On the acyclic choosability of graphs

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Abstract

A proper vertex coloring of a graph $G = (V, E)$ is acyclic if G contains no bicolored cycle. A graph G is L-list colorable if for a given list assignment $L = \{L(v) : v \in V\}$, there exists a proper coloring c of G such that $c(v) \in L(v)$ for all $v \in V$. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V$, then G is said k-choosable. A graph is said to be acyclically k-choosable if the obtained coloring is acyclic. In this paper, we study the links between acyclic k-choosability of G and $Mad(G)$ defined as the maximum average degree of the subgraphs of G and give some observations about the relationship between acyclic coloring, choosability and acyclic choosability.

1 Introduction

Let G be a graph. Let $V(G)$ be its set of vertices and $E(G)$ be its set of edges. A proper vertex coloring of G is an assignment f of integers (or labels) to the vertices of G such that $f(u) \neq f(v)$ if the vertices u and v are adjacent in G. A *k-coloring* is a proper vertex coloring using k colors. A proper vertex coloring of a graph is *acyclic* if there is no bicolored cycle. The *acyclic chromatic number* of G, $\chi_a(G)$, is the smallest integer k such that G is acyclically k-colorable. Acyclic colorings were introduced by Grünbaum in [Grü73] and studied by Mitchem [Mit74], Albertson, Berman [AB77], and Kostochka [Kos76]. In 1979, Borodin proved Grünbaum's conjecture :

Theorem 1 *[Bor79] Every planar graph is acyclically 5-colorable.*

This bound is best possible : In 1973, Grünbaum gave an example of a 4-regular planar graph [Grü73] which is not acyclically colorable with four colors. Moreover, there exist bipartite 2-degenerate planar graphs which are not acyclically 4-colorable [KM76] (see Figure 1).

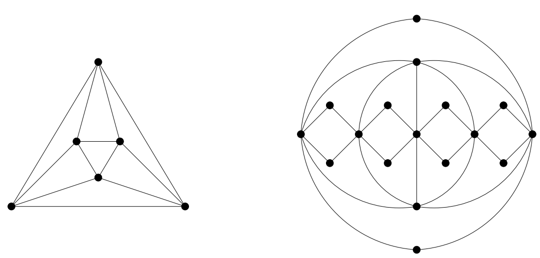

FIG. 1 – Grünbaum's example and Kostochka-Mel'nikov's example.

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Borodin, Kostochka and Woodall improved this bound for planar graphs with a given girth. We recall that the girth of a graph is the length of its shortest cycle.

Theorem 2 *[BKW99]*

1. *Every planar graph with girth at least 7 is acyclically 3-colorable.*
2. *Every planar graph with girth at least 5 is acyclically 4-colorable.*

A graph G is L-list colorable if for a given list assignment $L = \{L(v) : v \in V(G)\}$ there exists a coloring c of the vertices such that $c(v) \in L(v)$ and $c(v) \neq c(u)$ if u and v are adjacent in G. If G is L-list colorable for every list assignment with $|L(v)| \geq k$ for all $v \in V(G)$, then G is said k-choosable. In [Tho94], Thomassen proved that every planar graph is 5-choosable and Voigt proved that there are planar graphs which are not 4-choosable [Voi93]. In the following, we are interested in the acyclic choosability of graphs. In [BFDFK$^+$02], the following theorem is proved and the next conjecture is given :

Theorem 3 *[BFDFK$^+$02] Every planar graph is acyclically 7-choosable.*

This means that for any given list assignment L such that $\forall v \in V, |L(v)| \geq 7$, we can choose for each vertex v a color in $L(v)$ such that the obtained coloring of G is acyclic. The *acyclic list chromatic number* of G, $\chi_a^l(G)$, is the smallest integer k such that G is acyclically k-choosable.

Conjecture 1 *[BFDFK$^+$02] Every planar graph is acyclically 5-choosable.*

Conjecture 1 is very strong, since it implies the celebrated result of Borodin (Theorem 1), and we know that its proof is tough.

A first observation can be made concerning outerplanar graphs :

Proposition 1 *Every outerplanar graph is acyclically 3-choosable.*

Since outerplanar graphs are partial 2-trees, Proposition 1 follows from the following easy result :

Proposition 2 *Every k-tree is acyclically $(k+1)$-choosable.*

We can consider Proposition 2 as a counterpart for acyclic choosability of the following well-known fact :

Proposition 3 *Every k-degenerate graph is $(k+1)$-choosable.*

Now, we will prove that some sparse graphs verify the property of Conjecture 1. For this, we recall a graph invariant : the maximum average degree.

Definition 1 *Let G be a graph, the maximum average degree of G, denoted by $Mad(G)$ is :*

$$Mad(G) = \max\{2|E(H)|/|V(H)|, H \subseteqq G\}$$

Notice that the maximum average degree of a graph can be computed in polynomial time by using the Matroid Partitioning Algorithm due to Edmonds [Edm65, SU97].

Our main result is the following :

Theorem 4

1. *Every graph G with $Mad(G) < \frac{8}{3}$ is acyclically 3-choosable.*
2. *Every graph G with $Mad(G) < \frac{19}{6}$ is acyclically 4-choosable.*
3. *Every graph G with $Mad(G) < \frac{24}{7}$ is acyclically 5-choosable.*

We can apply these results to planar graphs by using the following well known observation based on the Euler's formula :

Observation 1 *If G is a planar graph with girth g, then $Mad(G) < \frac{2g}{g-2}$.*

Corollary 1

1. *Every planar graph with girth at least 8 is acyclically 3-choosable.*
2. *Every planar graph with girth at least 6 is acyclically 4-choosable.*
3. *Every planar graph with girth at least 5 is acyclically 5-choosable.*

In the following, we prove Theorem 4.1 in section 3, Theorem 4.2 in section 4 and Theorem 4.3 in section 5. In section 6, we give some hints for new directions of research and section 7 provides some observations about the relationship between χ_a, χ_l and χ_a^l.

2 Proof technique

In what follows, we call respectively k-vertex, $^{\geq}k$-vertex and $^{\leq}k$-vertex a vertex of degree k, $\geq k$, $\leq k$. We denote by $c(x)$ the color assigned to the vertex x. A $d(k)$-vertex is a d-vertex adjacent to at least k 2-vertices. The proof of Theorem 4 is based on the method of *reducible configurations* and on the *discharging method*, as used in [BKN$^+$99]. To obtain a result of the form "every graph G with $Mad(G) < q$ is acyclically n-choosable", we proceed as follows : We consider a graph H that is not acyclically n-choosable and is minimal for the subgraph partial order. This means that for every proper subgraph H' of H, $\chi_a^l(H') \leq n$. First, we provide a set S of configurations that H cannot contain due to its minimality property. To show that a configuration $C \in S$ is forbidden, we suppose that H contains C and we consider H together with a list assignment L witnessing that $\chi_a^l(H) > n$. We then argue that an acyclic coloring c (chosen from L) of some proper subgraph of H can be extended in a acyclic coloring (chosen from L) of the whole graph H, which is a contradiction. Now, we have to prove that any graph K avoiding S satisfies $Mad(K) \geq q$. We assume that every vertex v is assigned an initial charge equal to its degree $d(v)$ and define a suitable *discharging procedure* that preserves the total charge. We show that if the discharging procedure is applied to a graph K avoiding S, then the final charge $d^*(v)$ of every vertex $v \in V(K)$ satisfies $d^*(v) \geq q$. We thus have

$$Mad(K) \geq \frac{2|E(K)|}{|V(K)|} = \frac{\sum_{v \in V(K)} d(v)}{|V(K)|} = \frac{\sum_{v \in V(K)} d^*(v)}{|V(K)|} \geq \frac{q|V(K)|}{|V(K)|} = q.$$

In all the figures depicting forbidden configurations, all the neighbors of "white" vertices are drawn, whereas "black" vertices may have other neighbors in the graph. Two or more black vertices may coincide in a single vertex, provided they do not share a common white neighbor.

3 Proof of Theorem 4.1

We prove now that every graph G with $Mad(G) < 8/3$ is acyclically 3-choosable.

3.1 Forbidden configurations

Lemma 1 *Let $n \geq 3$ and let H be a minimal graph such that $\chi_a^l(H) > n$. Then H does not contain*

1. *a d-vertex adjacent to a clique of size d $(0 \leq d \leq n-1)$,*
2. *a $d(d)$-vertex $(2 \leq d \leq n^2 - 1)$,*
3. *a $d(d-1)$-vertex $(2 \leq d \leq (n-1)^2)$,*
4. *a $d(2)$-vertex $(2 \leq d \leq n)$,*
5. *a $d(1)$-vertex $(2 \leq d \leq n-1)$.*

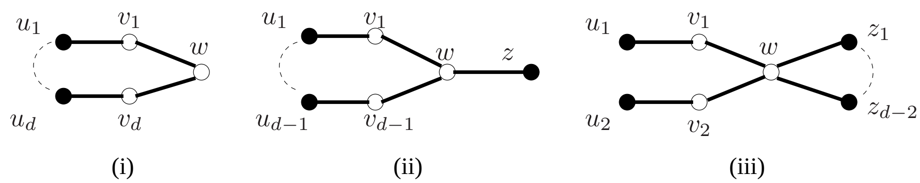

FIG. 2 – (i) : A $d(d)$-vertex. (ii) : A $d(d-1)$-vertex. (iii) : A $d(2)$-vertex.

Proof

1. Trivial.
2. Suppose that H contains a $d(d)$-vertex w adjacent to d 2-vertices $v_1, \dots, v_d$. Each vertex v_i is adjacent to w and to another vertex u_i, $1 \leq i \leq d$ (see Figure 2(i)). The vertices u_i are not necessarily distinct. Let c be a coloring of $H \setminus \{w, v_1, \dots, v_d\}$. Since $d \leq n^2 - 1$ and $|L(w)| = n$, the pigeonhole principle ensures that some $j \in L(w)$ is used at most $n-1$ times to color the u_i. We set $c(w) = j$. If $c(u_i) \neq j$, we can choose $c(v_i)$ in $L(v_i) \setminus \{c(u_i), j\}$ since $|L(v_i)| = n \geq 3$. The number of v_i such that $c(u_i) = j$ is at most $n-1$, so we can give these v_i distinct colors different from j.
3. Suppose that H contains a $d(d-1)$-vertex w adjacent to $(d-1)$ 2-vertices $v_1, \dots, v_{d-1}$ and to another vertex z. Each vertex v_i is adjacent to w and to another vertex u_i, $1 \leq i \leq d-1$ (see Figure 2(ii)). Let c be a coloring of $H \setminus \{w, v_1, \dots, v_{d-1}\}$. Note that we have $|L(w) \setminus \{c(z)\}| \geq n-1$ and $d-1 \leq (n-1)^2 - 1$. We set $c(w) = j$ where $j \in L(w) \setminus \{c(z)\}$ is used at most $n-2$ times to color the u_i. If $c(u_i) \neq j$, we can choose $c(v_i)$ in $L(v_i) \setminus \{c(u_i), j\}$ since $L(v_i) = n \geq 3$. The number of v_i such that $c(u_i) = j$ is at most $n-2$, so we can give these v_i distinct colors different from j and $c(z)$.
4. Suppose that H contains a $d(2)$-vertex w adjacent to $z_1, \dots, z_{d-2}$, and to two 2-vertices v_1, v_2 that are adjacent respectively to u_1, u_2 (see Figure 2(iii)). We assume $n \geq 4$ since the case $n = 3$ is implied by Lemma 1.3. Let c be a coloring of $H \setminus \{w, v_1, v_2\}$.
   4.1 If the $c(z_i)$ are pairwise distinct, we choose $c(w) \in L(w) \setminus \{c(z_1), \dots, c(z_{d-2}), c(u_1)\}$ and $c(v_1) \in L(v_1) \setminus \{c(w), c(u_1)\}$. If $c(w) = c(u_2)$, we choose $c(v_2) \in L(v_2) \setminus \{c(z_1), \dots, c(z_{d-2}), c(w)\}$; otherwise we choose $c(v_2) \in L(v_2) \setminus \{c(w), c(u_2)\}$.
   4.2 If the $c(z_i)$ are not pairwise distinct, we consider a coloring c of $H \setminus \{v_1, v_2\}$ and assume w.l.o.g. that $c(z_1) = c(z_2)$. If $c(w) = c(u_1)$, we choose $c(v_1) \in L(v_1) \setminus \{c(z_2), \dots, c(z_{d-2}), c(w)\}$, otherwise we choose $c(v_1) \in L(v_1) \setminus \{c(u_1), c(w)\}$. If $c(w) = c(u_2)$, we choose $c(v_2) \in L(v_2) \setminus \{c(z_2), \dots, c(z_{d-2}), c(v_1), c(w)\}$, otherwise we choose $c(v_2) \in L(v_2) \setminus \{c(u_2), c(w)\}$.
5. The proof is similar (and simpler) to that of Lemma 1.4.

$\square$

It follows that the minimum degree of H is at least 2 and that no 2-vertex is in a triangle.

3.2 Discharging procedure

We use the following discharging rule : Each vertex gives $\frac{1}{3}$ to each of its 2-neighbors. Let us check that for every $v \in V(H)$, $d^*(v) \geq \frac{8}{3}$:

- If $d(v) = 2$, then $d^*(v) = 2 + 2\frac{1}{3} = \frac{8}{3}$, since v has no 2-neighbor by Lemma 1.3 and v receives $\frac{1}{3}$ from each neighbor.
- If $d(v) = 3$, then $d^*(v) \geq 3 - \frac{1}{3} = \frac{8}{3}$, since v has at most one 2-neighbor by Lemma 1.3, so it gives at most $\frac{1}{3}$.
- If $d(v) = k \geq 4$, then $d^*(v) \geq k - k\frac{1}{3} = \frac{2k}{3} \geq \frac{8}{3}$ because v gives at most k times $\frac{1}{3}$.

4 Proof of Theorem 4.2

We prove now that every graph G with $Mad(G) < 19/6$ is acyclically 4-choosable.

4.1 Forbidden configurations

Lemma 2 *Let $n \geq 4$ and let H be a minimal graph such that $\chi_a^l(H) > n$. Then H does not contain*

1. *a 5(3)-vertex adjacent to a 3-vertex,*
2. *a 3-vertex adjacent to two 3-vertices.*

Proof

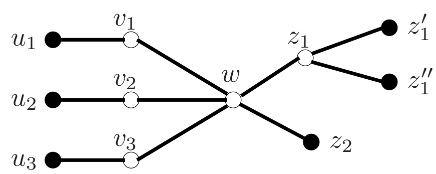

FIG. 3 – A 5(3)-vertex adjacent to a 3-vertex.

1. Suppose that H contains a 5(3)-vertex w adjacent to three 2-vertices v_1, v_2, v_3 (each adjacent to another vertex u_i), a 3-vertex z_1 (adjacent to z_1' and z_1'') and another vertex z_2 (see Figure 3). Let c be a coloring of $H \setminus \{v_1\}$. If $c(u_1) \neq c(w)$, we give a proper color to v_1. Now, we assume that $c(u_1) = c(w) = 1$:
   - 1.1 If $c(z_1) \neq c(z_2)$, we erase the colors of v_2, v_3 and we modify the color of w : In $L(w) \setminus \{c(z_1), c(z_2)\}$, there is a color which appears on at most one of u_1, u_2, u_3 ; we choose this color for w. Then, we give a color different from $c(z_1), c(z_2), c(w)$ to the vertex v_j (if it exists) whose neighbors have the same color ($c(w)$) and we give a proper color to the other v_i.
   - 1.2 If $c(z_1) = c(z_2)$ and w.l.o.g., $c(z_1) = 2$. Observe that $L(v_1)$ contains 1 and 2 ; otherwise, we can color v_1 with a color different from 1,2 and $c(v_2), c(v_3)$. We assume w.l.o.g. that $L(v_1) = \{1, 2, 3, 4\}$. If we cannot color v_1 this implies that $c(u_1) = c(u_2) = c(u_3) = 1$, $c(v_2) = 3$, $c(v_3) = 4$ and $c(z_1) = 2$.
     - 1.2.1 If $c(z_1') \neq c(z_1'')$, we modify the colors of z_1, w and give proper colors to v_1, v_2, v_3 : $c(z_1) \in L(z_1) \setminus \{c(z_1'), c(z_1''), 2\}$, $c(w) \in L(w) \setminus \{c(z_1), c(z_2), 1\}$.
     - 1.2.2 If $c(z_1') = c(z_1'')$, we modify the color of w with a color different from 1, 2, $c(z_1')$ and give proper colors to v_1, v_2, v_3.

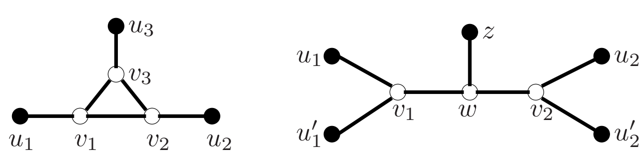

FIG. 4 – A 3-vertex having two 3-neighbors.

2. First suppose that H contains a 3-vertex adjacent to two adjacent 3-vertices (see Figure 4, left). Let c be a coloring of $H \setminus \{v_1, v_2, v_3\}$. We can choose $c(v_1)$ in $L(v_1) \setminus \{c(u_1), c(u_2), c(u_3)\}$, $c(v_2)$ in $L(v_2) \setminus \{c(v_1), c(u_2), c(u_3)\}$, and then $c(v_3)$ in $L(v_3) \setminus \{c(v_1), c(v_2), c(u_3)\}$. Now suppose that H contains a 3-vertex w adjacent to two 3-vertices v_1, v_2 (each adjacent to u_1, u_1' and u_2, u_2') and to another vertex z (see Figure 4, right). Let c be a coloring of $H \setminus \{w\}$. We have to consider the following cases :

2.1 $c(v_1)$, $c(v_2)$ and $c(z)$ are pairwise distinct. We color w with a proper color.

2.2 $c(v_1) = c(v_2) \neq c(z)$. W.l.o.g., suppose that $c(v_1) = c(v_2) = 2$ and $c(z) = 1$. Observe that $L(w)$ contains 1 and 2 ; otherwise, we color w with a color different from 1 or 2 and different from $c(u_1), c(u_1')$. Assume that $L(w) = \{1, 2, 3, 4\}$. If we cannot color w, this implies that $\{c(u_1), c(u_1')\} = \{c(u_2), c(u_2')\} = \{3, 4\}$. As well, observe that $L(v_1) = L(v_2) = \{1, 2, 3, 4\}$; otherwise, we modify the color of v_1 (or v_2) with a color different from 1,2,3,4 to get case 2.1. Hence, we recolor v_1 and v_2 with 1 and color w with 2.

2.3 $c(v_1) = c(z) \neq c(v_2)$. W.l.o.g., suppose that $c(v_1) = c(z) = 1$ and $c(v_2) = 2$. With the same argument as above, we can assume that $L(w) = \{1, 2, 3, 4\}$ and $L(v_1) = \{1, 2, 3, 4\}$. We recolor v_1 with 2 to get case 2.2.

2.4 $c(v_1) = c(v_2) = c(z)$. Observe that $c(u_1) = c(u_1')$; otherwise, we modify the color of v_1 to get a previous case. We have $c(u_2) = c(u_2')$ for the same reason and we can choose $c(w) \in L(w) \setminus \{c(u_1), c(u_2), c(z)\}$.

□

4.2 Discharging procedure

We use the following discharging rule : Each $^{\geq}4$-vertex gives $\frac{7}{12}$ to each of its 2-neighbors and $\frac{1}{12}$ to each of its 3-neighbors. Let us check that for every $v \in V(H)$, $d^*(v) \geq \frac{19}{6}$:

- If $d(v) = 2$, then v has two $^{\geq}4$-neighbors by Lemma 1.5, so $d^*(v) = 2 + 2\frac{7}{12} = \frac{19}{6}$.
- If $d(v) = 3$, then v has at least two $^{\geq}4$-neighbors by Lemma 1.5 and Lemma 2.2, so $d^*(v) \geq 3 + 2\frac{1}{12} = \frac{19}{6}$.
- If $d(v) = 4$, then v has at most one 2-neighbor by Lemma 1.4, so $d^*(v) \geq 4 - \frac{7}{12} - 3\frac{1}{12} = \frac{19}{6}$.
- If $d(v) = 5$, then v has at most three 2-neighbors by Lemma 1.3. If v is a $5(3)$-vertex, then it has no 3-neighbor by Lemma 2.1, so $d^*(v) = 5 - 3\frac{7}{12} = \frac{13}{4} > \frac{19}{6}$. Otherwise, $d^*(v) \geq 5 - 2\frac{7}{12} - 3\frac{1}{12} = \frac{43}{12} > \frac{19}{6}$.
- If $d(v) = k, 6 \leq k \leq 7$, then v has at most $(k-2)$ 2-neighbors by Lemma 1.3, so $d^*(v) \geq k - (k-2)\frac{7}{12} - 2\frac{1}{12} = \frac{5k}{12} + 1 \geq \frac{7}{2} > \frac{19}{6}$.
- If $d(v) = k \geq 8$, then $d^*(v) \geq k - k\frac{7}{12} = \frac{5k}{12} \geq \frac{10}{3} > \frac{19}{6}$.

5 Proof of Theorem 4.3

We prove now that every graph G with $Mad(G) < 24/7$ is acyclically 5-choosable. A vertex is said *weak* if it is either a 3-vertex or a 6(4)-vertex.

5.1 Forbidden configurations

Lemma 3 *Let $n \geq 5$ and let H be a minimal graph such that $\chi_a^l(H) > n$. Then H does not contain*

1. *a $d(d-2)$-vertex adjacent to a weak vertex, with $3 \leq d \leq 10$,*
2. *a 6(3)-vertex adjacent to three weak vertices,*
3. *a 6(4)-vertex adjacent to a $^{\leq}4$-vertex,*
4. *a 4-vertex adjacent to three 3-vertices.*

Proof

1. Suppose that H contains a $d(d-2)$-vertex w adjacent to $(d-2)$ 2-vertices v_i, $1 \leq i \leq d-2$ (each adjacent to another vertex u_i), a 3-vertex z (adjacent to two other vertices z_1, z_2) and a vertex y, with $3 \leq d \leq 10$ (see Figure 5).

   Let c be a coloring of $H \setminus \{v_i, 1 \leq i \leq d-2\}$.

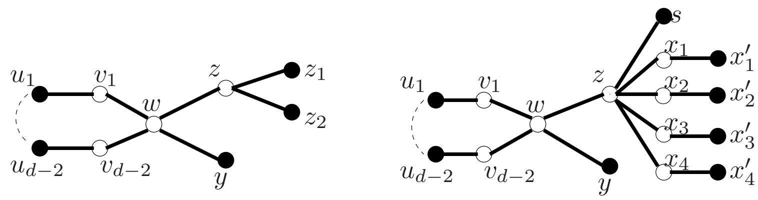

FIG. 5 – A $d(d-2)$-vertex adjacent to a weak vertex.

- $c(z) \neq c(y)$. We recolor w with a color, different from $c(z), c(y)$, which appears on at most two of the u_i, $1 \leq i \leq d-2$. If $c(u_i) \neq c(w)$, we color v_i with a proper color. At most two of the u_i (say u_1, u_2) satisfy $c(u_i) = c(w)$. We can choose $c(v_1) \in L(v_1) \setminus \{c(w), c(y), c(z)\}$ and $c(v_2) \in L(v_2) \setminus \{c(w), c(y), c(z), c(v_1)\}$.
- $c(z) = c(y)$. Observe that $c(z_1) = c(z_2)$; otherwise, we replace the color of z with a color different from $c(z_1), c(z_2), c(y), c(w)$ and we are in the previous case. Now, we recolor w with a color, different from $c(z), c(z_1)$, which appears on at most two of the u_i, $1 \leq i \leq d-2$. As above, it is easy then to color v_i, $1 \leq i \leq d-2$.

Now, we consider the case where the $d(d-2)$-vertex w is adjacent to a 6(4)-vertex z adjacent to four 2-vertices x_j, $1 \leq j \leq 4$ and another vertex s (see Figure 5). Observe that $x_i \neq u_j$ for all i, j since there is no 2(1)-vertex by Lemma 1.3. Let c be a coloring of $H \setminus \{x_1\}$.
- $c(w) \neq c(s)$. We erase the colors of the vertices z, x_2, x_3, x_4. We recolor z with a color, different from $c(s), c(w)$, which appears on at most one of $x_i', 1 \leq i \leq 4$. Then, we give a proper color to x_i for each index i such that $c(x_i') \neq c(z)$ and give a color different from $c(z), c(w), c(s)$ to the vertex x_j such that $c(z) = c(x_j')$.
- $c(w) = c(s)$. If $c(x_1') \neq c(z)$, we color x properly, which suffices. If $c(z) \neq c(x_i')$ for some i, we color x_1 avoiding $c(w), c(z)$, and all $c(x_j)$ for $j \neq i$, $j > 1$, which suffices.

Thus we may assume that $c(x_1') = c(x_2') = c(x_3') = c(x_4') = c(z) = 1$ and $c(s) = c(w) = 2$. Now, we erase the colors of the vertices x_i ($1 \leq i \leq 4$), v_j ($1 \leq j \leq d-2$), w and z. We recolor w with a color different from $c(y)$ and 2, which appears on at most two of the u_j. So, $c(s) \neq c(w)$ and we recolor z with a color different from 1,2, $c(w), c(y)$, then we color each x_i with a proper color. Finally, we recolor the v_i as in the case $c(z) \neq c(y)$.

2. Suppose that H contains a 6(3)-vertex w adjacent to three 2-vertices v_1, v_2, v_3 (each adjacent to another vertex u_i) and three weak vertices z_1, z_2, z_3. Let c be a coloring of $H \setminus \{v_1, v_2, v_3\}$.

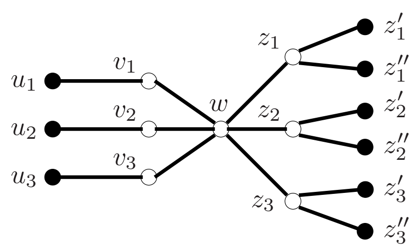

FIG. 6 – A 6(3)-vertex w adjacent to three 3-vertices.

First, observe that if $c(z_1), c(z_2), c(z_3)$ are all different, we can color v_1, v_2, v_3 : We recolor w with a color different from $c(z_1), c(z_2), c(z_3)$, which appears on at most one of u_1, u_2, u_3. Then, we give a proper color to v_i for each index i for which $c(u_i) \neq c(w)$ and a color different from $c(w), c(z_1), c(z_2), c(z_3)$ otherwise.

Second, observe that if $c(z_1) = c(z_2) = c(z_3)$, we can color v_1, v_2, v_3 : If $c(u_i) \neq c(w)$, we give a proper color to v_i. In the worst case, we have $c(u_1) = c(u_2) = c(u_3) = c(w)$ and we color v_1 with $c(v_1) \in L(v_1) \setminus \{c(w), c(z_1)\}$, v_2 with $c(v_2) \in L(v_2) \setminus \{c(w), c(z_1), c(v_1)\}$ and

v_3 with $c(v_3) \in L(v_3) \setminus \{c(w), c(z_1), c(v_1), c(v_2)\}$.

Consider now the case where two of z_1, z_2, z_3 have the same color. W.l.o.g., we assume that $c(w) = 1, c(z_1) = c(z_2) = 2, c(z_3) = 3$.

Third, observe that if $c(u_1) \neq 1$, we can color v_1, v_2, v_3 : We color v_1 and v_2 such that $c(v_1) \in L(v_1) \setminus \{1, c(u_1)\}$ and $c(v_2) \in L(v_2) \setminus \{1, 2, 3, c(u_2)\}$. Then if $c(u_3) = 1$, we choose $c(v_3)$ in $L(v_3) \setminus \{1, 2, 3, c(v_2)\}$ and otherwise we choose $c(v_3)$ in $L(v_3) \setminus \{1, c(u_3)\}$.

So, suppose now that $c(u_1) = c(u_2) = c(u_3) = c(w) = 1, c(z_1) = c(z_2) = 2, c(z_3) = 3$. The idea is to consider the neighborhood of the two vertices of z_1, z_2, z_3 which have the same color (z_1, z_2 in our case) and modify if necessary the color of one of these two vertices to get a previous case.

By permuting indices, we have only two cases to study :

2.1 z_1 is a 6(4)-vertex. The 6(4)-vertex z_1 is adjacent to w, to four 2-vertices x_i (each adjacent to another vertex x'_i) and another vertex s. Observe that since there is no 2(1)-vertex by Lemma 1.3, $x_i \neq u_j$ for all i, j. We erase the colors of $w, z_1, x_1, x_2, x_3, x_4$. We recolor z_1 with a color, different from 2, 3, $c(s)$, which appears on at most two of x'_i, $1 \leq i \leq 4$. We recolor now w with a color different from 1, 2, 3, $c(z_1)$ and give proper colors to v_1, v_2, v_3. Finally, we color the x_i, $1 \leq i \leq 4$: For two or fewer vertices whose neighbors have the same color, we give distinct colors different from $c(s), c(z_1)$ and give proper colors to the other vertices x_i.

2.2 z_1 and z_2 are 3-vertices. The vertex z_1 is adjacent to w and two other vertices z'_1, z''_1 and the vertex z_2 is adjacent to w and two other vertices z'_2, z''_2 (see Figure 6). It may be that z_i, z'_j, z''_k are not distinct, but it will not matter. If $c(z'_1) \neq c(z''_1)$ we can recolor z_1 and w such that $c(z_1) \in L(z_1) \setminus \{2, 3, c(z'_1), c(z''_1)\}$ and $c(w) \in L(w) \setminus \{1, 2, 3, c(z_1)\}$, and then give proper colors to the v_i, $1 \leq i \leq 3$. Thus $c(z'_1) = c(z''_1)$ and, for the same reason, $c(z'_2) = c(z''_2)$. Now we can recolor w with a color different from 1, 2, 3, $c(z'_1)$ and we give proper colors to the v_i, $1 \leq i \leq 3$.

3. Suppose that H contains a 6-vertex w adjacent to four 2-vertices v_1, v_2, v_3, v_4 (each adjacent to another vertex u_i), a $\leq$4-vertex z and another vertex y (see Figure 7). Notice that if $d(z) < 4$ then the configuration is forbidden by Lemma 2.1 and Lemma 1.3. So suppose z is a 4-vertex adjacent to z_1, z_2, z_3 (see Figure 7).

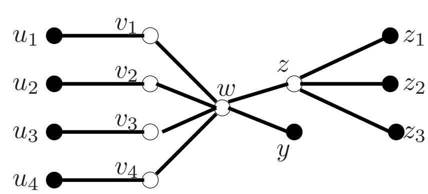

FIG. 7 – A 6(4)-vertex adjacent to a 4-vertex

Let c be a coloring of $H \setminus \{v_1, v_2, v_3, v_4\}$. If $c(y) \neq c(z)$, we recolor w with a color from $L(w) \setminus \{c(z), c(y)\}$ that appears on at most one u_i, then properly color each v_i avoiding $c(u_i), c(w), c(z)$, and $c(y)$. Suppose that $c(y) = c(z)$. If $c(u_i) \neq c(w)$, we properly color v_i and then may ignore it, so the worst case is $c(u_1) = c(u_2) = c(u_3) = c(u_4) = c(w)$. Assume that $c(u_1) = 1$ and $c(z) = 2$. Consider the following three cases :

3.1 If $c(z_1) \neq c(z_2) \neq c(z_3) \neq c(z_1)$, we modify the color of z, then we recolor w with a color different from 1, $c(z), c(y)$, then we color v_i ($i = 1, \ldots, 4$) with proper colors.

3.2 If $c(z_1) = c(z_2) \neq c(z_3)$, we recolor w such that $c(w) \in L(w) \setminus \{1, 2, c(z_1), c(z_3)\}$ and give proper colors to v_i.

3.3 If $c(z_1) = c(z_2) = c(z_3)$, we modify the color of w. We color w with $c(w) \in L(w) \setminus \{1, 2, c(z_1)\}$ and give proper colors to v_i.

4. Suppose that H contains a 4-vertex w adjacent to three 3-vertices x_1, x_2, x_3 (each adjacent to x'_i, x''_i) and to another vertex z (see Figure 8). Although x_i, x'_j, x''_k may not all be distinct, it will not matter.

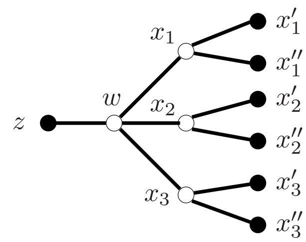

FIG. 8 – A 4-vertex adjacent to three 3-vertices

Let c be a coloring of $H \setminus \{w\}$. We consider the following cases :

4.1 If $c(x_1), c(x_2), c(x_3), c(z)$ are all different, then we color w with a proper color.

4.2 Suppose that two neighbors of w have the same color, and no color is shared by three neighbors of w.

4.2.1 Suppose that $c(x_1) = c(x_2) \neq c(x_3)$. W.l.o.g. we assume that $c(x_1) = 1$.

4.2.1.1 If $c(x_3) \neq c(z)$ and $c(x_3) \neq 1, c(z) \neq 1$, we assume that $c(x_3) = 2$ and $c(z) = 3$. Necessarily, $L(w)$ contains $1, 2, 3$; otherwise, we can color w with a color different from 1, 2, 3, $c(x'_1)$ and $c(x''_1)$. W.l.o.g., we suppose that $L(w) = \{1, 2, 3, 4, 5\}$. If we cannot color w, this implies that $\{c(x'_1), c(x''_1)\} = \{c(x'_2), c(x''_2)\} = \{4, 5\}$. Observe now that $L(x_1) = L(x_2) = \{1, 2, 3, 4, 5\}$; otherwise, we can recolor x_1 with a color different from 1, 2, 3, 4, 5 to get case 4.1. So, we recolor x_1 and x_2 with 3 and color w with 1.

4.2.1.2 If $c(x_3) = c(z)$ and $c(x_3) \neq 1$, we assume that $c(x_3) = 2$. Observe first that $c(x'_3) = c(x''_3)$; otherwise, we can recolor x_3 with a color different from $1, 2, c(x'_3), c(x''_3)$ to get case 4.2.1.1. So, suppose that $c(x'_3) = c(x''_3) = 3$ ($c(x'_3) = c(x''_3) = 1$ is an easier case). Necessarily, $L(w)$ contains 1, 2, 3 ; otherwise, we can color w with a color different from 1, 2, 3, $c(x'_1)$ and $c(x''_1)$. W.l.o.g., $L(w) = \{1, 2, 3, 4, 5\}$, and $\{c(x'_1), c(x''_1)\} = \{c(x'_2), c(x''_2)\} = \{4, 5\}$. So, we recolor x_1 and x_2 with a color different from 1, 2, 4, 5 and we color w with 1.

4.2.2 Suppose that $c(x_1) = c(z)$. W.l.o.g. we assume that $c(x_1) = 1$. Observe that $c(x_2) \neq c(x_3)$; otherwise, we get case 4.2.1.2. We assume that $c(x_2) = 2$ and $c(x_3) = 3$. Observe that $c(x'_1) = c(x''_1)$; otherwise we can recolor x_1 with a color different from $1, c(x'_1), c(x''_1)$ to get case 4.1 or 4.2.1.1. Hence, we color w with a color different from 1, 2, 3, $c(x'_1)$.

4.3 Suppose that exactly three neighbors of w have the same color.

4.3.1 We assume that $c(x_1) = c(x_2) = c(x_3) = 1$ and $c(z) = 2$. Observe that $c(x'_1) = c(x''_1)$; otherwise, we can recolor x_1 with a color different from 1, 2, $c(x'_1), c(x''_1)$ to get case 4.2.1.1. In the same way, $c(x'_i) = c(x''_i)$, $i = 1, 2, 3$. Then $L(w) = \{1, 2, c(x'_1), c(x'_2), c(x'_3)\}$ with $c(x'_1) \neq c(x'_2) \neq c(x'_3) \neq c(x'_1)$; otherwise, we color w with a color different from $1, 2, c(x'_1), c(x'_2), c(x'_3)$. So, we color w with $c(x'_1)$.

4.3.2 We assume that $c(z) = c(x_1) = c(x_2) = 1$ and $c(x_3) = 2$. As above, observe that $c(x'_1) = c(x''_1)$ and $c(x'_2) = c(x''_2)$; otherwise we can recolor x_1 or x_2 to obtain a previous case. Hence, we color w with a color different from 1, 2, $c(x'_1), c(x'_2)$.

4.4 All the neighbors of w have the same color. Suppose that $c(x_1) = c(x_2) = c(x_3) = c(z) = 1$. As above, for $i = 1, 2, 3$, $c(x'_i) = c(x''_i)$ (otherwise we can get a previous case). We color w with a color different from 1, $c(x'_1), c(x'_2), c(x'_3)$.

$\square$

5.2 Discharging procedure

We use the following discharging rule : Each $^{\geq}4$-vertex gives $\frac{5}{7}$ to each of its 2-neighbors, $\frac{3}{14}$ to each of its 3-neighbors and $\frac{1}{7}$ to each of its 6(4)-neighbors. Let us check that for every $v \in V(H)$, $d^*(v) \geq \frac{24}{7}$:

- If $d(v) = 2$, then v has two $^{\geq}5$-neighbors by Lemma 1.5, so $d^*(v) = 2 + 2\frac{5}{7} = \frac{24}{7}$.
- If $d(v) = 3$, then v has at least two $^{\geq}4$-neighbors by Lemma 1.5 and Lemma 2.2, so $d^*(v) \geq 3 + 2\frac{3}{14} = \frac{24}{7}$.
- If $d(v) = 4$, then v has no 2-neighbor by Lemma 1.5, no 6(4)-neighbor by Lemma 3.3, and at most two 3-neighbors by Lemma 3.4, so $d^*(v) \geq 4 - 2\frac{3}{14} = \frac{25}{7} > \frac{24}{7}$.
- If $d(v) = 5$, then v has at most one 2-neighbor by Lemma 1.4, so $d^*(v) \geq 5 - \frac{5}{7} - 4\frac{3}{14} = \frac{24}{7}$.
- If $d(v) = 6$, by Lemma 1.3, v has at most four 2-neighbors. If v is a 6(4)-vertex, then it has no weak neighbor by Lemma 3.1, so $d^*(v) = 6 - 4\frac{5}{7} + 2\frac{1}{7} = \frac{24}{7}$. If v has three 2-neighbors, then it has at most two weak neighbors by Lemma 3.2, so $d^*(v) \geq 6 - 3\frac{5}{7} - 2\frac{3}{14} = \frac{24}{7}$. Otherwise, v has at most two 2-neighbors, so $d^*(v) \geq 6 - 2\frac{5}{7} - 4\frac{3}{14} = \frac{26}{7} > \frac{24}{7}$.
- If $d(v) = k, 7 \leq k \leq 10$, then v has at most $(k-2)$ 2-neighbors by Lemma 1.3. If v is a $k(k-2)$-vertex, then it has no weak neighbor by Lemma 3.1 and $d^*(v) = k - (k-2)\frac{5}{7} = \frac{2k+10}{7} \geq \frac{24}{7}$. Otherwise, $d^*(v) \geq k - (k-3)\frac{5}{7} - 3\frac{3}{14} = \frac{4k+21}{14} \geq \frac{7}{2} > \frac{24}{7}$.
- If $d(v) = 11$, then v has at most nine 2-neighbors by Lemma 1.3, so $d^*(v) \geq 11 - 9\frac{5}{7} - 2\frac{3}{14} = \frac{29}{7} > \frac{24}{7}$.
- If $d(v) = k \geq 12$, then $d^*(v) \geq k - k\frac{5}{7} = \frac{2k}{7} \geq \frac{24}{7}$.

6 Optimality of Theorem 4

In order to study the tightness of Theorem 4, we introduce two measuring functions.

Definition 2 *Let $f : \mathbb{N} \rightarrow \mathbb{R}$ be the function defined by $f(n) = \inf\{Mad(H) \mid \chi_a(H) > n\}$.*

Definition 3 *Let $f_l : \mathbb{N} \rightarrow \mathbb{R}$ be the function defined by $f_l(n) = \inf\{Mad(H) \mid \chi_a^l(H) > n\}$.*

By Theorem 4, we have lower bounds on $f_l(3)$, $f_l(4)$ and $f_l(5)$. We now give graphs that provide upper bounds on these quantities.

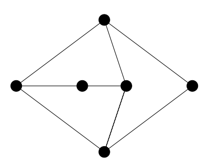

FIG. 9 – A graph G with $Mad(G) = \frac{8}{3}$ such that $\chi_a(G) = \chi_a^l(G) = 4$.

The graph G with $Mad(G) = \frac{8}{3}$ depicted in Figure 9 is acyclically 4-choosable by Theorem 4.2. To see that G is not acyclically 3-colorable, consider its four 3-vertices : Any two of them are either adjacent or have three common neighbors. Thus, different colors must be assigned to these four vertices in any acyclic 3-coloring of G. This contradiction shows that :

$$f_l(3) = f(3) = \frac{8}{3}$$

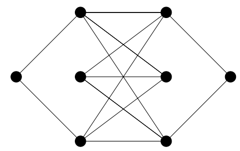

FIG. 10 – A graph G with $Mad(G) = \frac{13}{4}$ such that $\chi_a(G) = \chi_a^l(G) = 5$.

The graph G with $Mad(G) = \frac{13}{4}$ (see Figure 10) is acyclically 5-choosable : First, we assign five distinct colors to the four 4-vertices and to one of the 3-vertex, then we assign proper colors to the other vertices. To see that G is not acyclically 4-colorable, consider its four 4-vertices : Any two of them are either adjacent or have four common neighbors. Thus, different colors are assigned to the 4-vertices in any acyclic 4-coloring of G. Now, observe that properly coloring the 3-vertices produces a bicolored C_4 in every case. This contradiction shows that :

$$\frac{19}{6} \leq f_l(4) \leq f(4) \leq \frac{13}{4}$$

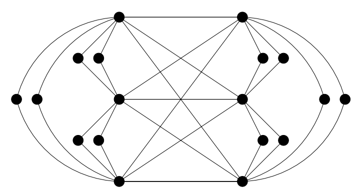

FIG. 11 – A graph G with $Mad(G) = \frac{11}{3}$ such that $\chi_a(G) = \chi_a^l(G) = 6$.

The graph G with $Mad(G) = \frac{11}{3}$ depicted by Figure 11 is acyclically 6-choosable : First, we assign distinct colors to the six 7-vertices, then we assign proper colors to the 2-vertices. To see that G is not acyclically 5-colorable, consider its six 7-vertices : Any two of them are either adjacent or have five common neighbors. Thus, different colors must be assigned to six vertices in any acyclic 5-coloring of G. This contradiction shows that :

$$\frac{24}{7} \leq f_l(5) \leq f(5) \leq \frac{11}{3}$$

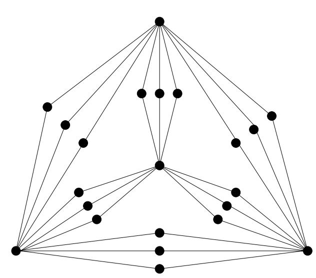

FIG. 12 – The graph G_n is such that $Mad(G_n) = 4 - \frac{8}{2+n^2}$ and $\chi_a(G_n) = \chi_a^l(G_n) = n + 1$.

We now use the construction proposed in [KM76] to obtain an asymptotic upper bound on $f(n)$. Let G_n be the graph defined as follows : G_n is a $(n+1)$-clique in which each edge is replaced by n paths with length 2 (see the graph G_3 depicted in Figure 12). It is easy to see that $Mad(G_n) = 4 - \frac{8}{2+n^2}$. The graph G_n is acyclically $(n+1)$-choosable : First, we assign distinct colors to the $^>2$-vertices, then we assign proper colors to the 2-vertices. To see that G_n is not acyclically n-colorable, consider its $^>2$-vertices : Any two of them have n common neighbors. Thus, different colors must be assigned to the $(n+1)$ $^>2$-vertices in any acyclic n-coloring of G_n. This contradiction shows that :

$$f(n) \leq 4 - \frac{8}{2+n^2}.$$

Problem 1
- *What are the values of $f_l(n)$ and $f(n)$ for $n > 3$?*
- *Does the equality $f_l(n) = f(n)$ hold also for every $n > 3$?*

We remark that we cannot reach the results of [BKW99] applied to the acyclic choosability without using some contraints of planarity : Indeed, to imply Theorem 2.2, we should have proven that every graph G with $Mad(G) < \frac{10}{3}$ is acyclically 4-choosable, which is not true, since there exists a graph G with $Mad(G) = \frac{13}{4} < \frac{10}{3}$ which is not acyclically 4-colorable (see Figure 10). Similarly, it is impossible to prove that every graph G with $Mad(G) < \frac{14}{5}$ is acyclically 3-choosable to imply Theorem 2.1, since there exists a graph G with $Mad(G) = \frac{8}{3} < \frac{14}{5}$ which is not acyclically 3-colorable (see Figure 9).

Problem 2 *Prove that planar graphs with girth at least 4 are acyclically 6-choosable.*

As the graph G_n shows, we cannot solve Problem 2 with techniques using $Mad(G)$ only.

7 Relationship between χ_a, χ_l and χ_a^l

We first consider the relationship between χ_a and χ_l. The graph G_n above satisfies $\chi_a(G_n) = \chi_a^l(G_n) = n+1$ and $\chi_l(G_n) = 3$, thus we cannot bound $\chi_a(G)$ by a function of $\chi_l(G)$ for a general graph G. On the other hand, we can show that $\chi_l(G) \leq 2\chi_a(G) - 2$ by using the following lemma :

Lemma 4 *[Xu04] Every maximal acyclically k-colorable graph with n vertices has exactly $(k-1)(n-\frac{k}{2})$ edges.*

Suppose $k \geq 2$: Lemma 4 implies that if a graph G is acyclically k-colorable, then G has arboricity $k-1$, so G is $(2k-3)$-degenerate, and thus G is $(2k-2)$-choosable. (For more details on arboricity, see [NW61])

The previous result is best possible for $k = 2$ since $\chi_a(K_2) = \chi_l(K_2) = 2$. The next statement implies in particular that it is also best possible for $k = 3$.

Claim 1 *There exist acyclically 3-colorable planar graphs without cycles of length 4 and 5 which are not 3-choosable.*

Proof
In [Voi03], Voigt gives a planar graph without cycles of length 4 and 5 which is not 3-choosable. This graph is acyclically 3-colorable. See Figure 13. □

Claim 2 *There exist acyclically 4-colorable planar graphs which are not 4-choosable.*

Proof
Let p_1 and p_2 be adjacent vertices with lists $L_{p_1} = L_{p_2} = \{1,2,3,4\}$. Then, for each pair (a,b) of colors of $\{1,2,3,4\}^2$, $a \neq b$, take the corresponding copy of the graph $F_{a,b}$ depicted in Figure 14. We identify all the vertices v_1 (resp. v_2) to the vertex p_1 (resp. p_2). It is easy to see that the obtained graph is acyclically 4-colorable and not 4-choosable. □

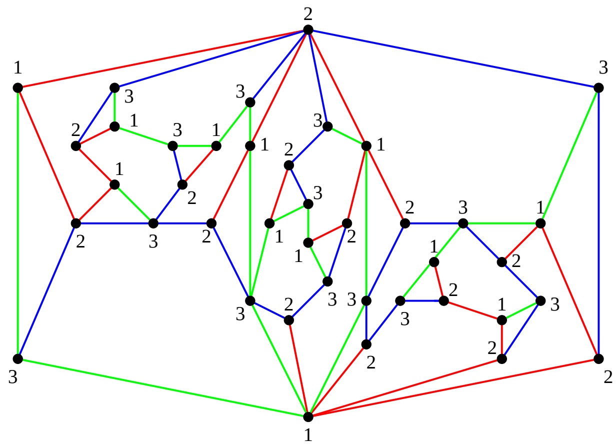

FIG. 13 – The graph of Voigt without cycles of length 4 and 5 which is not 3-choosable is constructed as follows : We take nine copies of the drawn graph and we identify all nine top vertices to a vertex v_1 and all nine bottom vertices to a vertex v_2. The given acyclic 3-coloring of the drawn graph applied to each copy gives an acyclic 3-coloring of the whole graph.

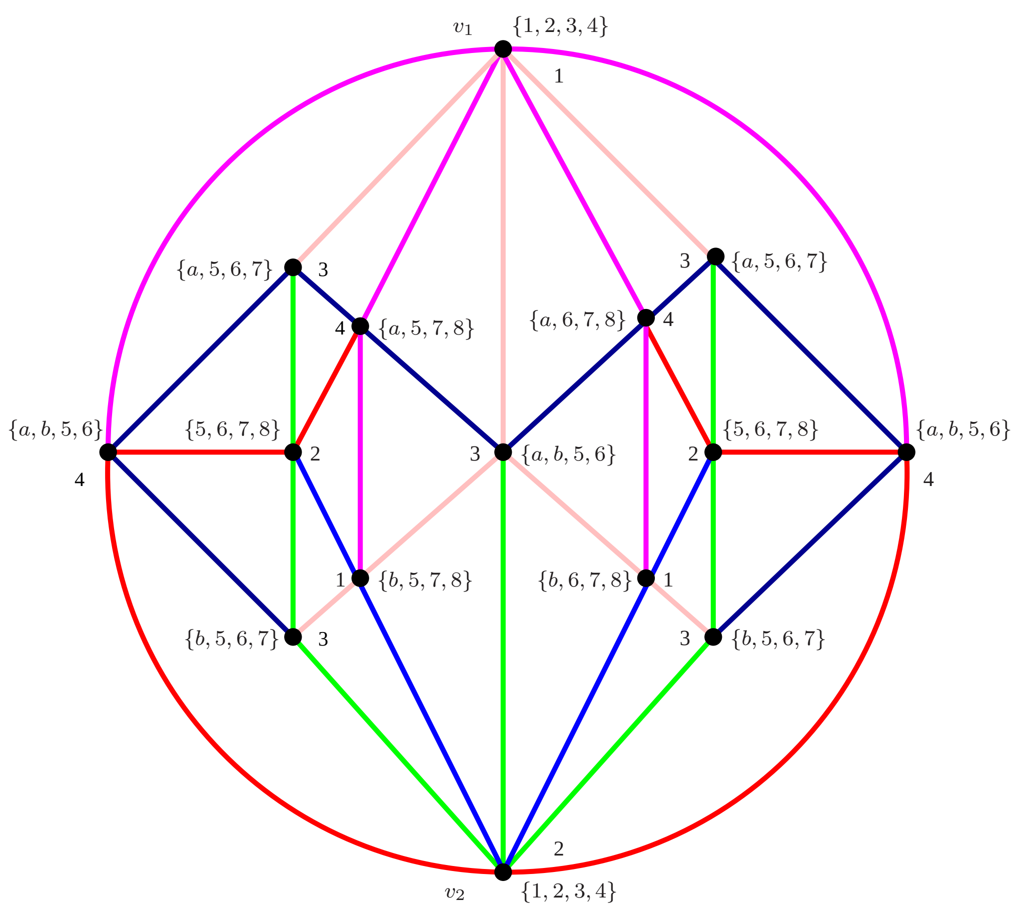

FIG. 14 – The graph $F_{a,b}$: 4-list assignment and acyclic 4-coloring.

We now consider the relationship between χ_a and χ_a^l.

Lemma 5 *Let G be a properly p-colorable graph which is not l-choosable. Let G' be the graph obtained by replacing every edge uv of G by l 2-vertices, each adjacent to u and v. Then G' is bipartite, 2-degenerate and acyclically $(\max(3,p))$-colorable, but not acyclically l-choosable.*

Proof
The graph G' is clearly bipartite and 2-degenerate. A vertex of G' that is also in G is called *old*, and for each edge uv of G, the non old vertices of G' adjacent to u and v are called (u,v)-vertices. We now give an acyclic coloring of G' using a set S of $\max(3,p)$ colors. Since $|S| \geq p$, we can take a proper coloring of G using p colors in S which colors the old vertices of G'. To color the (u,v)-vertices, we use a color of S distinct from $c(u)$ and $c(v)$: Such a color exists since $|S| \geq 3$. We check easily that this coloring is acyclic. Finally we have to show that $\chi_a^l(G') > l$. Let L be a list assignment of the old vertices with lists of size l. For each edge uv of G, pick one endpoint u, and assign the list $L(u)$ to every (u,v)-vertex. Suppose $c(u) = c(v)$. To avoid a bicolored C_4, no two (u,v)-vertices can get the same color. There are l such vertices but only $l-1$ colors in the set $L(u) \setminus c(u)$. This contradiction shows that $c(u) \neq c(v)$. Given a non-colorable list assignment of $V(G)$ with lists of size l, we can thus produce a list assignment of $V(G')$ with lists of size l that is not acyclically colorable. $\square$

It is well known that, for any fixed k, there exist bipartite graphs which are not k-choosable. There also exist 3-colorable non-4-choosable planar graphs, see [VW97, Mir96]. We can use Lemma 5 with these graphs to obtain the following claim.

Claim 3
- *For any fixed k, there exist bipartite 2-degenerate graphs which are acyclically 3-colorable but not acyclically k-choosable.*
- *There exist bipartite 2-degenerate planar graphs which are acyclically 3-colorable but not acyclically 4-choosable.*

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