [5], it is proven that every graph G with $\mathrm{mad}(G) < \frac{10k+22}{3k+9}$ is $(k,1)$-colorable for $k \geqslant 2$.

(k,j)-coloring.
A first step was made by Havet and Sereni [21] who showed that, for every $k \geqslant 0$, every graph G with $\mathrm{mad}(G) < \frac{4k+4}{k+2}$ is (k,k)-colorable (in fact (k,k)-choosable). More generally, they studied k-improper l-choosability and proved that every graph G with mad(G) $< l + \frac{lk}{l+k}$ $(l \geqslant 2, k \geqslant 0)$ is k-improper l-choosable; this implies that such graphs are $(k,\dots,k)$-colorable (where the number of partite sets is l). Borodin, Ivanova, Montassier, and Raspaud [6] gave some sufficient conditions of (k,j)-colorability depending on the density of the graphs using linear programming. Finally, Borodin and Kostochka [8]

solved the problem for a wide range of j and k: let $j \geqslant 0$ and $k \geqslant 2j+2$; every graph G with $\mathrm{mad}(G) \leqslant 2(2 - \frac{k+2}{(j+2)(k+1)})$ is (k,j)-colorable. This result is tight in terms of the maximum average degree and improves some results in [4, 5, 6].

Using the fact that every planar graph G with girth $g(G)$ has $\mathrm{mad}(G) < 2g(G)/(g(G)-2)$, the previous results give results for planar graphs. They are summarized in Table 1, which also shows the recent results that planar graphs with girth 5 are $(5,3)$-colorable (Choi and Raspaud [12]) and $(10,1)$-colorable (Choi, Choi, Jeong, and Suh [11]).

girth	$(k,0)$	$(k,1)$	$(k,2)$	$(k,3)$	$(k,4)$
3,4	$\times$	$\times$	$\times$	$\times$	$\times$
5	$\times$	$(10,1)$ [11]	$(6,2)$ [8]	$(5,3)$ [12]	$(4,4)$ [21]
6	$\times$ [4]	$(4,1)$ [8]	$(2,2)$ [21]		
7	$(4,0)$ [8]	$(1,1)$ [9]			
8	$(2,0)$ [8]				
11	$(1,0)$ [22]				

Table 1: The girth and the (k,j)-colorability of planar graphs. The symbol "$\times$" means that there exist non-(k,j)-colorable planar graphs for every k.

From the previous discussion, the following questions are natural:

Question 1.

1. *Are planar graphs with girth 10* $(1,0)$*-colorable?*
2. *Are planar graphs with girth 7* $(3,0)$*-colorable?*
3. *Are planar graphs with girth 6* $(1,1)$*-colorable?*
4. *Are planar graphs with girth 5* $(4,1)$*-colorable?*
5. *Are planar graphs with girth 5* $(2,2)$*-colorable?*

$(d_1,\dots,d_k)$-coloring.

Finally we would like to mention two studies. Chang, Havet, Montassier, and Raspaud [10] gave some approximation results to Steinberg's Conjecture using (k,j,i)-colorings. Dorbec, Kaiser, Montassier, and Raspaud [14] studied the particular case of $(d_1,\dots,d_k)$-coloring where the value of each d_i $(1 \leqslant i \leqslant k)$ is either 0 or some value d, making the link between $(d,0)$-coloring [8] and $(d,\dots,d)$-coloring [21].

The aim of this paper is to provide complexity results on the subject and to obtain non-colorable planar graphs showing that some above-mentioned results are optimal.

Claim 2. *There exist 2-degenerate planar graphs that are:*

1. *non-*(k,k)*-colorable with girth 4, for every* $k \geqslant 0$*,*
2. *non-*$(3,1)$*-colorable with girth 5,*
3. *non-*$(k,0)$*-colorable with girth 6,*
4. *non-*$(2,0)$*-colorable with girth 7.*

Claim 2.4 shows that the $(2,0)$-colorability of planar graphs with girth at least 8 [8] is a tight result. Claim 2.3 has been obtained in [?] and the corresponding graph is depicted in Figure 1.

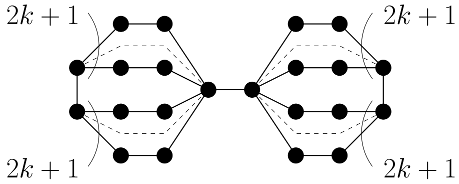

Figure 1: A non-$(k,0)$-colorable planar graph with girth 6 [4].

Theorem 3. *Let* k*,* j*, and* g *be fixed integers such that* $(k,j) \neq (0,0)$ *and* $g \geqslant 3$*. Either every planar graph with girth at least* g *is* (k,j)*-colorable or it is NP-complete to determine whether a planar graph with girth at least* g *is* (k,j)*-colorable.*

Theorem 4. *Let* k *be a fixed integer. It is NP-complete to determine whether a 3-degenerate planar graph that is either* $(0,0,0)$*-colorable or non-*$(k,k,1)$*-colorable is* $(0,0,0)$*-colorable.*

We construct a non-(k,k)-colorable planar graph with girth 4 in Section 2, a non-$(3,1)$-colorable planar graph with girth 5 in Section 3, and a non-$(2,0)$-colorable planar graph with girth 7 in Section 4. We prove Theorem 3 in Section 5 and we prove Theorem 4 in Section 6.

Notation.

In the following, when we consider a $(d_1,\dots,d_k)$-coloring of a graph G, we color the vertices of V_i with color d_i for $1 \leqslant i \leqslant k$: for example in a $(3,0)$-coloring, we will use color 3 to color the vertices of V_1 inducing a subgraph with maximum degree 3 and use color 0 to color the vertices of V_2 inducing a stable set. A vertex is said to be *colored* i^j if it is colored i and has j neighbors colored i, that is, it has degree j in the subgraph induced by its color. A vertex is *saturated* if it is colored i^i, that is, if it has maximum degree in the subgraph induced by its color. A cycle (resp. face) of length k is called a k-cycle (resp. k-face). A k-vertex (resp. k^--vertex, k^+-vertex) is a vertex of degree k (resp. at most k, at least k). The minimum degree of a graph G is denoted by $\delta(G)$.

2 A non-(k,k)-colorable planar graph with girth 4

For every $k \geqslant 0$, we construct a non-(k,k)-colorable planar graph J_4 with girth 4. Let $H_{x,y}$ be a copy of $K_{2,2k+1}$, as depicted in Figure 2. In any (k,k)-coloring of $H_{x,y}$, the vertices x and y must receive the same color. We obtain J_4 from a vertex u and a star S with center v_0 and $k+1$ leaves $v_1, \ldots, v_{k+1}$ by linking u to every vertex v_i with a copy H_{u,v_i} of $H_{x,y}$. The graph J_4 is not (k,k)-colorable: by the property of $H_{x,y}$, every vertex v_i should get the same color as u. This gives a monochromatic S, which is forbidden. Notice that J_4 is a planar graph with girth 4 and is 2-degenerate.

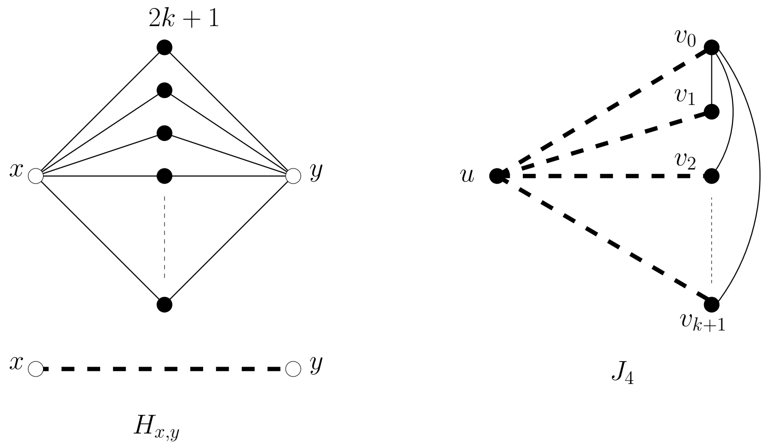

Figure 2: A non-(k,k)-colorable planar graph with girth 4.

3 A non-$(3,1)$-colorable planar graph with girth 5

We construct a non-$(3,1)$-colorable planar graph J_5 with girth 5. Consider the graph $H_{x,y}$ depicted in Figure 3. If x and y are colored 3 but have no neighbor colored 3, then it is not possible to extend this partial coloring to $H_{x,y}$. Now, we construct the graph S_z as follows. Let z be a vertex and $t_1t_2t_3$ be a path on three vertices. Take 21 copies H_{x_i,y_j} of $H_{x,y}$ with $1 \leqslant i \leqslant 7$ and $1 \leqslant j \leqslant 3$. Identify every x_i with z and identify every y_i with t_i. Finally, we obtain J_5 from three copies S_{z_1}, S_{z_2}, and S_{z_3} of S_z by adding the edges z_1z_2 and z_2z_3 (Figure 3). Notice that J_5 is planar with girth 5 and is 2-degenerate. Let us show that J_5 is not $(3,1)$-colorable. In every $(3,1)$-coloring of J_5, the path $z_1z_2z_3$ contains a vertex z colored 3. In the copy of S_z corresponding to z, the path $t_1t_2t_3$ contains a vertex t colored 3. Since z (resp. t) has at most 3 neighbors colored 3, one of the seven copies of $H_{x,y}$ between z and t, does not contain a neighbor of z or t colored 3. This copy of $H_{x,y}$ is not $(3,1)$-colorable, and thus J_5 is not $(3,1)$-colorable.

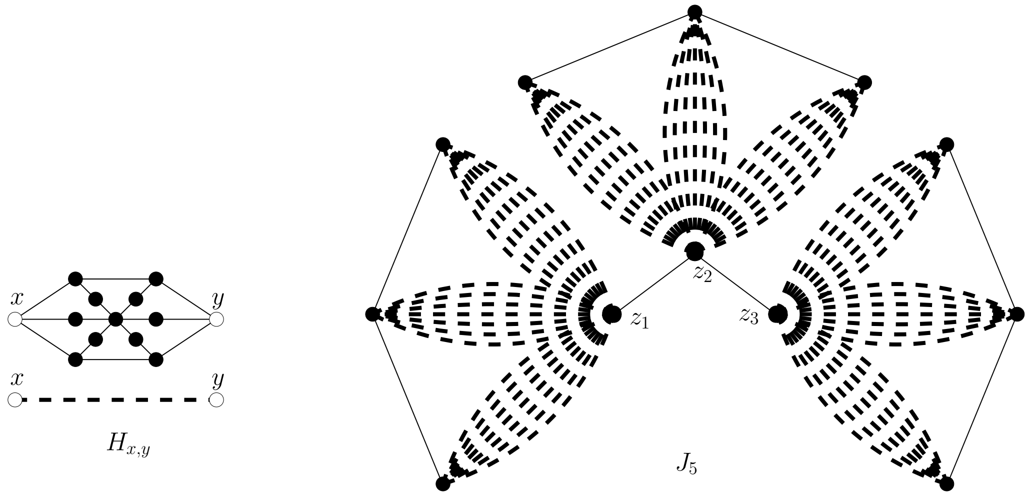

Figure 3: A non-$(3, 1)$-colorable planar graph with girth 5.

4 A non-$(2, 0)$-colorable planar graph with girth 7

We construct of a non-$(2, 0)$-colorable planar graph J_7 with girth 7. Consider the graphs $T_{x,y,z}$ and S in Figure 4. If the vertices x, y, and z of $T_{x,y,z}$ are colored 2 and have no neighbor colored 2, then w is colored 2^2. Suppose that the vertices a, b, c, d, e, f, g of S are respectively colored 2, 0, 2, 2, 2, 2, 0, and that a has no neighbor colored 2. Using successively the property of $T_{x,y,z}$, we have that w_1, w_2, and w_3 must be colored 2^2. It follows that w_4 is colored 0, w_5 is colored 2, and so w_6 is colored 2^2. Again, by the property of $T_{x,y,z}$, w_7 must be colored 2^2. Finally, w_8 must be colored 0 and there is no choice of color for w_9. Hence, such a coloring of the outer 7-cycle $abcdefg$ cannot be extended.

The graph H_z depicted on the left of Figure 5 is obtained as follows. We link a vertex z to every vertex of a 7-cycle $v_1 \dots v_7$ with a path of three edges. Then we embed the graph S in every 7-face F_i $(1 \leqslant i \leqslant 7)$ incident to z by identifying the outer 7-cycle of S with the 7-cycle of F_i such that a is identified to z. Finally, the graph J_7 depicted on the right of Figure 5 is obtained from two adjacent vertices u and v and six copies $H_{z_1}, \dots, H_{z_6}$ of H_z by identifying z_1, z_2, z_3 with u and z_4, z_5, z_6 with v. Notice that J_7 is planar with has girth 7. Let us prove that J_7 is not $(2, 0)$-colorable.

- We assume that u is colored 2 since u and v cannot be both colored 0.
- In one of the three copies of H_z attached to u, say H_{z_1}, u has no neighbor colored 2.
- Since a 7-cycle is not 2-colorable, the 7-cycle $v_1 \dots v_7$ of H_{z_1} contains a monochromatic edge colored 2, say v_1v_2.
- The coloring of the face F_2 cannot be extended to the copy of S embedded in F_2.

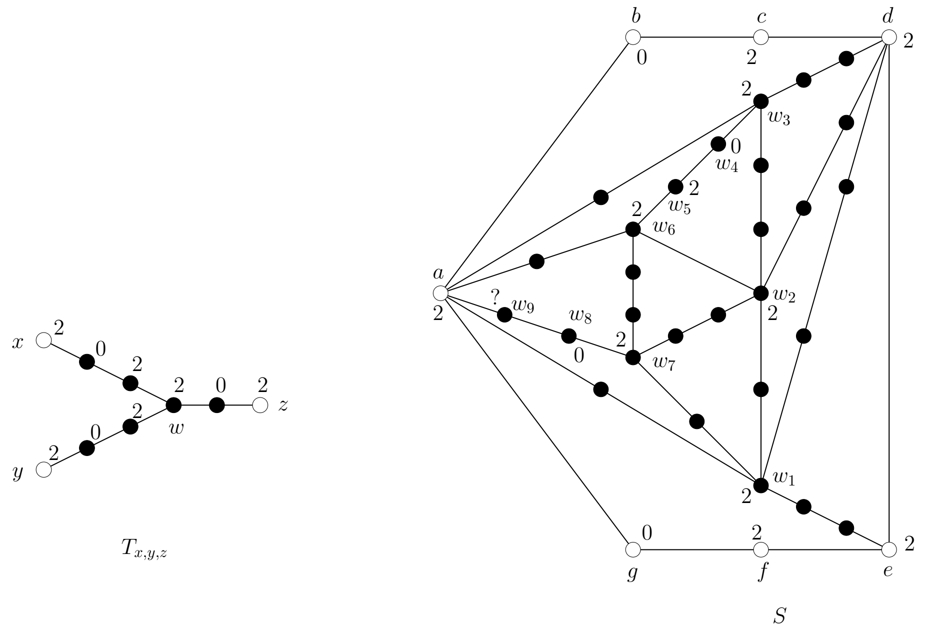

Figure 4: The graphs $T_{x,y,z}$ and S.

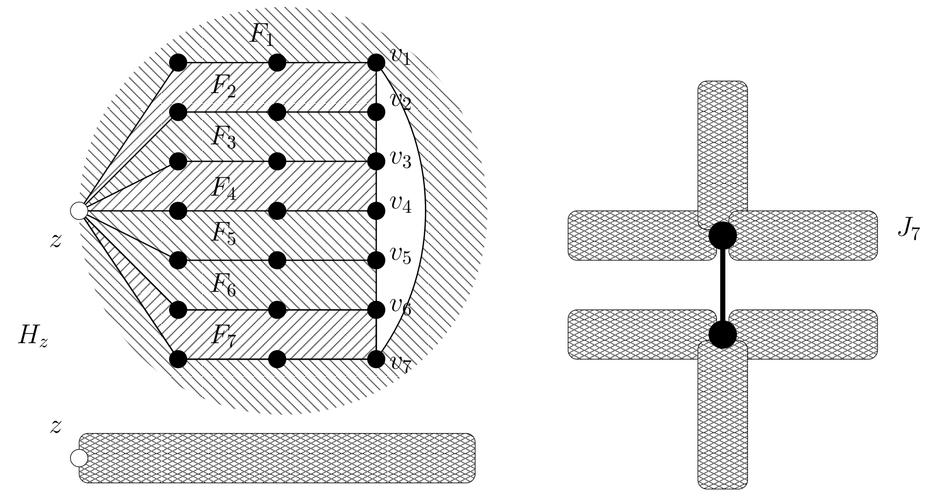

Figure 5: The graphs H_z and J_7.

5 NP-completeness of (k, j)-colorings

Let $g_{k,j}$ be the largest integer g such that there exists a planar graph with girth g that is not (k, j)-colorable. Because of large odd cycles, $g_{0,0}$ is not defined. For $(k, j) \neq (0, 0)$, we have $4 \leqslant g_{k,j} \leqslant 10$ by the example in Figure 2 and the result that planar graphs with girth at least 11 are $(0, 1)$-colorable [22]. We prove this equivalent form of Theorem 3:

Theorem 5. *Let k and j be fixed integers such that $(k, j) \neq (0, 0)$. It is NP-complete to determine whether a planar graph with girth $g_{k,j}$ is (k, j)-colorable.*

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of 3^+-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions holds:

- $|V(G_1)| < |V(G_2)|$ and $n_3(G_1) \leqslant n_3(G_2)$.
- $n_3(G_1) < n_3(G_2)$.

Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the subgraph poset. The following lemma is useful.

Lemma 6. *Let k and j be fixed integers such that $(k, j) \neq (0, 0)$. There exists a planar graph $G_{k,j}$ with girth $g_{k,j}$, minimally non-(k, j)-colorable for the subgraph order, such that $\delta(G_{k,j}) = 2$.*

Proof. We have $\delta(G_{k,j}) \geqslant 2$, since a non-$(k, j)$-colorable graph that is minimal for the subgraph order does not contain a 1^--vertex. Supppose that for some pair (k, j), we construct a 2-degenerate non-(k, j)-colorable planar graph with girth $g_{k,j}$. Then this graph contains a (not necessarily proper) minimally non-(k, j)-colorable subgraph with minimum degree 2. Thus, we can prove the lemma for the following pairs (k, j) by using Claim 2.

- Pairs (k, j) such that $g_{k,j} \leqslant 4$: We actually have $g_{k,j} = 4$ by Claim 2.1.
- Pairs (k, j) such that $g_{k,j} \geqslant 6$: Indeed, a planar graph with girth at least 6 is 2-degenerate. In particular, Claim 2.3 shows that $g_{k,0} \geqslant 6$, so the lemma is proved for all pairs $(k, 0)$.
- Pairs $(k, 1)$ such that $1 \leqslant k \leqslant 3$: If $g_{k,j} \geqslant 6$, then we are in a previous case. Otherwise, we have $g_{k,j} = 5$ by Claim 2.2.

The remaining pairs satisfy $g_{k,j} = 5$. There are two types of remaining pairs (k, j):

- Type 1: $k \geqslant 4$ and $j = 1$.
- Type 2: $2 \leqslant j \leqslant k$.

Consider a planar graph G with girth 5 that is non-(k, j)-colorable and is minimal for the order $\preceq$. Suppose for contradiction that G does not contain a 2-vertex. Also, suppose that G contains a 3-vertex a adjacent to three 4^--vertices a_1, a_2, and a_3. For colorings of type 1, we can extend to G a coloring of $G \setminus \{a\}$ by assigning to a the color of improperty at least 4. For colorings of type 2, we consider the graph G' obtained from $G \setminus \{a\}$ by adding three 2-vertices b_1, b_2, and b_3 adjacent to, respectively, a_2 and a_3, a_1 and a_3, a_1 and a_2. Notice that $G' \preceq G$, so G' admits a coloring c of type 2. We can extend to

G the coloring of $G \setminus \{a\}$ induced by c as follows. If a_1, a_2, and a_3 have the same color, then we assign to a the other color. Otherwise, we assign to a the color that appears at least twice among the colors of b_1, b_2, and b_3. Now, since G does not contain a 2-vertex nor a 3-vertex adjacent to three 4^--vertices, we have $\mathrm{mad(G)} \geqslant \frac{10}{3}$. This can be seen using the discharging procedure such that the initial charge of each vertex is its degree and every 5^+-vertex gives $\frac{1}{3}$ to each adjacent 3-vertex. The final charge of a 3-vertex is at least $3 + \frac{1}{3} = \frac{10}{3}$, the final charge of a 4-vertex is $4 > \frac{10}{3}$, and the final charge of a k-vertex with $k \geqslant 5$ is at least $k - k \times \frac{1}{3} = \frac{2k}{3} \geqslant \frac{10}{3}$. Now, $\mathrm{mad(G)} \geqslant \frac{10}{3}$ contradicts the fact that G is a planar graph with girth 5, and this contradiction shows that G contains a 2-vertex. □

We are ready to prove Theorem 5. The case of $(1,0)$-coloring is proved in a stronger form which takes into account restrictions on both the girth and the maximum degree of the input planar graph [16].

Proof of the case $(k,0)$, $k \geqslant 2$.
We consider a graph $G_{k,0}$ as described in Lemma 6, which contains a path uxv where x is a 2-vertex. By minimality, any $(k,0)$-coloring of $G_{k,0} \setminus \{x\}$ is such that u and v get distinct saturated colors. Let G be the graph obtained from $G_{k,0} \setminus \{x\}$ by adding three 2-vertices x_1, x_2, and x_3 to create the path $ux_1x_2x_3v$. So any $(k,0)$-coloring of G is such that x_2 is colored k^1. To prove the NP-completeness, we reduce the $(k,0)$-coloring problem to the $(1,0)$-coloring problem. Let I be a planar graph with girth $g_{1,0}$. For every vertex s of I, add $(k-1)$ copies of G such that the vertex x_2 of each copy is adjacent to s, to obtain the graph I'. By construction, I' is $(k,0)$-colorable if and only if I is $(1,0)$-colorable. Moreover, I' is planar, and since $g_{k,0} \leqslant g_{1,0}$, the girth of I' is $g_{k,0}$.

Proof of the case $(1,1)$.
By Claim 2.2 and [9], $g_{1,1}$ is either 5 or 6. There exist two independent proofs [17, 19] that $(1,1)$-coloring is NP-complete for triangle-free planar graphs with maximum degree 4. We use a reduction from that problem to prove that $(1,1)$-coloring is NP-complete for planar graphs with girth $g_{1,1}$. We consider a graph $G_{1,1}$ as described in Lemma 6, which contains a path uxv where x is a 2-vertex. By minimality, any $(1,1)$-coloring of $G_{1,1} \setminus \{x\}$ is such that u and v get distinct saturated colors. Let G be the graph obtained from $G_{1,1} \setminus \{x\}$ by adding a vertex u' adjacent to u and a vertex v' adjacent to v. So any $(1,1)$-coloring of G is such that u' and v' get distinct colors and u' (resp. v') has a color distinct from the color of its (unique) neighbor. We construct the graph $E_{a,b}$ from two vertices a and b and two copies of G such that a is adjacent to the vertices u' of both copies of G and b is adjacent to the vertices v' of both copies of G. There exists a $(1,1)$-coloring of $E_{a,b}$ such that a and b have distinct colors and neither a nor b is saturated. There exists a $(1,1)$-coloring of $E_{a,b}$ such that a and b have the same color. Moreover, in every $(1,1)$-coloring

of $E_{a,b}$ such that a and b have the same color, both a and b are saturated.

The reduction is as follows. Let I be a planar graph. For every edge (p,q) of I, replace (p,q) by a copy of $E_{a,b}$ such that a is identified with p and b is identified with q, to obtain the graph I'. By the properties of $E_{a,b}$, I is $(1,1)$-colorable if and only if I' is $(1,1)$-colorable. Moreover, I' is planar with girth $g_{1,1}$.

Proof of the case (k,j).
We consider a graph $G_{k,j}$ as described in Lemma 6, which contains a path uxv where x is a 2-vertex. By minimality, any (k,j)-coloring of $G_{k,j} \setminus \{x\}$ is such that u and v get distinct saturated colors. Let G be the graph obtained from $G_{k,j} \setminus \{x\}$ by adding a vertex u' adjacent to u and a vertex v' adjacent to v. So any (k,j)-coloring of G is such that u' and v' get distinct colors and u' (resp. v') has a color distinct from the color of its (unique) neighbor. Let $t = \min(k-1,j)$. To prove the NP-completeness, we reduce the (k,j)-coloring to the $(k-t,j-t)$-coloring. Thus the case (k,k) reduces to the case $(1,1)$ which is NP-complete, and the case (k,j) with $j<k$ reduces to the case $(k-j,0)$ which is NP-complete. The reduction is as follows. Let I be a planar graph with girth $g_{k-t,j-t}$. For every vertex s of I, add t copies of G such that the vertices u' and v' of each copy is adjacent to s, to obtain the graph I'. By construction, I is $(k-t,j-t)$-colorable if and only if I' is (k,j)-colorable. Moreover, I' is planar, and since $g_{k,j} \leqslant g_{k-t,j-t}$, the girth of I' is $g_{k,j}$.

6 NP-completeness of (k,j,i)-colorings

In this section, we prove Theorem 4 using a reduction from 3-colorability, i.e. $(0,0,0)$-colorability, which is NP-complete for planar graphs [18].

Let E be the graph depicted in Fig 6. The graph E' is obtained from $2k-1$ copies of E by identifying the edge ab of all copies. Take 4 copies E'_1, E'_2, E'_3, and E'_4 of E' and consider a triangle T formed by the vertices y_0, x_0, x_1 in one copy of E in E'_1. The graph E'' is obtained by identifying the edge y_0x_0 (resp. y_0x_1, x_0x_1) of T with the edge ab of E'_2 (resp. E'_3, E'_4). The edge ab of E'_1 is then said to be the edge ab of E''.

Lemma 7.

1. *E'' admits a $(0,0,0)$-coloring.*

2. *E' does not admit a $(k,k,1)$-coloring such that a and b have a same color of impropriety k.*

3. *E'' does not admit a $(k,k,1)$-coloring such that a and b have the same color.*

Proof.

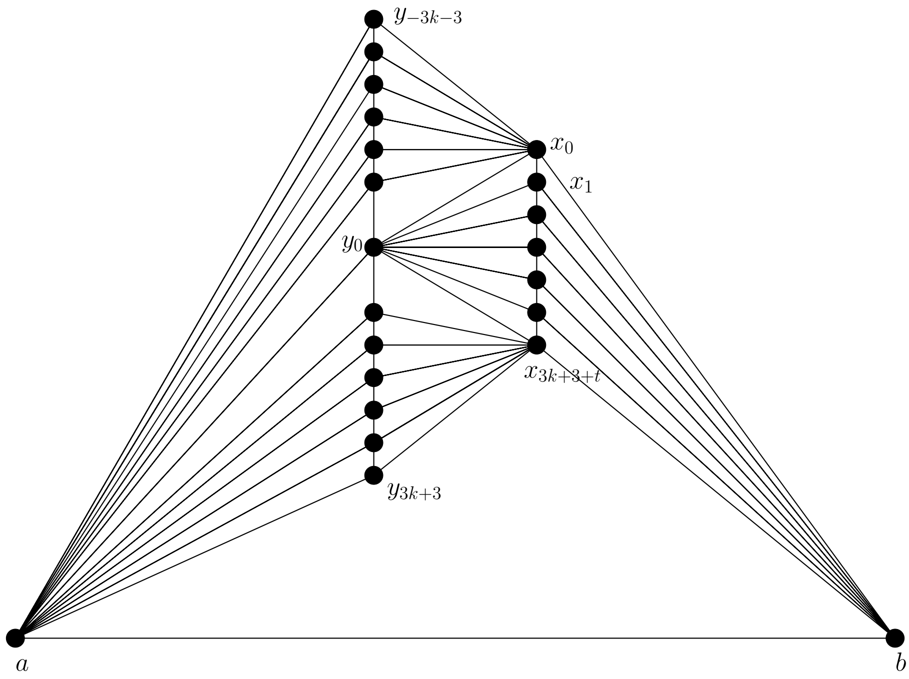

Figure 6: The graph E. We take $t = 0$ if k is odd and $t = 1$ if k is even, so that $3k + 3 + t$ is even.

1. The following $(0, 0, 0)$-coloring c of E is unique up to permutation of colors: $c(a) = c(x_i) = 1$ for even i, $c(b) = c(y_i) = 2$ for even i, and $c(x_i) = c(y_i) = 3$ for odd i. This coloring can be extended into a $(0, 0, 0)$-coloring of E' and E''.

2. Let k_1, k_2, and 1 denote the colors in a potential $(k, k, 1)$-coloring c of E' such that $c(a) = c(b) = k_1$. By the pigeon-hole principle, one of the $2k-1$ copies of E in E', say E^*, is such that a and b are the only vertices with color k_1. So, one of the vertices x_0, y_0, and x_{3k+3+t} in E^* must get color k_2 since they cannot all get color 1. We thus have a vertex $v_1 \in \{a, b\}$ colored k_1 and vertex $v_2 \in \{x_0, y_0, x_{3k+3+t}\}$ colored k_2 in E^* which both dominate a path on at least $3k + 3$ vertices. This path contains no vertex colored k_1 since it is in E^*. Moreover, it contains at most k vertices colored k_2. On the other hand, every set of 3 consecutive vertices in this path contains at least one vertex colored k_2, so it contains at least $\frac{3k+3}{3} = k + 1$ vertices colored k_2. This contradiction shows that E' does not admit a $(k, k, 1)$-coloring such that a and b have a same color of improperty k.

3. By the previous item and by construction of E'', if E'' admits a $(k, k, 1)$-coloring c such that $c(a) = c(b)$, then $c(a) = c(b) = 1$. We thus have that $\{c(y_0), c(x_0), c(x_1)\} \subset \{k_1, k_2\}$. This implies that at least one edge of the triangle T is monochromatic with a color of improperty k. By

the previous item, the coloring c cannot be extended to the copy of E' attached to that monochromatic edge. This shows that E'' does not admit a $(k, k, 1)$-coloring such that a and b have the same color.

□

For every fixed integer k, we give a polynomial construction that transforms every planar graph G into a planar graph G' such that G' is $(0, 0, 0)$-colorable if G is $(0, 0, 0)$-colorable and G' is not $(k, k, 1)$-colorable otherwise. The graph G' is obtained from G by identifying every edge of G with the edge ab of a copy of E''. If G is $(0, 0, 0)$-colorable, then this coloring can be extended into a $(0, 0, 0)$-coloring of G' by Lemma 7.1. If G is not $(0, 0, 0)$-colorable, then every $(k, k, 1)$-coloring G contains a monochromatic edge uv, and then the copy of E'' corresponding to uv (and thus G') does not admit a $(k, k, 1)$-coloring by Lemma 7.3. The instance graph G in the proof that $(0, 0, 0)$-coloring is NP-complete [18] is 3-degenerate. Then by construction, G' is also 3-degenerate.

References

[1] K. Appel and W. Haken. Every planar map is four colorable. Part I. Discharging. *Illinois J. Math.*, 21:429–490, 1977.

[2] K. Appel and W. Haken. Every planar map is four colorable. Part II. Reducibility. *Illinois J. Math.*, 21:491–567, 1977.

[3] O.V. Borodin and A.O. Ivanova. Near proper 2-coloring the vertices of sparse graphs. *Diskretn. Anal. Issled. Oper.*, 16(2):16–20, 2009.

[4] O.V. Borodin, A.O. Ivanova, M. Montassier, P. Ochem, and A. Raspaud. Vertex decompositions of sparse graphs into an edgeless subgraph and a subgraph of maximum degree at most k. *J. Graph Theory*, 65(2):83–93, 2010.

[5] O.V. Borodin, A.O. Ivanova, M. Montassier, and A. Raspaud. $(k, 1)$-coloring of sparse graphs. *Discrete Math.*, 312(6):1128–1135, 2012.

[6] O.V. Borodin, A.O. Ivanova, M. Montassier, and A. Raspaud. (k, j)-coloring of sparse graphs. *Discrete Appl. Math.*, 159(17):1947–1953, 2011.

[7] O.V. Borodin and A.V. Kostochka. Vertex partitions of sparse graphs into an independent vertex set and subgraph of maximum degree at most one. *Sibirsk. Mat. Zh.*, 52(5):1004–1010, 2011. (in Russian.)

[8] O.V. Borodin and A.V. Kostochka. Defective 2-coloring of sparse graphs. *J. Combin. Theory S. B*, 104:72–80, 2014.

[9] O.V. Borodin, A.V. Kostochka, and M. Yancey. On 1-improper 2-coloring of sparse graphs. *Discrete Math.*, 313(22):2638–2649, 2013.

[10] G.J. Chang, F. Havet, M. Montassier, and A. Raspaud. Steinberg's Conjecture and near-colorings. *Research Report* RR-7669, INRIA, 2011.

[11] H. Choi, I. Choi, J. Jeong, and G. Suh. $(1, k)$-coloring of graphs with girth at least 5 on a surface. arXiv:1412.0344

[12] I. Choi and A. Raspaud. Planar graphs with girth at least 5 are (3,5)-colorable. *Discrete Math.*, 318(4):661–667, 2015.

[13] L.J. Cowen, R.H. Cowen, and D.R. Woodall. Defective colorings of graphs in surfaces: partitions into subgraphs of bounded valency. *J. Graph Theory*, 10(2):187–195, 1986.

[14] P. Dorbec, T. Kaiser, M. Montassier, and A. Raspaud. Limits of near-coloring of sparse graphs. *J. Graph Theory*, 75(2):191–202, 2014.

[15] N. Eaton and T. Hull. Defective list colorings of planar graphs. *Bull. Inst. Combin. Appl.*, 25:79–87, 1999.

[16] L. Esperet, M. Montassier, P. Ochem, and A. Pinlou. A complexity dichotomy for the coloring of sparse graphs. *J. Graph Theory*, 73(1):85–102, 2013.

[17] J. Fiala, K. Jansen, V.B. Le, and E. Seidel. Graph subcolorings: complexity and algorithms. *SIAM J. Discrete Math.*, 16(4):635–650, 2003.

[18] M.R. Garey, D.S. Johnson, and L.J. Stockmeyer, Some simplified NP-complete graph problems. *Theor. Comput. Sci.*, 1:237–267, 1976.

[19] J. Gimbel and C. Hartman. Subcolorings and the subchromatic number of a graph. *Discrete Math.*, 272:139–154, 2003.

[20] A.N. Glebov, D.Zh. Zambalaeva. Path partitions of planar graphs. *Sib. Elektron. Mat. Izv.*, http://semr.math.nsc.ru, 4:450–459, 2007. (in Russian.)

[21] F. Havet and J.-S. Sereni. Improper choosability of graphs and maximum average degree. *J. Graph Theory*, 52:181–199, 2006.

[22] J. Kim, A.V. Kostochka, and X. Zhu. Improper coloring of sparse graphs with a given girth, I: (0,1)-colorings of triangle-free graphs. *European J. Combin.*, 42:26–48, 2014.

[23] R. Škrekovski. List improper coloring of planar graphs. *Combin. Probab. Comput.*, 8:293–299, 1999.