Vertex Decompositions of Sparse Graphs into an Edgeless Subgraph and a Subgraph of Maximum Degree at Most *k*

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Abstract: A graph G is $(k,0)$-colorable if its vertices can be partitioned into subsets V_1 and V_2 such that in $G[V_1]$ every vertex has degree at most k, while $G[V_2]$ is edgeless. For every integer $k \geq 0$, we prove that every graph with the maximum average degree smaller than $(3k+4)/(k+2)$ is $(k,0)$-colorable. In particular, it follows that every planar graph with girth at least 7 is (8, 0)-colorable. On the other hand, we construct planar graphs with girth 6 that are not $(k,0)$-colorable for arbitrarily large k. J Graph Theory 65: 83–93, 2010

Keywords: *vertex decompositions; sparse graphs*

1. INTRODUCTION

A graph G is called *improperly* $(d_1,\dots,d_k)$*-colorable*, or just $(d_1,\dots,d_k)$*-colorable*, if the vertex set of G can be partitioned into subsets $V_1,\dots,V_k$ such that the graph $G[V_i]$ induced by the vertices of V_i has maximum degree at most d_i for all $1 \leq i \leq k$. This notion generalizes those of proper k-coloring (when $d_1 = \dots = d_k = 0$) and d-improper k-coloring (when $d_1 = \dots = d_k = d \geq 1$).

Proper and d-improper colorings have been widely studied. As shown by Appel and Haken [1, 2], every planar graph is 4-colorable, i.e. $(0,0,0,0)$-colorable. Eaton and Hull [11] and independently Škrekovski [15] proved that every planar graph is 2-improperly 3-colorable (in fact, 2-improper 3-choosable), i.e. $(2,2,2)$-colorable. This latter result was extended by Havet and Sereni [14] to not necessarily planar sparse graphs.

Theorem 1 (Havet and Sereni [14]). *For every $k \geq 0$, every graph G with $mad(G) < (4k+4)/(k+2)$ is k-improperly 2-colorable (in fact k-improperly 2-choosable), i.e. (k,k)-colorable.*

Recall that

$$mad(G) = \max\left(\frac{2|E(H)|}{|V(H)|}, H \subseteq G\right)$$

is the maximum average degree of a graph G.

In this article, we focus on $(k,0)$-colorability of graph. So, a graph G is $(k,0)$-*colorable* if its vertices can be partitioned into subsets V_1 and V_2 such that in $G[V_1]$ every vertex has degree at most k, while $G[V_2]$ is edgeless.

Let $g(G)$ denote the girth of graph G (the length of a shortest cycle in G). Glebov and Zambalaeva [12] proved that every planar graph G is $(1,0)$-colorable if $g(G) \geq 16$. This was strengthened by Borodin and Ivanova [6] by proving that every graph G is $(1,0)$-colorable if $mad(G) < \frac{7}{3}$, which implies that every planar graph G is $(1,0)$-colorable if $g(G) \geq 14$.

The purpose of our article is to extend the result in [6] as follows:

Theorem 2. *Let $k \geq 0$ be an integer. Every graph with maximum average degree smaller than $(3k+4)/(k+2)$ is $(k,0)$-colorable.*

On the other hand, we construct non-$(k,0)$-colorable graphs whose maximum average degree exceeds $(3k+4)/(k+2)$ not much (in fact, by less than $1/(k+3)$, see Section 3).

Since each planar graph G satisfies

$$mad(G) < \frac{2g(G)}{g(G)-2},$$

from Theorem 2 we have:

Corollary 1. *Every planar graph G is*:

1. $(1,0)$-*colorable if* $g(G) \geq 14$,
2. $(2,0)$-*colorable if* $g(G) \geq 10$,
3. $(3,0)$-*colorable if* $g(G) \geq 9$,
4. $(4,0)$-*colorable if* $g(G) \geq 8$, *and*
5. $(8,0)$-*colorable if* $g(G) \geq 7$.

The key concepts in the proof of our main result, Theorem 2, are those of soft components and feeding areas. These further develop those of soft cycles and feeding paths introduced by Borodin et al. in [7] and used in [5, 7] to improve results in [8, 10] about homomorphisms of sparse graphs to the circulant $C(5;1,2)$ and the cycle C_5. A distinctive feature of the discharging in [5, 7] is that charge can be transferred along "feeding paths" to unlimited distances. Note that similar ideas of "global discharging" were also used by Havet and Sereni in [14]. In fact, the proof of Theorem 2 below further develops the argument in [6].

An induced cycle $C = v_1 v_2 \ldots v_{2k}$ in a graph G is called *2-alternating* if $d(v_1) = d(v_3) = \cdots = d(v_{2k-1}) = 2$. This is perhaps the first and simplest example of a "global" reducible configuration in some graph-theoretic problems. The feeding area in this case is a tree consisting of edges incident with vertices of degree 2. This notion, introduced by Borodin in [3], as well as its subsequent variations, turned out to be useful in some coloring and edge-decomposition problems on sparse and quasiplanar graphs (see, for example, [4, 9, 13, 16, 17]).

Section 2 is dedicated to the proof of Theorem 2. Section 3 contains results about the $(k,0)$-colorability of outerplanar and series-parallel graphs.

2. PROOF OF THEOREM 2

A vertex of degree k (resp. at least k, at most k) is called a *k-vertex* (resp. *$\geq k$-vertex*, *$\leq k$-vertex*).

The case $k=0$ in Theorem 2 is equivalent to the fact that forests are bipartite graphs and the case $k=1$ is already proven [6], so we can assume that $k \geq 2$. Among all counterexamples to Theorem 2 with the minimum number of ≥ 3-vertices, we consider a counterexample G having the minimum $|V(G)|$.

Clearly, G is connected and its minimum degree $\delta(G)$ is at least 2 (it follows that $mad(G) \geq 2$). By definition, we have

$$\sum_{v \in V} \left(d(v) - \frac{3k+4}{k+2} \right) < 0,$$

where $d(v)$ is the degree of a vertex v. This can be rewritten as

$$\sum_{v \in V} \left(\left(2+\frac{4}{k}\right) d(v) - \left(6+\frac{8}{k}\right) \right) < 0. \qquad (1)$$

Let the *charge* $\mu(v)$ of each vertex v of G be $(2+4/k)d(v)-(6+8/k)$. Since $\delta(G) \geq 2$, it follows that in G only 2-vertices have negative charge (equal to -2). We shall describe a number of structural properties of G (Section A) which make it possible to vary the charges so that the new charge μ^* of every vertex becomes non-negative (Section B). Since the sum of charges does not change, we shall get a contradiction with (1), which will complete the proof of Theorem 2.

In what follows, by a *k-path* we mean a path consisting of precisely k vertices of degree 2, while by a $(k_1, k_2, \ldots)$-*vertex* we mean a vertex that is incident with k_1-, k_2-, … paths. We will color the vertices of the independent set by the color 0, and the vertices of the subgraph of maximum degree at most k by the color k.

A vertex of degree at least $k+2$ is called *senior*. Note that if a vertex v is senior, then

$$\mu(v) = \left(2+\frac{4}{k}\right) d(v) - \left(6+\frac{8}{k}\right) \geq 2d(v) + \frac{4}{k}(k+2) - 6 - \frac{8}{k} = 2d(v) - 2.$$

A. Structural Properties

Lemma 1. *Every $\leq (k+1)$-vertex is adjacent to at least one senior vertex.*

Proof. Suppose to the contrary that G contains a $\leq (k+1)$-vertex v adjacent only to $\leq (k+1)$-vertices. We delete such a vertex v and extend a coloring c of the graph obtained to G as follows. If v has all its neighbors colored with k, then we are done by putting $c(v)=0$. Now, if v has at most k neighbors colored with k, then we put $c(v)=k$. We are in trouble only if v is adjacent to a $(k+1)$-vertex x which is colored with k together with all its neighbors. But then we can recolor x with 0. The same argument is then applicable to every other neighbor of v. ■

Corollary 2. *G has no ≥ 3-paths.*

A triangle is *special* if it has at least two 2-vertices. Since $G \neq C_3$, a special triangle actually has just two vertices of degree 2.

Corollary 3. *There is no d-vertex with $3 \leq d \leq k+1$ incident with a 2-path or a special triangle.*

Lemma 2. *Every d-vertex with $3 \le d \le k+1$ is adjacent to at least two senior vertices.*

Proof. Suppose to the contrary that G contains a d-vertex with $3 \le d \le k+1$, say u, adjacent to a senior vertex s (by Lemma 1) and to $\le (k+1)$-vertices $v_1, \ldots, v_{d-1}$. Let G' be the graph obtained from $G \setminus u$ by adding 1-paths sz_iv_i for $1 \le i \le d-1$.

Note that $mad(G') \le mad(G)$ since $mad(G) \ge 2$. Indeed, suppose $H' \subset G'$. We consider two cases: (1) H' contains no z_i with $d_{H'}(z_i) = 2$ and (2) some z_i has $d_{H'}(z_i) = 2$. In the former case, H' is obtained from a subgraph H of G (which does not contain u) by adding β vertices of degree at most 1; we have

$$\frac{2|E(H')|}{|V(H')|} = \frac{2|E(H)| + 2\beta}{|V(H)| + \beta} \le \frac{|V(H)| + 2\beta / mad(G)}{|V(H)| + \beta} mad(G) \le mad(G).$$

In the later case, assume that H' contains α vertices z_i with $d_{H'}(z_i) = 2$. Let H' be obtained from a subgraph H of G (which contains u) by adding β vertices of degree at most 1: we have

$$\frac{2|E(H')|}{|V(H')|} = \frac{2|E(H)| + 2(\alpha - 1) + 2\beta}{|V(H)| + \beta} \le \frac{|V(H)| + (2(\alpha - 1) + 2\beta) / mad(G)}{|V(H)| + \alpha + \beta} mad(G)$$

$$\le mad(G)$$

By the minimality of G, G' has a $(k,0)$-coloring c. The coloring c yields a $(k,0)$-coloring of G. If all the neighbors of u have a same color, then we color u properly. Otherwise we color u with k (followed by recoloring the v_i's with 0 if necessary). ■

Lemma 3. *G has no $(2,2,\ldots,2)$-vertices.*

Proof. Indeed, otherwise we delete such a vertex v and all its neighbors. To extend a coloring c of the graph obtained to G, we color the vertex v with 0 and color all neighbors of v by k. ■

By a *soft vertex* we mean a senior vertex each of whose neighbors have degree 2.

By a *feeding area*, denoted by FA, we mean a maximal subgraph (by inclusion) in G consisting of:

(i) soft vertices mutually accessible from each other along 1-paths, and
(ii) 2-vertices adjacent to soft vertices of FA.

See Figure 1. Note that according to this definition, every edge xy joining $x \in FA$ with $y \notin FA$ has the following properties: $d(x) = 2$ and y is not soft.

By a *soft component* we mean a feeding area FA such that every edge from FA to $G \setminus FA$ leads to a non-senior vertex.

Lemma 4. *G has no soft components.*

Proof. Let FA be a soft component. (It is not excluded that $FA = G$.) We first take a coloring c of $G \setminus FA$. Now for each edge xy such that $x \in FA$ and $y \notin FA$, we color x with k.

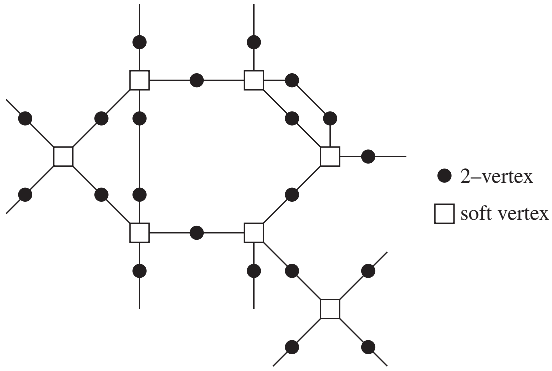

FIGURE 1. Example of a feeding area.

Now if y is a $(k+1)$-vertex colored k together with all its neighbors, we recolor y with 0. Then we color every soft vertex of FA with 0. Finally, we color all yet uncolored vertices of FA (namely, 2-vertices) with k to get a desired coloring of G. ■

Corollary 4. *For each feeding area FA of G there exists a* 1*-path xyz such that* $x \in FA$ *is a soft vertex*, $y \in FA$ *is a* 2*-vertex*, *and* $z \notin FA$ *is a senior vertex incident with at least one* 0*-path.*

A feeding area is *weak*, and denoted *WFA*, if it has only one 1-path xyz such that $x \in FA$ is a soft vertex, while $z \notin FA$, where z is a senior vertex incident with at least one 0-path.

By a *special vertex* we mean a senior vertex z such that:

(i) z is incident with precisely one 0-path, and
(ii) every 1-path from z leads to a *WFA*.

The notion of a special soft component, denoted *SSC*, is very close to that of a soft component. The only difference is that *SSC* includes just one vertex, called its special vertex, that *SC* fails to include. Namely, a *special soft component SSC* consists of a special vertex z and the vertices of all those *WFA*'s joined to z along their unique outgoing 1-paths. Informally speaking, a *SSC* is a collection of *WFA*'s joined by a special vertex.

Lemma 5. *Each* $(1,0,0)$*-vertex w is adjacent to a*(*t least one*) *senior non-special vertex.*

Proof. Suppose a $(1,0,0)$-vertex w is adjacent to two $\geq k+2$-vertices z_1, z_2 (by Lemma 2) such that each z_i is the special vertex of SSC_i. Delete w and SSC_i (together with z_i). We first take a coloring c of the graph obtained and recolor the 2-vertex adjacent to w with a color different from the color of its undeleted neighbor. We color each SSC_i as in the proof of Lemma 4; in particular, each z_i is colored with 0. Next, we color w with k. ■

B. Discharging Procedure

Our rules of discharging are:

R1. Every 2-vertex that belongs to a 1-path gets charge 1 from each of its ends, while each 2-vertex that belongs to a 2-path gets charge 2 from the neighbor vertex of degree greater than 2.
R2. Each $(1,0,0)$-vertex gets charge 1 from its senior non-special neighbor.
R3. Every weak FA gets charge 1 along its only 1-path that leads to $G\setminus FA$.

Lemma 6. *The total charge $\mu^*(FA)$ of all soft vertices in each feeding area FA after applying rules R1 and R2 is non-negative if FA is not weak and is at least -1 otherwise.*

Proof. We perform a series of transformations, each of which makes a feeding area FA into another, "more standard", feeding area FA' of the same type (weak or otherwise) such that $\mu^*(FA') \le \mu^*(FA)$. Eventually, any original feeding area FA will be transformed into a(n ultimate) feeding area FA_0 which consists of a $(k+2)$-vertex v all of whose neighbors have degree 2. Furthermore, if FA_0 is weak then precisely one of the neighbors of v will belong to a 1-path, which means that $\mu^*(FA_0) \ge -1$ in this case; if FA_0 is not weak then the number of 1-paths incident with v will be at least 2, so that $\mu^*(FA_0) \ge 0$. We reduce FA to FA_0 as follows:

Step 1. If FA has a cycle of 1-paths, then we replace one of its 1-paths by a 2-path and get a feeding area FA' such that $\mu^*(FA') = \mu(FA)^* - 2$ due to R1. We repeat this procedure until all the cycles of 1-paths disappear.
Step 2. If FA has a 2-path P joining vertices u, w of FA, then we replace P by two 2-paths one of which is incident only with u, while the other only with w, so that the other ends of these new 2-paths "are loose" (do not belong to FA'). As a result we have $\mu^*(FA') = \mu^*(FA)$.

From now on, our FA is a tree consisting of senior (soft) vertices and 2-vertices with the property that each path between two senior vertices is a 1-path.

Step 3. Let $P_i = x_i y_i z_i$, $1 \le i \le t$ be all 1-paths such that $x_i \in FA$ while $\{z_i\} \cap FA = \emptyset$. (Recall that $t \ge 1$ and FA is weak if and only if $t = 1$.) If $t \ge 3$ then we replace each P_i with $i \ge 3$ by a loose 2-path incident to x_i. We have $\mu^*(FA') \le \mu^*(FA)$.

Note that in none of Steps 1–3 we changed the status of FA to be weak or non-weak. The same is true for the remaining step.

Step 4. Suppose FA has at least two senior vertices. Let v be a pendant senior vertex in FA, i.e. joined by 1-path with precisely one senior vertex $w \in FA$. Suppose v is incident with p outgoing 1-paths. Since FA has at least two pendant vertices and at most two outgoing 1-paths, we can assume that $p \le 1$. If $p = 0$ then we replace v and its neighbors by a loose 2-path incident with w, which implies $\mu^*(FA') = \mu^*(FA)$. If $p = 1$, we replace v and and its neighbors by a loose 1-path going out of w, which again implies $\mu^*(FA') = \mu^*(FA)$ by R1.

Finally, FA_0 contains a unique $\geq(k+2)$-vertex incident to p outgoing 1-paths ($p=1$ if FA is weak and $p=2$ otherwise) and $\mu^*(FA_0)=p-2$. Hence, after applying R1 and R2, $\mu^*(FA)=-1$ if FA is weak and $\mu^*(FA)\geq 0$ otherwise. ■

We now check that after applying R1–R3, the new charge μ^* of each non-soft vertex v and of each feeding area is non-negative.

Indeed, if $d(v)=2$ then $\mu^*(v)=-2+2=0$ by R1 due to Corollary 2.

Suppose $d(v)=3$. By Lemmas 1 and 2, v is not incident with a 2-path and is adjacent to at least two senior vertices. If v is adjacent to a 2-vertex, then $\mu^*(v)=(4/k)-1+1>0$ by R1 and R2 due to Lemma 5. Otherwise, $\mu^*(v)=4/k>0$.

Now suppose $4\leq d(v)\leq k+1$. By Lemma 2, v is adjacent to at least two senior vertices. By Corollary 3, v gives charge 1 to each adjacent 2-vertex. Hence $\mu^*(v)\geq \mu(v)-(d(v)-2)\times 1>2(d(v)-3)-(d(v)-2)=d(v)-4\geq 0$.

Finally, let v be senior, i.e. having $d(v)\geq k+2$. Recall that $\mu(v)\geq 2d(v)-2$.

Suppose v is not soft. If v is special, then v does not give any charge along its (only) 0-path, which implies that v only gives away charge along its $\geq$1-paths by R1 and R3. Thus, $\mu^*(v)\geq\mu(v)-2(d(v)-1)\geq 0$. If v is not special, then v has at least two 0-paths and $\mu^*(v)\geq\mu(v)-2(d(v)-2)-2\times 1\geq 0$.

By Lemma 6, after applying rules R1–R3, the total charge $\mu^*(FA)$ of all soft vertices in each feeding area FA of G, both weak and non-weak, is nonnegative. Since the feeding areas are disjoint, it follows that the total μ^*-charge of all soft vertices in G is non-negative.

This contradiction with (1) completes the proof of Theorem 2.

3. OPTIMALITY AND RELATED RESULTS

In this section, we give some results concerning the optimality of Theorem 2 and Corollary 1, as well as other results about $(k,0)$-colorings of outerplanar and series-parallel graphs with given girth.

Claim 1.

1. *Outerplanar graphs with girth* 4 *are* $(2,0)$*-colorable.*
2. *Outerplanar graphs with girth* 5 *are* $(1,0)$*-colorable.*
3. *Series-parallel graphs with girth* 7 *are* $(1,0)$*-colorable.*

Proof. It is well know that outerplanar (resp. series-parallel) graphs with minimum degree two and girth g contain a $(g-2)$-path (resp. a $(\lfloor(g-1)/2\rfloor)$-path). So, outerplanar graphs with girth 5 and series-parallel graphs with girth 7 contain either a $\leq$1-vertex or a 3-path, and are thus $(1,0)$-colorable. To obtain the first statement, we check that outerplanar graphs with girth 4 admit a vertex partition into a forest of paths (getting color 2) and a stable set, such that every edge on the outerface is incident to a vertex which is the extremity of a path colored 2. ■

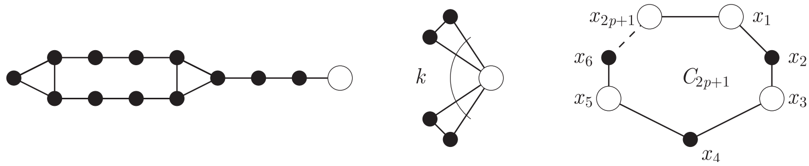

FIGURE 2. Non-(k,0)-colorable outerplanar graphs with small maximum average degree.

Claim 2. *There exist*:

1. *non*-$(1,0)$-*colorable outerplanar graphs with mad arbitrarily close to* $\frac{17}{7}$,
2. *for any* $k \geq 2$, *non*-$(k,0)$-*colorable outerplanar graphs with mad arbitrarily close to* $(3k+2)/(k+1)$,
3. *a non*-$(1,0)$-*colorable outerplanar graph G with girth* 4,
4. *a non*-$(1,0)$-*colorable planar graph with girth* 7.
5. *for any* $k \geq 1$, *a non*-$(k,0)$-*colorable series-parallel graph with girth* 6,

Proof. Let us first prove Claims 2.1 and 2.2. For $p,k \geq 1$, let $G_{p,k}$ be the graph obtained from $p+1$ copies of the graph on the left of Figure 2 if $k=1$, or from $p+1$ copies of the graph in the middle of Figure 2 if $k \geq 2$, by identifying the big vertex of one copy with a big vertex x_i (i.e. with odd i) of the odd cycle C_{2p+1} depicted on the right of Figure 2. In both cases $k=1$ and $k \geq 2$, the big vertex of a copy is adjacent to at least k vertices colored k in any $(k,0)$-coloring of the copy. Also, in any $(k,0)$-coloring of the C_{2p+1}, at least one big vertex x_i is colored k and is adjacent to another vertex colored k. In the whole graph $G_{p,k}$, this x_i would be colored k and have $k+1$ neighbors colored k, this contradiction shows that $G_{p,k}$ is not $(k,0)$-colorable.

It is easy to check that the maximum average degree of $G_{p,k}$ is equal to its average degree, so for $k=1$,

$$mad(G_{p,1}) = \frac{2|E(G_{p,1})|}{|V(G_{p,1})|} = \frac{2(17(p+1)-1)}{14(p+1)-1} \xrightarrow[p\to\infty]{} \frac{17}{7},$$

and for $k \geq 2$,

$$mad(G_{p,k}) = \frac{2|E(G_{p,k})|}{|V(G_{p,k})|} = \frac{3k+2-\dfrac{1}{p+1}}{k+1-\dfrac{1}{2(p+1)}},$$

$$\lim_{p\to\infty} mad(G_{p,k}) = \frac{3k+2}{k+1} = 3-\frac{1}{k+1} < \frac{3k+4}{k+2} + \frac{1}{k+3}.$$

Graphs proving Claims 2.3, 2.4, and 2.5 are, respectively, given on the left of Figure 3, on the right of Figure 3, and in Figure 4. ■

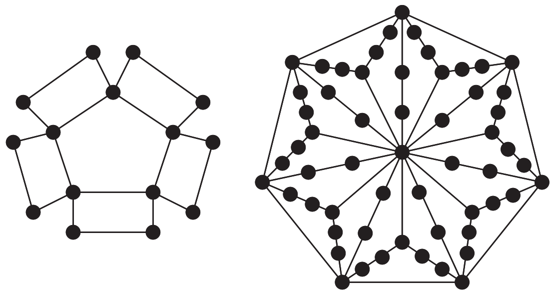

FIGURE 3. Non-(1,0)-colorable graphs: outerplanar with girth 4 and planar with girth 7.

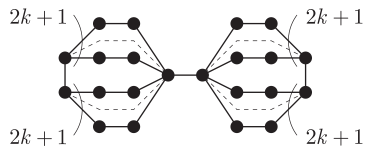

FIGURE 4. A non-(k,0)-colorable series-parallel graph G with girth 6.

4. CONCLUSION

Claims 1 and 2 give the exact value of the minimal k such that the classes of outerplanar and series-parallel graphs with girth g admit a $(k,0)$-coloring. For outerplanar graphs, k is unbounded for $g=3$, $k=2$ for $g=4$, and $k=1$ for $g\geq 5$. For series-parallel graphs, k is unbounded for $g\leq 6$ and $k=1$ for $g\geq 7$.

The case of planar graphs is more challenging: we know that k is unbounded for $g\leq 6$ and $k=1$ for $g\leq 14$, but we have only partial results, namely Corollary 1 and Claim 2.4, for $7\leq g\leq 13$.

From our main result about the maximum average degree, we have that the minimum $m=mad(G)$ of a non-$(k,0)$-colorable graph G satisfies $\frac{7}{3}\leq m\leq \frac{17}{7}+\varepsilon$ for $k=1$ and $(3k+4)/(k+2)\leq m\leq (3k+2)/(k+1)+\varepsilon$ for $k\geq 2$.

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