Negative results on acyclic improper colorings

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May 19, 2010

Abstract

Raspaud and Sopena showed that the oriented chromatic number of a graph with acyclic chromatic number k is at most $k2^{k-1}$. We prove that this bound is tight for $k \geq 3$. We also consider acyclic improper colorings on planar graphs and partial k-trees. Finally, we show that some improper and/or acyclic colorings are NP-complete on restricted subclasses of planar graphs, in particular ACYCLIC 3-COLORABILITY on bipartite planar graphs with maximum degree 4, and ACYCLIC 4-COLORABILITY on bipartite planar graphs with maximum degree 8.

1 Introduction

Oriented graphs are directed graphs without opposite arcs. In other words an oriented graph is an orientation of an undirected graph, obtained by assigning to every edge one of the two possible orientations. If G is a graph, $V(G)$ denotes its vertex set, $E(G)$ denotes its set of edges (or arcs if G is an oriented graph). A homomorphism from an oriented graph G to an oriented graph H is a mapping φ from $V(G)$ to $V(H)$ which preserves the arcs, that is $(x, y) \in E(G) \Longrightarrow (\varphi(x), \varphi(y)) \in E(H)$. We say that H is a *target graph* of G if there exists a homomorphism from G to H. The oriented chromatic number $\chi_o(G)$ of an oriented graph G is defined as the minimum order of a target graph of G. The oriented chromatic number $\chi_o(G)$ of an undirected graph G is then defined as the maximum oriented chromatic number of its orientations. Finally, the oriented chromatic number $\chi_o(\mathcal{C})$ of a graph class $\mathcal{C}$ is the maximum of $\chi_o(G)$ taken over every graph $G \in \mathcal{C}$. We use in this paper the following notations:

$\mathcal{P}_k$ denotes the class of planar graphs with girth at least k.

$out(k)$ denotes the class of graphs k-outerplanar graphs.

$\mathcal{T}_k$ denotes the class of partial k-trees.

$\mathcal{S}_k$ denotes the class of graphs with maximum degree at most k.

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$\mathcal{D}_k$ denotes the class of k-degenerate graphs.

bip denotes the class of bipartite graphs.

A vertex coloring c of a graph G is *acyclic* if for every two distinct colors i and j, the edges uv such that $c(u) = i$ and $c(v) = j$ induce a forest. A cycle or a path is said to be *alternating* if it is properly colored with two colors. Notice that only even cycles can be alternating and that a coloring is acyclic if and only if there exists no alternating cycle. The acyclic chromatic number $\chi_a(G)$ is the minimum number of colors needed in an acyclic proper coloring of the graph G. Similarly, the acyclic chromatic number $\chi_a(\mathcal{C})$ of a graph class $\mathcal{C}$ is the maximum of $\chi_a(G)$ taken over every graph $G \in \mathcal{C}$. Raspaud and Sopena [9] proved that:

Proposition 1 *[9] For every graph G such that $\chi_a(G) = k$, $\chi_o(G) \leq k2^{k-1}$.*

Since Borodin [2] proved that planar graphs are acyclically 5-colorable (i.e., $\chi_a(\mathcal{P}_3) = 5$), this implies that the oriented chromatic number of a planar graph is at most 80 (i.e., $\chi_o(\mathcal{P}_3) \leq 80$), which is yet the best known upper bound. In order to get a better upper bound on $\chi_o(\mathcal{P}_3)$, if possible, it is interesting to study the tightness of Proposition 1, in particular for $k = 5$. The previously best known lower bound on the maximum value of $\chi_o(G)$ in terms of $\chi_a(G)$ was given by Vignal [13] with a family of graphs G_k, $k \geq 1$ such that $\chi_a(G_k) = k$ and $\chi_o(G_k) = 2^k - 1$.

Boiron et al. [1] introduced the notion of acyclic improper coloring. Let $\mathcal{C}_0, \ldots, \mathcal{C}_{k-1}$ be graph classes. A graph G belongs to the class $\mathcal{C}_0 \circ \cdots \circ \mathcal{C}_{k-1}$ (resp. $\mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-1}$) if and only if G has a k-coloring (resp. an acyclic k-coloring) such that the i^{th} color class induces a graph in $\mathcal{C}_i$, for $0 \leq i \leq k-1$. For brevity, if $\mathcal{C}_0 = \cdots = \mathcal{C}_{k-1} = \mathcal{C}$ we will denote by $\mathcal{C}^k$ the class $\mathcal{C}_0 \circ \cdots \circ \mathcal{C}_{k-1}$ and by $\mathcal{C}^{(k)}$ the class $\mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-1}$. The main motivation in the study of acyclic improper colorings is the following generalization of Proposition 1.

Proposition 2 *[1] Let $\mathcal{C}_0, \ldots, \mathcal{C}_{k-1}$ be graph classes such that $\chi_o(\mathcal{C}_i) = n_i$, for $0 \leq i < k$. Every graph $G \in \mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-1}$ satisfies $\chi_o(G) \leq 2^{k-1} \sum_{i=0}^{i<k} n_i$.*

The bound of Proposition 2 is shown to be tight for $k \geq 3$ under mild assumptions in Section 2. Sections 3, 4, and 5 provide results about acyclic improper colorings on, respectively, the classes of planar graphs, k-outerplanar graphs, and partial k-trees. In Section 6, we prove the NP-completeness of some coloring problems where the input graph is planar with some large girth and low maximum degree.

2 Acyclic improper coloring versus oriented coloring

Theorem 1 *Let $k \geq 3$. Let $\mathcal{C}_0, \ldots, \mathcal{C}_{k-1}$ be hereditary graph classes closed under disjoint union, and such that $\chi_o(\mathcal{C}_i) = n_i$. Then $\chi_o(\mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-1}) = 2^{k-1} \sum_{i=0}^{i<k} n_i$.*

Proof. We construct an oriented graph $G \in \mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-1}$ such that $\chi_o(G) = 2^{k-1} \sum_{i=0}^{i<k} n_i$. Let u_1, u_2, u_3 be a directed 2-path with arcs u_1u_2 and u_2u_3, or u_3u_2 and u_2u_1. We say that u_1 and u_3 are the *endpoints* of the directed 2-path. By definition, the endpoints of the directed 2-path get distinct colors in any oriented coloring. Since $\chi_o(\mathcal{C}_i) = n_i$, there exists a witness oriented graph W^i such that $\chi_o(W^i) = n_i$. The graph G_i contains $k-1$ independent vertices

v_j^i, $0 \leq j < k-1$ and 2^{k-1} disjoint copies W_l^i, $0 \leq l < 2^{k-1}$ of W^i. We consider the binary representation $l = \sum_{n=0}^{n<k-1} 2^n x_n(l)$ of l. For every two vertices v_j^i and $u_l^i \in W_l^i$, we put the arc $v_j^i u_l^i$ (resp. $u_l^i v_j^i$) if $x_j(l) = 1$ (resp. $x_j(l) = 0$). If $l \neq l'$, their binary representations differ at the n^{th} digit, thus $u_l^i \in W_l^i$ $u_{l'}^i \in W_{l'}^i$ are the endpoints of a directed 2-path $u_l^i, v_n^i, u_{l'}^i$. So the same color cannot be used in distinct copies of W^i, which means that at least $2^{k-1}n_i$ colors are needed to color the copies of W^i in any oriented coloring of G_i. We acyclically color G_i as follows. The $k-1$ vertices v_j^i get pairwise distinct colors in $\{0, \ldots, k-1\} \setminus \{i\}$ and every vertex u_l^i get color i (that is why we need the "closed under disjoint union" assumption). Let S_i denote the set of colors in some oriented coloring of the vertices u_l^i of G_i. Now we take one copy of each graph G_i and finish the construction of G. For every two vertices $u_l^i \in W_l^i$ and $u_{l'}^{i'} \in W_{l'}^{i'}$, such that $i \neq i'$, we add a new vertex l and create a directed 2-path $u_l^i, l, u_{l'}^{i'}$. So, for $i \neq i'$, we have $S_i \cap S_{i'} = \emptyset$, which means that at least $2^{k-1} \sum_{i=0}^{i<k} n_i$ colors are needed in any oriented coloring of G. To obtain an acyclic coloring of G, the new vertex l adjacent to u_l^i and $u_{l'}^{i'}$ gets a color in $\{0, \ldots, k-1\} \setminus \{i, i'\}$, which is non-empty if $k \geq 3$. □

Notice that Theorem 1 cannot be extended to the case $k = 2$ in general. By setting $k = 2$ and $\mathcal{C}_0 = \mathcal{C}_1 = \mathcal{S}_0$, we obtain the class of forests $\mathcal{S}_0^{(2)} = \mathcal{D}_1$. Proposition 2 provides the bound $\chi_o(\mathcal{S}_0^{(2)}) \leq 4$. This is not a tight bound, since oriented forests have a homomorphism to the oriented triangle, and thus $\chi_o(\mathcal{S}_0^{(2)}) = 3$.

3 Acyclic improper colorings of planar graphs

The proof of Theorem 1 is constructive, but it does not help for the problem of determining $\chi_o(\mathcal{P}_3)$. Indeed, the graph corresponding to the proper acyclic 5-coloring (i.e., $k = 5$ and $\mathcal{C}_0 = \cdots = \mathcal{C}_4 = \mathcal{S}_0$) is highly non-planar (it contains, e.g., $K_{32,48}$ as a minor). We now consider acyclic improper colorings of planar graphs using at most four colors.

Theorem 2 *Let $2 \leq k \leq 4$. If $\chi_o(\mathcal{C}_{k-1}) \leq 15$, then*
$\mathcal{P}_3 \subset \mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-2} \odot \mathcal{C}_{k-1} \Longleftrightarrow \mathcal{P}_3 \subset \mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-2} \odot S_0$.

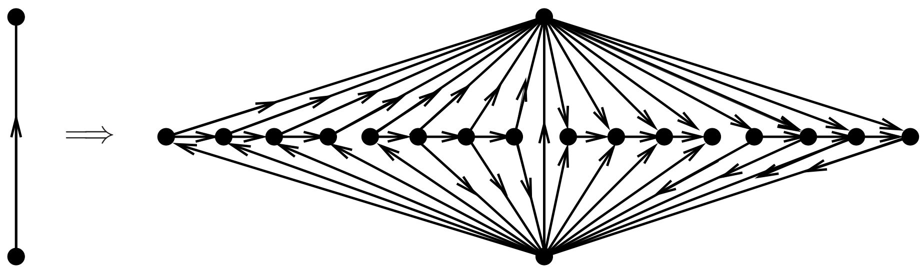

Figure 1: The graph we add to an arc.

Proof. Let G be an oriented graph. The oriented graph $f(G)$ is obtained from G by adding to every arc 16 vertices as described in Figure 1. We also define $f_n(G)$ such that $f_0(G) = G$ and $f_{n+1}(G) = f(f_n(G))$. Notice that if G is planar, then $f(G)$ is planar too. We can check that the oriented planar graph with oriented chromatic number 16 described in [12] is a subgraph of $f_5(K_2)$. Let us now consider any acyclic improper k-coloring c of $f_n(K_2)$ such that

$c(v) = c(w) = 0$. To avoid an alternating cycle $vxwy$ for some vertices x and y, $f_k(K_2)$ must contain a monochromatic copy of $f_1(K_2)$. By induction, $f_{i\times k}(K_2)$ must contain a monochromatic copy of $f_i(K_2)$ for $i \geq 1$. The "$\Longleftarrow$" implication of Theorem 2 holds by definition. We now prove the "$\Longrightarrow$" implication by contradiction. Suppose that there exists an oriented planar witness graph W such that $W \in \mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-2} \odot \mathcal{C}_{k-1}$ and $W \notin \mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-2} \odot S_0$. This means that any $\mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-2} \odot \mathcal{C}_{k-1}$ coloring of W contains a monochromatic edge vw colored $k-1$. So, by previous discussions, the graph $f_{20}(W)$ contains a monochromatic copy of G_4 colored $k-1$, which contradicts the requirement $\chi_o(\mathcal{C}_{k-1}) \leq 15$. □

Theorem 2 allows us to study which statement of the form "every planar graph belongs to $\mathcal{C}_0 \odot \cdots \odot \mathcal{C}_{k-1}$" may improve the upper bound $\chi_o(\mathcal{P}_3) \leq 80$. If $k = 4$, a "least" candidate class would be $\mathcal{C}_0 \odot S_0 \odot S_0 \odot S_0$ with $\chi_o(\mathcal{C}_0) = 16$, but the corresponding bound is too large: $2^{4-1}(16+1+1+1) = 152 > 80$. If $k = 3$, there must be exactly one improper color, otherwise a least candidate, $\mathcal{C}_0 \odot \mathcal{C}_1 \odot S_0$ with $\chi_o(\mathcal{C}_0) = \chi_o(\mathcal{C}_1) = 16$, would yield a too large bound: $2^{3-1}(16 + 16 + 1) = 132 > 80$. Sopena [11, 12] proved that $\chi_o(\mathcal{T}_3) = \chi_o(\mathcal{T}_3 \cap \mathcal{P}_3) = 16$. Thus, Boiron et al. [1] pointed out that:

1. $\mathcal{P}_3 \subset \mathcal{T}_3 \odot \mathcal{S}_0 \odot \mathcal{S}_0$ would imply that $\chi_o(\mathcal{P}_3) \leq 72$,
2. $\mathcal{P}_3 \subset \mathcal{T}_3 \odot \mathcal{S}_1 \odot \mathcal{S}_0$ would imply that $\chi_o(\mathcal{P}_3) \leq 76$.

We now see that the second point is meaningless, since $\mathcal{P}_3 \subset \mathcal{T}_3 \odot \mathcal{S}_1 \odot \mathcal{S}_0 \Longleftrightarrow \mathcal{P}_3 \subset \mathcal{T}_3 \odot \mathcal{S}_0 \odot \mathcal{S}_0$ by Theorem 2.

A theorem of Boiron et al. [1] states that some planar graphs have no acyclic $\mathcal{T}_3^{(2)}$ coloring, actually they even showed that $out(3) \not\subset \mathcal{D}_3^{(2)}$. This result on acyclic improper colorings of planar graphs still holds with larger color classes.

Theorem 3 *$out(3) \not\subset \mathcal{F}^{(2)}$, where $\mathcal{F} = \mathcal{D}_3 \cup \mathcal{T}_4 \cup out(2)$.*

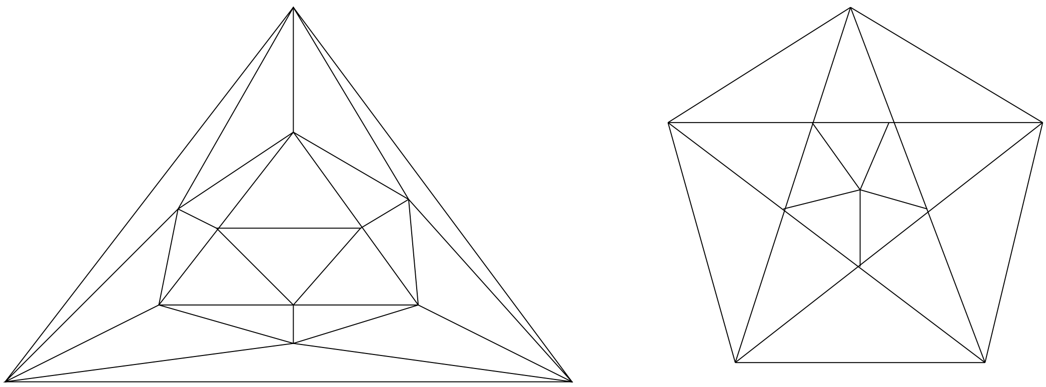

Figure 2: The graph I (the icosahedron) and the graph I^-.

Let I denote the icosahedron graph depicted in Figure 2 (left) and let I^- denote the graph depicted in Figure 2 (right) obtained by deleting one vertex from I. The next lemma considers improper acyclic 2-colorings of I without restriction on the color classes. Let $\mathcal{G}$ denote the class of all simple graphs.

Lemma 1 *Up to symetries, there are only two types of $\mathcal{G}^{(2)}$ coloring of the icosahedron:*

(i) At most one vertex is colored 1 and all others are colored 2.

(ii) Two vertices at distance 3 are colored 1 and all others are colored 2.

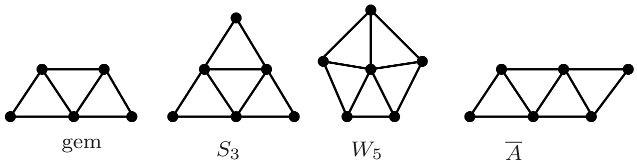

Figure 3: Small graphs.

Proof. We assume without loss of generality that at most 6 vertices are colored 1. Suppose first that two adjacent vertices are colored 1. They have two common neighbors, so at least one of them must be colored 1 to avoid an alternating C_4. Thus we have a 1-monochromatic K_3. Three vertices outside of this K_3 are adjacent to two vertices of the K_3, so at least one of them must be colored 1 to avoid an alternating C_6. Thus we have a 1-monochromatic K_4^-. Four vertices outside of this K_4^- are adjacent to two vertices of the K_4^-, and at least one of them must be colored 1 to avoid an alternating C_8. Thus we have a 1-monochromatic gem (see Figure 3). Four vertices outside of this gem are adjacent to at least two vertices of the gem, and at least one of them must be colored 1 to avoid an alternating C_8. Thus we have a 1-monochromatic subgraph S, which is either S_3, W_5, or $\overline{A}$ (see Figure 3). Since $|S| = 6$, $I \setminus S$ must be 2-monochromatic. We easily check that, for each S, there exists an alternating cycle in I. Suppose now that two vertices at distance two are colored 1. They have two common neighbors that must be colored 2 by the previous case. This creates an alternating C_4. □

Lemma 2 *I^- is neither 3-degenerate, 2-outerplanar, nor a partial 4-tree.*

Proof. Since the minimum degree of I^- is four, it is not 3-degenerate. The graph I^- is 3-connected, so it has a unique embedding on the sphere. Notice that I^- contains four distinct non-equivalent types of faces: one of degree five and three types of triangles. For every face F, the graph obtained by removing the vertices incident to F is not outerplanar, thus I^- is not 2-outerplanar. Finally, to prove that I^- is not a partial 4-tree, we show that we cannot obtain the empty graph from I^- by repeatedly deleting a $^{\leq}4$-vertex and placing a clique on its neighbors [10]. Any such vertex elimination ordering must start with one of the 4-vertices of the outerface of I^-, which all play the same role. Deleting a 4-vertex of I^- and placing a clique on its neighbors gives a graph J. Now J has two $^{\leq}4$-vertices playing the same role. Deleting one of them and placing a clique on its neighbors gives a graph K. Since K has minimum degree 5, I^- has no elimination ordering and thus is not a partial 4-tree. □

Lemmas 1 and 2 prove that the only $\mathcal{G}^{(2)}$ colorings of I are of type (ii).

Observation 1 *If I has a coloring of type (ii), then for every 2-monochromatic triangle t, there exists a vertex colored 1 adjacent to two vertices of t.*

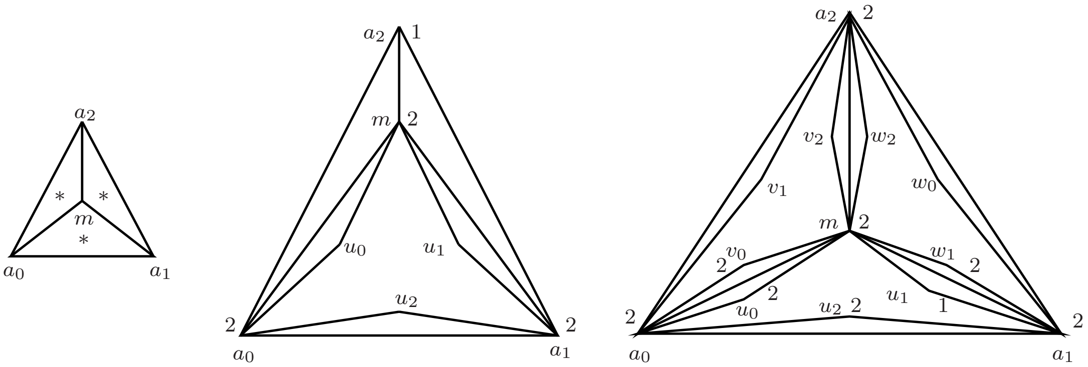

Figure 4: The graph G considered in Lemma 3.

Consider now the graph G depicted in Fig 4 (left) obtained from K_4 by identifying each of the 3 marked faces with the outerface of a copy of an icosahedron.

Lemma 3 *If G is acyclically 2-colored such that every copy of I has a coloring of type (ii), then the outer-face is monochromatic and there is an alternating path between a_0 and a_1.*

Proof. Suppose the first part of the statement is false and assume that $c(a_0) = c(a_1) = 2$ and $c(a_2) = 1$ (see Fig 4 (middle)). We have $c(m) = 2$ to avoid an alternating cycle $a_2a_0ma_1$. By Observation 1, one u_i must be colored 1, and this creates an alternating C_4, a contradiction. Now we check the last part of statement (see Fig 4 (right)). By the previous discussion, the a_i's are colored 2, and $c(m) = c(u_2) = 2$ to avoid an alternating path between a_0 and a_1. By Observation 1, one of the u_i's must be colored 1 and we suppose without loss of generality that $c(u_1) = 1$. Now $c(w_1) = 2$ to avoid an alternating cycle $mu_1a_1w_1$ and $c(u_0) = c(v_0) = 2$ to avoid an alternating path between a_0 and a_1. By Observation 1, either v_1 or v_2 (resp. w_0 or w_2) must be colored 1. In these four cases we have either an alternating cycle or an alternating path between a_0 and a_1. □

To finish the proof of Theorem 3, we take two copies G' and G'' of G and we identify a'_0 and a''_0 (resp. a'_1 and a''_1) to obtain the 3-outerplanar graph G^*. By the previous lemmas, if G' and G'' are both colored as in Lemma 3, then there exists an alternating path between a'_0 and a'_1 in G' and another one in G''. This creates an alternating cycle in G^*.

4 Acyclic improper colorings of k-outerplanar graphs

We obtain the following result on acyclic improper colorings of k-outerplanar graphs.

Theorem 4 $out(k+1) \subset \mathcal{S}_0 \odot \mathcal{S}_0 \odot \mathcal{S}_0 \odot out(k)$

Proof. We show that there exists a coloring of a $(k+1)$-outerplanar graph such that any vertex of the outerface gets one of the first three colors and all other vertices are colored with the last color. Let us characterize a counter-example T for the specified coloring with minimal number of vertices. The special type of coloring considered allows us to assume without loss of generality that the inner-vertices of T induce an indepedent set, since a potential alternating cycle contains no monochromatic edge. We add a maximal number of edges connecting outer-vertices of T inside the outerface. This way, the neighborhood of any inner-vertex induces a single vertex, a K_2, or a cycle of the outerface. The outerface cannot contain a vertex cut of size two. These two vertices would be adjacent, therefore they would get different colors in a valid coloring of "one part" of T. Thus we could extend this coloring to the whole graph, since a potential minimal alternating cycle cannot lie on both parts. The only remaining possibility is that T is a wheel, and a wheel has the specified coloring. □

Theorem 4 implies that $\chi_o(out(2)) \leq 2^{4-1}(1+1+1+7) = 80$. This does not improve upon the bound of 80 that holds for every planar graph, but it gives another target graph on 80 vertices for the class of 2-outerplanar graphs. However, Esperet and the author [3] recently showed that $\chi_o(out(2)) \leq 67$ via an homomorphism to the Paley tournament QR_{67}.

5 Acyclic improper colorings of partial k-trees

We now consider acyclic improper colorings of partial k-trees and show that the equality $\chi_a(\mathcal{T}_k) = k+1$ is best possible in this context.

Theorem 5 *For every $k \in \mathbb{N}^*$ and for every $G \in \mathcal{T}_k$, $\mathcal{T}_k \not\subset$ (G-free)$^{(k)}$.*

Proof. The case $k=1$ is obvious, so assume $k \geq 2$ is a fixed integer in the following. Let us call *good* a clique c such that $2 \leq |c| \leq k$. Now we define the graphs $U_{k,n},\ n \geq 1$, such that:

1. $U_{k,1} = K_2$.
2. For each good clique c of $U_{k,n}$, we add a new vertex adjacent to every vertex of c to obtain $U_{k,n+1}$.

Clearly, every graph in $\mathcal{T}_k$ is a subgraph of $U_{k,n}$ for some n. To finish the proof, we will show that in any improper acyclic k-coloring, $U_{k,n\times k}$ contains a monochromatic copy of $U_{k,n}$. For $n=1$, we have that $U_{k,k}$ contains a clique K_{k+1}, and thus contains a monochromatic K_2. Now assume that $U_{k,n\times k}$ contains a monochromatic copy of $U_{k,n}$. For every good clique c of that copy there are k new vertices adjacent to c in $U_{k,n\times k+k}$, and one of these k new vertices must get the same color as c. This implies that $U_{k,(n+1)\times k}$ contains a monochromatic copy of $U_{k,n+1}$. □

6 NP-complete colorings

If $\mathcal{C}_1$ and $\mathcal{C}_2$ are graph classes, then $(\mathcal{C}_1 : \mathcal{C}_2)$ denotes the problem of deciding whether a given graph $G \in \mathcal{C}_1$ belongs to $\mathcal{C}_2$. If P_1 and P_2 are decision problems, we note $P_1 \propto P_2$ if there is a polynomial reduction from P_1 to P_2.

Kratochvíl proved that PLANAR $(3, {}^{\leq}4)$-SAT is NP-complete [7]. In this restricted version of SAT, the graph of incidences variable-clause of the input formula must be planar, every clause is a disjonction of exactly three literals, and every variable occurs in at most four clauses. A subcoloring is a partition of the vertex set into disjoint cliques. The problem 2-SUBCOLORABILITY is NP-complete on triangle-free planar graphs with maximum degree 4 [4, 6]. Notice that on triangle-free graphs, a 2-subcoloring corresponds to a vertex partition into two graphs with maximum degree 1, i.e., a $\mathcal{S}_1^2$ coloring. Finally, the problem 3-COLORABILITY is NP-complete on planar graphs with maximum degree 4 [5].

6.1 $\mathcal{S}_0 \circ \mathcal{S}_1$ coloring

Theorem 6 PLANAR $(3, {}^{\leq}4)$-SAT $\propto (\mathcal{P}_7 \cap \mathcal{S}_3 : \mathcal{S}_0 \circ \mathcal{S}_1)$

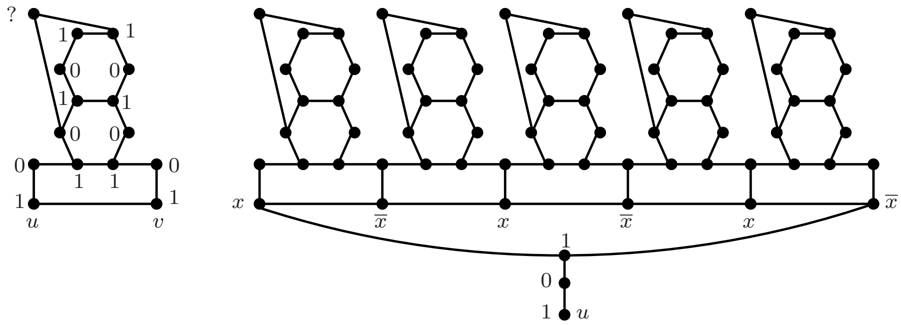

Figure 5: The forcing gadget for the reduction of Theorem 6.

In a $\mathcal{S}_0 \circ \mathcal{S}_1$ or $\mathcal{S}_0 \odot \mathcal{S}_1$ coloring c, a vertex v gets color $c(v) = i$ if v is in the color class $\mathcal{S}_i$, $0 \leq i \leq 1$. We observe that the graph depicted in Figure 5(i) has no $\mathcal{S}_0 \circ \mathcal{S}_1$ coloring such that $c(u) = c(v) = 1$. This implies that the vertex u in the graph depicted in Figure 5(ii) must be colored 1.

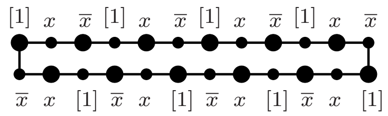

Figure 6: The variable gadget for the reduction of Theorems 6 and 7.

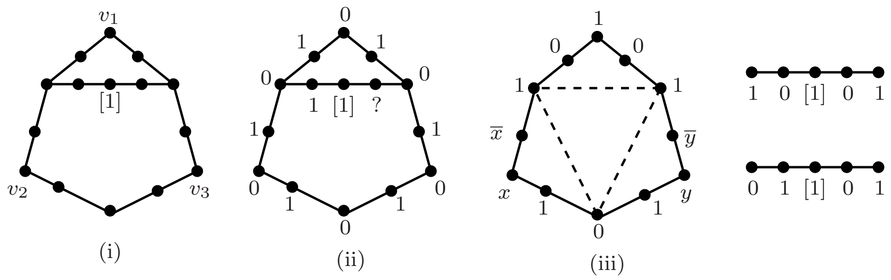

Figure 7: The clause gadget for the reduction of Theorems 6

Proof. Given an instance I of PLANAR $(3, {}^{\leq}4)$-SAT, we build a graph G as follows. We replace every variable of I by a copy the variable gadget depicted in Figure 6. We replace every clause of I by a copy the clause gadget depicted in Figure 7(i). The way we link variables to clauses is best explained with an example: for a clause gadget $C = (x, \overline{y}, z)$ and variables gadgets X, Y, Z, we add an edge between a big vertex $\overline{x}$ of X and the vertex v_1 of C, between a big vertex y of Y and the vertex v_2 of C, and between a big vertex $\overline{z}$ of Z and the vertex v_3 of C. The boolean value true (resp. false) is associated with the color 0 (resp. 1). We see in figure 7(ii) that an unsatisfied clause is not $\mathcal{S}_0 \circ \mathcal{S}_1$ colorable, whereas any satisfied clause (i.e., such that at least one v_1, v_2, v_3 is colored 1) is colorable, see figure 7(ii). This means that I is satisfiable if and only if G belongs to $\mathcal{S}_0 \circ \mathcal{S}_1$. We easily check that G is indeed planar, with girth 6, and maximum degree 3. □

Notice that on triangle-free graphs, the $\mathcal{S}_0 \circ \mathcal{S}_1$ coloring corresponds to the $(1, r)$-subcoloring defined in [8]. Theorem 6 improves a result of Le and Le [8] stating that $(\mathcal{P}_4 \cap \mathcal{S}_3 : \mathcal{S}_0 \circ \mathcal{S}_1)$ is NP-complete.

6.2 $\mathcal{S}_0 \odot \mathcal{S}_1$ coloring

Theorem 7 PLANAR $(3, {}^{\leq}4)$-SAT $\propto (\mathcal{P}_{10} \cap \mathcal{S}_3 \cap bip : \mathcal{S}_0 \odot \mathcal{S}_1)$

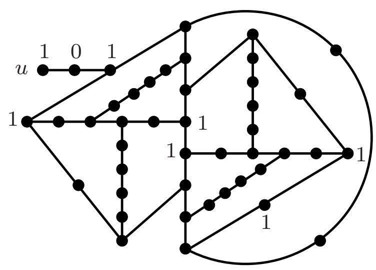

Figure 8: The forcing gadget for the reduction of Theorem 7.

Proof. The proof is similar to the previous one, with the following two changes. We use another forcing gadget depicted in Figure 8. The clause gadget is obtained from the one in Figure 7(i) by deleting the vertex forced to be colored 1 and its two 2-neighbors. In any $\mathcal{S}_0 \odot \mathcal{S}_1$ coloring of the forcing gadget, the vertex u must be colored 1. If a clause is unsatisfied, then its clause gadget is not colorable (an alternating cycle C_{12} is forbidden). If a clause is unsatisfied, then its clause gadget is colorable (the coloring in Figure 7(iii) is acyclic). □

6.3 $\mathcal{S}_1^{(2)}$ coloring

Theorem 8 $(\mathcal{P}_4 \cap \mathcal{S}_4 : \mathcal{S}_1^2) \propto (\mathcal{P}_8 \cap \mathcal{S}_4 \cap bip : \mathcal{S}_1^{(2)})$

Proof. Consider the graph depicted in Figure 9 (left). Any $\mathcal{S}_1^2$ coloring such that the vertex u is colored 1 and has no neighbor colored 1 contains an alternating cycle C_8. So in every $\mathcal{S}_1^{(2)}$ coloring, both u and one neighbor of u must get the same color. Now we use three copies of this graph in the forcing gadget depicted in Figure 9 (right). Since the cycle of this forcing gadget cannot be alternating, the dotted edge has to be monochromatic. Given a planar graph G, we construct the graph G' as follows. We replace every vertex v of G by a copy of the vertex gadget depicted in Figure 10 and for every edge vw we link a big vertex u_i in the

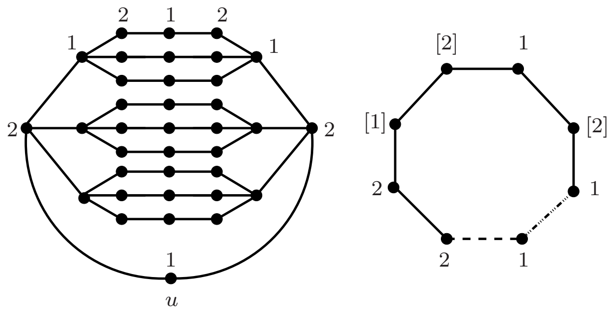

Figure 9: The forcing gadget for the reduction of Theorem 8.

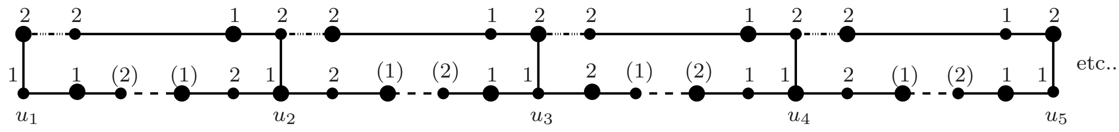

Figure 10: The vertex gadget for the reduction of Theorem 8.

gadget of v to a small vertex u_i in the gadget of w. In any $\mathcal{S}_1^{(2)}$ coloring of the vertex gadget, all u_i's get the same color, say 1, and there exists no alternating path between distinct u_i's. This common color in the gadget of a vertex v corresponds to the color of v in a $\mathcal{S}_1^2$ coloring of G. Notice that if one of the u_i, say u_2, has a neighbor colored 1 not in the gadget, then every other u_i has a neighbor colored 1 in the gadget. Thus we can obtain a $\mathcal{S}_1^{(2)}$ coloring of G' from a $\mathcal{S}_1^2$ coloring of G and vice-versa. □

6.4 $\mathcal{S}_0^{(3)}$ coloring

Theorem 9 $(\mathcal{P}_3 \cap \mathcal{S}_4 : \mathcal{S}_0^3) \propto (\mathcal{P}_4 \cap \mathcal{S}_4 \cap bip \cap \mathcal{D}_2 : \mathcal{S}_0^{(3)})$

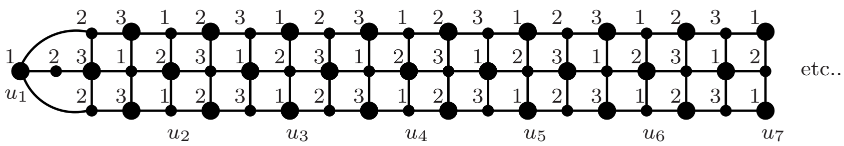

Figure 11: The vertex gadget for the reduction of Theorem 9.

Proof. Given a planar graph G, we construct the graph G' as follows. We replace every vertex v of G by a copy of the vertex gadget depicted in Figure 11 and for every edge vw we link a big vertex u_i in the gadget of v to a small vertex u_i in the gadget of w. The given acyclic 3-coloring of the vertex gadget is the unique one up to permutation of colors. Notice that all u_i's get the same color and there exists no alternating path between distinct u_i's. This common color in the gadget of a vertex v corresponds to the color of v in a 3-coloring of G. Thus G' is acyclically 3-colorable if and only if G is 3-colorable. □

6.5 $\mathcal{S}_0^{(4)}$ coloring

Theorem 10 $(\mathcal{P}_3 \cap \mathcal{S}_4 : \mathcal{S}_0^3) \propto (\mathcal{P}_4 \cap \mathcal{S}_8 \cap bip \cap \mathcal{D}_2 : \mathcal{S}_0^{(4)})$

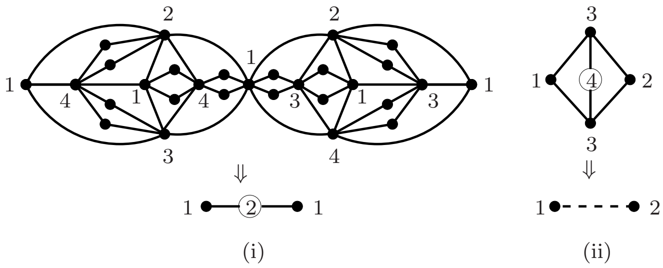

Figure 12: The forcing gadgets for the reduction of Theorem 10.

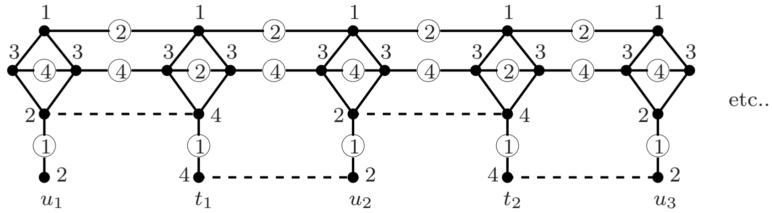

Figure 13: The vertex gadget for the reduction of Theorem 10.

Proof. Consider the graph depicted in Figure 12(i). Any $\mathcal{S}_0^{(4)}$ coloring is such that x and y get the same color. Moreover, this gadget has a coloring such that there exists only one alternating path between x and y (the path colored 1 and 2 in Figure 12(i)). In the graph depicted in Figure 12(ii), the vertices x and y must have distinct colors and there is no alternating path between x and y. Given a planar graph G, we construct the graph G' as follows. We replace every vertex v of G by a copy of the vertex gadget depicted in Figure 13 and for every edge vw we connect a vertex u_i (resp. t_i) in the gadget of v to a vertex u_i (resp. t_i) in the gadget of w using the forcing gadget (ii) (resp. (i)). In any $\mathcal{S}_0^{(4)}$ coloring of the vertex gadget, all u_i's get the same color and all t_i's get a same color distinct from the color of the u_i's. The color of the u_i's in the gadget of a vertex v corresponds to the color of v in a 3-coloring of G. The color of the t_i's is common to every vertex gadget in G', assuming that G is connected. Thus G' is acyclically 4-colorable if and only if G is 3-colorable. □

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