Oriented colorings of triangle-free planar graphs

Pascal Ochem

LaBRI, Université Bordeaux 1
351, cours de la Libération
33405 Talence Cedex, France

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1 Introduction

Oriented graphs are directed graphs without opposite arcs. In other words an oriented graph is an orientation of an undirected graph, obtained by assigning to every edge one of the two possible orientations. If G is a graph, $V(G)$ denotes its vertex set, $E(G)$ denotes its set of edges (or arcs if G is an oriented graph) and $F(G)$ denotes its set of faces if G is planar. A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from $V(G)$ to $V(H)$ which preserves the arcs, that is $(x,y) \in E(G) \Longrightarrow (\varphi(x), \varphi(y)) \in E(H)$. We say that H is a *target graph* of G if there exists a homomorphism from G to H. The oriented chromatic number $\chi_o(G)$ of an oriented graph G is defined as the minimum order of a target graph of G. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is then defined as the maximum oriented chromatic number of its orientations. Finally, the oriented chromatic number $\chi_o(C)$ of a graph class C is defined as the maximum of $\chi_o(G)$ taken over all graphs $G \in C$. We will say that a graph G is *H-colorable* if H is a target graph of G and the vertices of H will be called *colors*.

The problem of bounding the oriented chromatic number has already been investigated for various graph classes [7]. In this note, we focus on planar graphs and we use the notation P_k for the class of planar graphs of girth at least k. In particular, it has been shown that $\chi_o(P_5) \leq 19$ [2] and $\chi_o(P_3) \leq 80$ [6]. The proofs of these results involve an auxiliary graph parameter, respectively the maximum average degree and the acyclic chromatic number. The maximum

Email address: `ochem@labri.fr` (Pascal Ochem).

average degree $mad(G)$ of a graph G is defined as the maximum of the average degrees $ad(H) = 2|E(H)|/|V(H)|$ taken over all the subgraphs H of G. Recall that a k-coloring of an undirected graph G is said to be *acyclic* if every cycle in G uses at least three colors. The acyclic chromatic number $\chi_a(G)$ of G is then defined as the minimum number k such that G has an acyclic k-coloring. More precisely, we have:

$$G \in P_5 \Longrightarrow mad(G) < 10/3,\ mad(G) < 10/3 \Longrightarrow \chi_o(G) \leq 19 \text{ [2]}$$

$$G \in P_3 \Longrightarrow \chi_a(G) \leq 5 \text{ [1]},\ \chi_a(G) \leq 5 \Longrightarrow \chi_o(G) \leq 80 \text{ [6]}$$

Unfortunately, these graph parameters cannot directly give us a good upper bound for graphs in P_4, that is triangle-free planar graphs:

- There are bipartite planar graphs whose maximum average degree is arbitrarily close to 4 (consider for instance large grids or $K_{2,n}$), and the oriented chromatic number of a graph with maximum average degree strictly less than 4 can be arbitrarily large [2].
- There are bipartite planar graphs with acyclic chromatic number 5 [4], thus we cannot improve this way upon our current upper bound of 80.

Raspaud and Nešetřil introduced in [5] the *strong oriented chromatic number* of an oriented graph G (denoted $\chi_s(G)$), which definition differs from that of $\chi_o(G)$ by requiring that the target graph is an oriented Cayley graph. Thus In this note, we prove the following:

Theorem 1 $11 \leq \chi_o(P_4) \leq \chi_s(P_4) \leq 59$

In section 2, we exhibit an oriented triangle-free planar graph with oriented chromatic number at least 11 to prove the lower bound of Theorem 1.
In section 3, we introduce the tournament QR_{59} and some of its properties.
In section 4, we prove the upper bound of Theorem 1 by showing that every triangle-free planar graph has a homomorphism to the Cayley graph QR_{59}.

2 The lower bound

Sopena showed that $\chi_o(P_3) \geq 16$ [8]. Let $N^+(x)$ and $N^-(x)$ be respectively the out-neighborhood and in-neighborhood of the vertex x. We say that a pair (x, y) of distinct vertices forms a *good pair* if the sets $N^+(x) \cap N^+(y)$, $N^+(x) \cap N^-(y)$, $N^-(x) \cap N^+(y)$ and $N^-(x) \cap N^-(y)$ are all of size at least 2. A triplet (x, y, z) is a *good triplet* if (x, y), (y, z) and (z, x) are all good pairs. Consider the graph G in Figure 1. Let a *i-vertex* (resp. $\geq$ *i-vertex*) be a vertex of degree i (resp. at least i). We remark that every two distinct ≥ 3-vertices are joined by an arc or a directed 2-path. Therefore every two $\geq$ 3-vertices must be

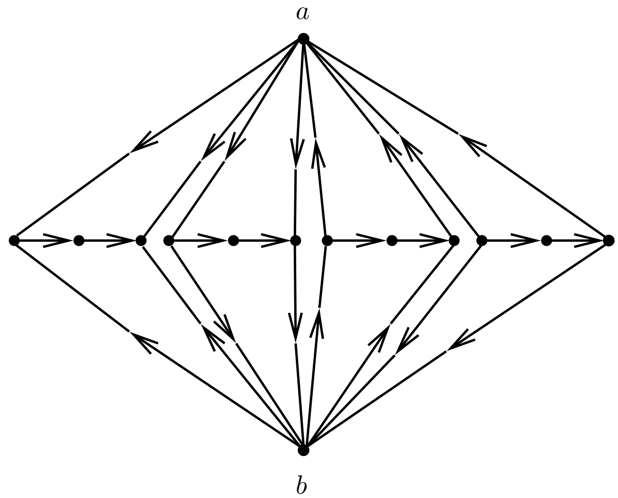

Fig. 1. How to force a good pair in a target graph.

assigned distinct colors in any oriented coloring of G. This provides a simple proof of $\chi_o(P_4) \geq 10$ since G is clearly in P_4. It also implies that the colors of a and b form a good pair in any target graph of G. We now construct the graph G^* by taking 3 copies G_1, G_2, G_3 of G and identifying a_1 and b_2, a_2 and b_3, a_3 and b_1. Similarly, the colors of a_1, a_2, a_3 form a good triplet in any target graph of G^*. A computer check shows that no tournament of order 10 contains a good triplet. The lower bound of Theorem 1 is thus proved since G^* is a 2-degenerate 2-outerplanar bipartite graph having no target graph of order at most 10.

3 The target graph

For a prime $p \equiv 3 \pmod 4$, the Paley tournament QR_p is defined as the oriented graph whose vertices are the integers modulo p and such that (i,j) is an arc if and only if $j-i$ is a non-zero quadratic residue of p. Paley tournaments are clearly Cayley graphs. Another important property is that they are *arc-transitive* [3], which means that for every two arcs (v,w) and (x,y) there exists an automorphism mapping (v,w) to (x,y). An *orientation k-vector* is a sequence $\alpha = \{\alpha_1, \alpha_2, \dots, \alpha_k\}$ in $\{0,1\}^k$. If G is an oriented graph, a *k-sequence* of G is a sequence $X = (x_1, x_2, \dots, x_k)$ of k pairwise distinct vertices of G. A vertex y of G is said to be an *α-successor of X* if for every i, $1 \leq i \leq k$, we have $\alpha_i = 1 \implies (x_i, y) \in E(G)$ and $\alpha_i = 0 \implies (y, x_i) \in E(G)$. An oriented graph G satisfies property $S_{k,n}$ if for every k-sequence X of G and for every orientation k-vector α, there exist at least n vertices in $V(G)$ which are α-successors of X.

Lemma 2 *The tournament QR_{59} satisfies property $S_{3,5}$.*

PROOF. The first part of the proof holds on any statement of the form "QR_p satisfies $S_{3,n}$". Note that the order of the vertices in a sequence does not matter. Thus, since any oriented triangle contains a directed 2-path, we only have to consider sequences (s_1, s_2, s_3) in which (s_1, s_2) and (s_2, s_3) are arcs of QR_p. Now, by the arc-transitivity of QR_p, we only need to check the property on sequences of the form $(0, 1, v)$ such that $2 \leq v \leq p-1$ and $v-1$ is a quadratic residue of p. Let us write $\langle v_1, v_2, v_3 \rangle$ if and only if $v_1 \neq v_2$ and there are automorphisms of QR_p mapping $(0, 1, v_1)$ to $(v_2, 0, 1)$ and $(1, v_3, 0)$. We easily see that if $\langle v_1, v_2, v_3 \rangle$ and $(0, 1, v_1)$ is checked, then $(0, 1, v_2)$ and $(0, 1, v_3)$ need no check. In the case of QR_{59}, we have $\langle 2, 58, 30 \rangle$, $\langle 6, 47, 50 \rangle$, $\langle 8, 42, 23 \rangle$, $\langle 10, 13, 54 \rangle$ and $\langle 18, 52, 37 \rangle$. For every remaining sequence and for every orientation vector, 5 α-successors are listed in the appendix. □

4 The upper bound

We use the well-known method of *reducible configurations* to show that every triangle-free planar graph is QR_{59}-colorable.

PROOF. We define the partial order $\prec$ for the set of all graphs. Let $n_3(G)$ be the number of ≥ 3-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if at least one of the following conditions hold:

- G_1 is a proper subgraph of G_2.
- $n_3(G_1) < n_3(G_2)$.

Note that this partial order is well-defined, since if G_1 is a proper subgraph of G_2, then $n_3(G_1) \leq n_3(G_2)$. So $\prec$ is a partial linear extension of the subgraph poset. Consider a potential counter-example to Theorem 1 which is minimal according to $\prec$. We first remark that such a graph G must be 2-connected since QR_{59} is a circular tournament. We now show that G cannot contain any of the configurations depicted in Figure 2. In a figure representing a forbidden configuration, all the neighbors of "white" vertices are drawn, whereas "black" vertices may have other neighbors in the graph. For every configuration, we give both a subgraph of G and a QR_{59}-coloring of this subgraph (such a coloring exists since G is a minimal counter-example). The coloring is chosen so that it can be extended to G thanks to Lemma 2, contradicting the fact that G is a counter-example.

(i) Let f be any QR_{59}-coloring of $G \setminus \{x\}$. By Lemma 2, we can choose f such that $f(c) \neq f(v)$.

(ii) Since (i) is forbidden, u_1, u_2 and u_3 are $\geq$ 3-vertices. We now consider the graph G' obtained from $G \setminus \{x\}$ by adding directed 2-paths joining

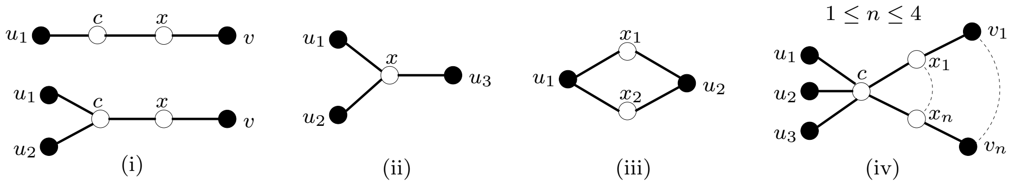

Fig. 2. Unavoidable set of configurations for triangle-free planar graphs.

respectively u_1 and u_2, u_2 and u_3, u_3 and u_1, Note that $G' \prec G$ since $n_3(G') = n_3(G) - 1$. Any QR_{59}-coloring f of G' induces a coloring of $G \setminus \{x\}$ such that $f(u_1) \neq f(u_2) \neq f(u_3) \neq f(u_1)$.

(iii) Let G' be the graph obtained from $G \setminus \{x_1, x_2\}$ by adding a directed 2-path joining u_1 and u_2. Any QR_{59}-coloring f of G' induces a coloring of $G \setminus \{x_1, x_2\}$ such that $f(u_1) \neq f(u_2)$.

(iv) Let f be any QR_{59}-coloring of $G \setminus \{x_1, \ldots, x_n\}$. By Lemma 2, we can choose f such that $f(c) \notin \{f(v_1), \ldots, f(v_n)\}$.

Euler's formula $|V(G)| + |F(G)| = |E(G)| + 2$ and

$$\sum_{v \in V(G)} d(v) = \sum_{f \in F(G)} d(f) = 2|E(G)|$$

show that

$$\sum_{v \in V(G)} (d(v) - 4) + \sum_{f \in F(G)} (d(f) - 4) = -8.$$

We set an *initial charge ch* to every vertex and every face:

$$\forall x \in V(G) \cup F(G),\ ch(x) = d(x) - 4$$

Then we use a discharging procedure consisting of the following two rules, and we get a *final charge* ch^*.

Rule 1. Every ≥ 4-vertex v gives $\frac{1}{2}$ to each face f incident to both v and a 2-neighbor of v.
Rule 2. Every face f gives 1 to each 2-vertex incident to f.

Since the above procedure preserves the total charge, we have:

$$\sum_{x \in V(G) \cup F(G)} ch(x) = \sum_{x \in V(G) \cup F(G)} ch^*(x) = -8.$$

We now prove the following to get a contradiction:

$$\forall x \in V(G) \cup F(G),\ ch^*(x) \geq 0.$$

case $x \in V(G)$

$d(x) = 2$: by **Rule 2**, x receives exactly 1 from each of the two faces incident to x and thus $ch^*(x) = -2 + 2 \times 1 = 0$.

$d(x) = 3$: Since (ii) is forbidden, G contains no 3-vertex.

$d(x) = k$, $4 \leq k \leq 7$: Since (iv) is forbidden, x has at most $(k-4)$ 2-neighbors, so x gives $\frac{1}{2}$ to at most $2 \times (k-4)$ faces and thus $ch^*(x) \geq k - 4 - 2 \times (k-4) \times \frac{1}{2} = 0$.

$d(x) = k \geq 8$: x gives $\frac{1}{2}$ to at most k faces and thus $ch^*(x) \geq k - 4 - k \times \frac{1}{2} \geq 0$.

case $x \in F(G)$

$d(x) = 3$: G is triangle-free, so it contains no face of degree 3.

$d(x) = 4$: Since (i) and (iii) are forbidden, x is incident to at most one 2-vertex. If x is incident to a 2-vertex then $ch^*(x) = 0 + 2 \times \frac{1}{2} - 1 = 0$, otherwise $ch^*(x) = ch(x) = 0$.

$d(x) = k \geq 5$: Let n be the number of 2-vertices incident to x. Since (iv) is forbidden, $n \leq \lfloor \frac{k}{2} \rfloor$ and x receives $\frac{1}{2}$ from at least n vertices, thus $ch^*(x) \geq k - 4 + n \times \frac{1}{2} - n \times 1 = k - 4 - \frac{n}{2} \geq k - 4 - \lfloor \frac{k}{4} \rfloor = \lceil \frac{3k}{4} \rceil - 4 \geq 0$.

□

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Appendix

The quadratic residues of 59:
0 1 3 4 5 7 9 12 15 16 17 19 20 21 22 25 26 27 28 29 35 36 41 45 46 48 49 51 53 57

The table of α-successors:

$(0,1,v)$	{0,0,0}	{0,0,1}	{0,1,0}	{0,1,1}
	{1,0,0}	{1,0,1}	{1,1,0}	{1,1,1}
(0,1,02)	32.33.34.39.40	11.14.24.31.38	08.10.13.42.52	06.18.23.30.37
	12.15.25.35.41	03.07.09.19.48	04.16.20.26.36	05.17.21.22.27
(0,1,04)	14.34.38.43.44	11.24.31.32.33	06.10.18.37.42	02.08.13.23.30
	03.12.15.35.41	07.09.19.25.45	17.22.27.28.36	05.16.20.21.26
(0,1,05)	11.38.39.43.44	14.24.31.32.33	02.13.18.23.37	06.08.10.30.50
	07.15.19.35.45	03.09.12.25.41	04.16.28.29.36	17.20.21.22.26
(0,1,06)	14.24.38.39.40	11.31.32.33.34	02.08.30.37.50	10.13.18.23.42
	03.12.19.45.48	07.09.15.25.35	05.16.17.20.29	04.21.22.26.27
(0,1,08)	14.31.32.38.39	11.24.33.34.43	10.18.42.47.50	02.06.13.23.30
	03.07.19.41.45	09.12.15.25.35	04.05.16.21.22	17.20.27.28.29
(0,1,10)	24.33.34.40.43	11.14.31.32.38	06.18.23.42.47	02.08.13.30.37
	03.07.09.12.41	15.19.25.35.45	05.16.20.21.28	04.17.22.26.27
(0,1,16)	11.24.34.39.40	14.31.32.33.38	13.18.30.47.50	02.06.08.10.23
	07.09.12.15.48	03.19.25.35.41	04.22.26.27.29	05.17.20.21.28
(0,1,17)	14.31.40.55.56	11.24.32.33.34	02.08.10.13.23	06.18.37.42.52
	12.19.25.35.41	03.07.09.15.45	05.16.27.28.49	04.20.21.22.26
(0,1,18)	11.14.24.31.32	33.34.38.39.40	02.06.13.42.50	08.10.23.30.37
	03.09.15.41.48	07.12.19.25.35	17.20.26.28.29	04.05.16.21.22
(0,1,20)	11.31.33.34.38	14.24.32.39.40	08.13.30.50.52	02.06.10.18.23
	03.15.19.51.53	07.09.12.25.35	04.05.16.17.22	21.27.29.36.46
(0,1,21)	14.31.32.34.39	11.24.33.38.40	02.06.18.23.52	08.10.13.30.37
	09.12.35.45.51	03.07.15.19.25	04.05.16.17.20	22.26.28.36.46
(0,1,22)	24.32.33.40.55	11.14.31.34.38	02.06.10.13.18	08.23.37.42.47
	03.07.15.19.35	09.12.25.41.48	05.17.21.28.36	04.16.20.26.27
(0,1,26)	11.14.32.34.39	24.31.33.38.43	06.10.23.37.50	02.08.13.18.30
	07.09.19.25.57	03.12.15.35.41	04.05.17.21.22	16.20.27.29.46
(0,1,27)	11.24.33.38.40	14.31.32.34.39	02.06.08.10.18	13.30.42.47.52
	07.12.15.35.41	03.09.19.25.48	05.20.22.26.29	04.16.17.21.28
(0,1,28)	11.24.34.38.39	14.31.32.33.40	02.06.08.13.23	10.18.37.47.50
	03.07.09.12.19	15.35.45.48.53	16.21.27.36.46	04.05.17.20.22
(0,1,29)	14.24.31.39.40	11.32.33.34.38	02.08.10.13.37	06.18.23.30.50
	03.07.09.12.25	15.19.41.45.48	04.17.20.22.26	05.16.21.27.36
(0,1,36)	11.14.24.31.32	34.39.40.43.55	08.10.42.47.50	02.06.13.18.23
	07.09.15.19.35	03.12.25.41.45	16.17.20.21.27	04.05.22.26.28

(0,1,46)	11.24.31.34.39	14.32.33.38.40	10.18.30.37.42	02.06.08.13.23
	19.25.41.45.48	03.07.09.12.15	05.17.20.21.26	04.16.22.28.36
(0,1,49)	14.24.32.33.34	11.31.38.39.43	08.13.23.30.37	02.06.10.18.47
	03.45.48.51.57	07.09.12.15.19	04.20.21.22.27	05.16.17.26.36
(0,1,50)	14.24.31.33.34	11.32.39.40.44	02.23.30.47.52	06.08.10.13.18
	09.15.25.35.41	03.07.12.19.48	04.05.21.22.28	16.17.20.26.27