Vertex partitions of \((C_3, C_4, C_6)\)-free planar graphs

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Abstract

A graph is \((k_1, k_2)\)-colorable if its vertex set can be partitioned into a graph with maximum degree at most \(k_1\) and a graph with maximum degree at most \(k_2\). We show that every \((C_3, C_4, C_6)\)-free planar graph is \((0, 6)\)-colorable. We also show that deciding whether a \((C_3, C_4, C_6)\)-free planar graph is \((0, 3)\)-colorable is NP-complete.

1. Introduction

A graph is \((k_1, k_2)\)-colorable if its vertex set can be partitioned into a graph with maximum degree at most \(k_1\) and a graph with maximum degree at most \(k_2\). Choi, Liu, and Oum [1] have established that there exists exactly two minimal sets of forbidden cycle length such that every planar graph is \((0, k)\)-colorable for some absolute constant \(k\).

- planar graphs without odd cycles are bipartite, that is, \((0, 0)\)-colorable.
- planar graphs without cycles of length 3, 4, and 6 are \((0, 45)\)-colorable.

The aim of this paper is to improve this last result. Notice that forbidding cycles of length 3, 4, and 6 as subgraphs or as induced subgraphs result in the same graph class. For every \(n \geq 3\), we denote by \(C_n\) the cycle on \(n\) vertices. So we are interested in the class \(\mathcal{C}\) of \((C_3, C_4, C_6)\)-free planar graph.

We will prove the following two theorems in the next two sections.

**Theorem 1.** Every graph in \(\mathcal{C}\) is \((0, 6)\)-colorable.

**Theorem 2.** For every \(k \geq 1\), either every graph in \(\mathcal{C}\) is \((0, k)\)-colorable, or deciding whether a graph in \(\mathcal{C}\) is \((0, k)\)-colorable is NP-complete.

In addition, we construct a graph in \(\mathcal{C}\) that is not \((0, 3)\)-colorable in Section 4. This graph and Theorem 2 imply the following.

**Corollary 3.** Deciding whether a graph in \(\mathcal{C}\) is \((0, 3)\)-colorable is NP-complete.

Since we deal with \((0, k)\)-colorings for some \(k \geq 2\), we denote by the letter 0 the color of the vertices that induce the independent set and we denote by the letter \(k\) the color of the vertices that induce the graph with maximum degree \(k\).
2. Proof of Theorem 1

The proof will be using the discharging method. For every plane graph \( G \), we denote by \( V(G) \) the set of vertices of \( G \), by \( E(G) \) the set of edges of \( G \), and by \( F(G) \) the set of faces of \( G \).

Let us define the partial order \( \preceq \). Let \( n_3(G) \) be the number of \( 3^+ \)-vertices in \( G \). For any two graphs \( G_1 \) and \( G_2 \), we have \( G_1 \prec G_2 \) if and only if at least one of the following conditions holds:

- \(|V(G_1)| < |V(G_2)| \) and \( n_3(G_1) \leq n_3(G_2) \).
- \( n_3(G_1) < n_3(G_2) \).

Note that the partial order \( \preceq \) is well-defined and is a partial linear extension of the subgraph poset.

We suppose for contradiction that \( G \) is a graph in \( C \) that is not \((0, 6)\)-colorable and is minimal according to \( \preceq \). Let \( n \) denote the number of vertices, \( m \) the number of edges and \( f \) the number of faces of \( G \). For every vertex \( v \), the degree of \( v \) in \( G \) is denoted by \( d(G) \). For every face \( \alpha \), the degree of \( \alpha \), denoted \( d(\alpha) \), is the number of edges that are shared between this face and another face, plus twice the number of edges that are entirely in \( \alpha \). More generally, when counting the number of edges of a certain type in a face, we will always count twice the edges that are only in this face. For all \( d \), let us call a vertex of \( G \) of degree \( d \), at most \( d \), and at least \( d \) a \( d \)-vertex, \( d^- \)-vertex, and \( d^+ \)-vertex respectively. For every vertex \( v \), a \( d \)-neighbor, a \( d^- \)-neighbor, and a \( d^+ \)-neighbor of \( v \) is a neighbor of \( v \) that is a \( d \)-vertex, a \( d^- \)-vertex, and a \( d^+ \)-vertex respectively. For all \( d \), let us call a face of \( G \) of degree \( d \), at most \( d \), and at least \( d \) a \( d \)-face, a \( d^- \)-face, and a \( d^+ \)-face respectively. For all set \( S \) of vertices, \( S \)-vertex is a vertex that belongs to \( S \), and an \( S \)-neighbor of a vertex \( v \) is a neighbor of \( v \) that belongs to \( S \). For all set \( S \) of vertices, let \( G[S] \) denote the set of vertices induced by \( S \), and \( G - S = G[V(G) \setminus S] \). For convenience, we will note \( G - v \) for \( G - \{v\} \).

Let us first prove some results on the structure of \( G \), and then we will prove that \( G \) cannot exist, thus proving the theorem.

**Lemma 4.** \( G \) is connected.

**Proof.** If \( G \) is not connected, then every connected component of \( G \) is smaller than \( G \) and thus admits a \((0, 6)\)-coloring. The union of these \((0, 6)\)-colorings gives a \((0, 6)\)-coloring of \( G \), a contradiction. \( \Box \)

**Lemma 5.** \( G \) has no \( 1 \)-vertex.

**Proof.** Let \( v \) be a \( 1 \)-vertex and \( w \) be the neighbor of \( v \). The graph \( G - v \) admits a \((0, 6)\)-coloring since \( G - v \prec G \). We get a \((0, 6)\)-coloring \( G \) by assigning to \( v \) the color distinct from the color of \( w \), a contradiction. \( \Box \)

**Lemma 6.** Every \( 7^- \)-vertex of \( G \) has a \( 8^+ \)-neighbor.
**Proof.** Let $v$ be a $7^-$-vertex with no $8^+$-neighbors. The graph $G - v$ admits a $(0,6)$-coloring since $G - v \prec G$. If there is a neighbor $w$ of $v$ with no neighbor colored 0, then we color $w$ with 0. Thus, we can assume that every neighbor of $v$ that is colored $k$ has a neighbor colored 0 in $G - v$, and thus at most 5 neighbors colored $k$ in $G - v$. Also, we can assume that $v$ has at least one neighbor colored 0, since otherwise $v$ can be colored 0. Thus, $v$ has at most 6 neighbors colored $k$ and $v$ can be colored $k$, a contradiction. \hfill \Box

**Lemma 7.** Every vertex with degree at least 3 and at most 7 has two $8^+$-neighbors.

**Proof.** Suppose for contradiction that $G$ contains a $d$-vertex $v$ such that $3 \leq d \leq 7$ and such that $v$ has at most one $8^+$-neighbor. By Lemma 6, $v$ has exactly one $8^+$-neighbor $w$. Let $w_1, \ldots, w_{d-1}$ be the other neighbors of $v$. Let $H$ be the graph obtained from $G - v$ by adding $d - 1$ 2-vertices $v_1, \ldots, v_{d-1}$, such that for every $i \in \{1, d - 1\}$, $v_i$ is adjacent to $w$ and $w_i$.

Notice that $H \prec G$ since $n_3(H) = n_3(G) - 1$. Moreover, every cycle of length $\ell$ in $H$ is associated a cycle of length $\ell$ or $\ell - 2$ in $G$. Therefore $H \in C$, so $H$ has a $(0,6)$-coloring.

If $w$ is colored 0, then every $v_i$ is colored $k$, coloring $v$ with $k$ leads to a $(0,6)$-coloring of $G$, a contradiction. Therefore $w$ is colored $k$.

While at least one of the $w_i$'s has no neighbor colored 0 in $G - v$, we color it 0, and color the corresponding $v_i$ with $k$ if it was colored 0. By doing this, we keep a $(0,6)$-coloring of $H$. We can thus assume that in $G - v$, every $w_i$ that is colored $k$ has a neighbor colored 0 and thus at most five neighbors colored $k$. If at least one of the $v_i$'s is colored $k$, then $w$ has at most five neighbors colored $k$ in $G - v$, and assigning $k$ to $v$ gives a $(0,6)$-coloring of $G$. Otherwise, every $v_i$ is colored 0, every $w_i$ is colored $k$, and $w$ is colored $k$. Thus we assign 0 to $v$ to obtain a $(0,6)$-coloring of $G$, a contradiction. \hfill \Box

**Lemma 8.** No 3-vertex is adjacent to a 2-vertex.

**Proof.** Let $w$ be a 3-vertex adjacent to a 2-vertex $v$, let $x_1$ and $x_2$ be the other two neighbors of $w$, and let $u$ be the other neighbor of $v$. Let $H$ be the graph obtained from $G - \{v, w\}$ by adding five 2-vertices $v_1, v_2, w_1, w_2$, and $x$ which form the 8-cycle $w_1 v_1 w_2 x_1 x_2 w_2 v_2$. It is easy to check that $H$ is in $C$. By Lemmas 6 and 7, $u, x_1$, and $x_2$ are 8$^+$-vertices in $G$ and thus are 9$^+$-vertices in $H$. Since $w$ is in $G$ but not in $H$, $n_3(H) = n_3(G) - 1$, so $H \prec G$. Therefore $H$ has a $(0,6)$-coloring.

Suppose that $v_1$ and $v_2$ are both colored 0. Then $w_1, w_2$, and $u$ are colored $k$. We color $v$ with 0 and $w$ with $k$. The number of neighbors of $x_1$ (resp. $x_2$) colored $k$ in $G$ is at most the number of neighbors of $x_1$ (resp. $x_2$) colored $k$ in $H$. Thus we have a $(0,6)$-coloring of $G$, a contradiction. Now we assume without loss of generality that $v_1$ is colored $k$. We color $w$ with the color of $x$ and we color $v$ with $k$. The number of neighbors of $u$ (resp. $x_1, x_2$) colored $k$ in $G$ is at most the number of neighbors of $u$ (resp. $x_1, x_2$) colored $k$ in $H$. Thus we have a $(0,6)$-coloring of $G$, a contradiction. \hfill \Box
A special face is a 5-face with three 2-vertices and two non-adjacent \(8^+\)-vertices. See figure 1, left. A special configuration is three 5-faces sharing a common 3-vertex adjacent to three \(8^+\)-vertices, such that all the other vertices of these faces are 2-vertices. See figure 1, right. We say special structure to speak indifferently about a special face or a special configuration.

Let us define a hypergraph \(\hat{G}\) whose vertices are the \(8^+\)-vertices of \(G\) and the hyperedges correspond to the sets of \(8^+\)-vertices contained in the same special structure. For every vertex \(v\) of \(\hat{G}\), let \(\hat{d}(v)\) denote the degree of \(v\) in \(\hat{G}\), that is the number of hyperedges containing \(v\).

**Lemma 9.** Let \(\alpha\) be a special structure, with the notations of Figure 1. Consider a \((0, 6)\)-coloring of \(\alpha\).

We can change the color of the \(x_i\)'s, \(y_i\)'s and \(u\) such that the \(v_i\)'s have no more neighbors colored \(k\) than before, and for all \(i\), if \(v_i\) is colored \(k\), then \(v_i\) has a neighbor colored 0.

**Proof.** If all of the \(v_i\)'s are colored 0, then there is noting to do. If they are all colored \(k\), then we assign 0 to \(u\). If one of the \(v_i\)'s, say \(v_0\), is colored 0 and another one, say \(v_1\), is colored \(k\), then \(u\) and \(x_0\) are colored \(k\) and we assign 0 to \(y_0\). Moreover, if \(\alpha\) is a special configuration and \(v_2\) is colored \(k\), then \(y_2\) is colored \(k\) and we assign 0 to \(x_2\).

**Lemma 10.** For every vertex \(v\) in \(\hat{G}\), \(d(v) - \hat{d}(v) \geq 7\).

**Proof.** Let \(v\) be a vertex that does not verify the lemma, i.e. such that \(d(v) - \hat{d}(v) \leq 6\). As \(v\) is an \(8^+\)-vertex, \(d(v) \geq 1\). Let \(\alpha\) be a special structure incident to \(v\) in \(\hat{G}\). We use the notations of Figure 1, with say \(v = v_0\). The graph \(G - x_0\) is smaller than \(G\), thus it admits a \((0, 6)\)-coloring. Since \(G\) does not admit a \((0, 6)\)-coloring, \(v_0\) is colored \(k\) and \(y_0\) is colored 0. By Lemma 9, we can assume that \(v\) has a neighbor colored 0 in each of its special structures distinct from \(\alpha\). If \(v_1\) is colored 0, then \(y_0\) is colored \(k\), a contradiction. Thus \(v_1\) is colored \(k\). If \(\alpha\) is a special face, or if \(v_2\) is colored \(k\), then we assign 0 to \(u\). If \(\alpha\) is a special configuration and \(v_2\) is colored 0, then \(x_2\) is colored \(k\) and we
assign 0 to \( y_2 \). In both cases, \( v \) has at least \( \hat{d}(v) \) neighbors colored 0. Thus \( v \) has at most \( d(v) - \hat{d}(v) \leq 6 \) neighbors colored \( k \) and we can assign \( k \) to \( v \), a contradiction.

\[ \square \]

**Lemma 11.** Every component of \( \hat{G} \) has at least one vertex \( v \) such that \( d(v) - \hat{d}(v) \geq 8 \).

**Proof.** Suppose the lemma is false, and let \( C \) be a component of \( \hat{G} \) that does not verify the lemma. If \( C \) has only one vertex, then this vertex is an \( 8^+ \)-vertex, which verifies \( d(v) - \hat{d}(v) \geq 8 \). Therefore \( C \) has at least one hyperedge, which corresponds to a special structure \( \alpha \) of \( G \). By Lemma 10, every vertex of \( C \) verifies \( d(v) - \hat{d}(v) = 7 \). We use the notations of Figure 1. The graph \( G - \{x_0, y_0\} \) is smaller than \( G \), thus it admits a \((0, 6)\)-coloring. Since \( G \) admits no \((0, 6)\)-coloring, \( v_0 \) and \( v_1 \) are colored \( k \). If \( \alpha \) is a special configuration and \( v_2 \) is colored 0, then \( x_2 \) and \( y_1 \) are colored \( k \) and we can color \( y_2 \) and \( x_1 \) with 0. Otherwise, we can color \( u \) with 0. Note that \( v_0 \) and \( v_1 \), as well as \( v_2 \) if it exists and is colored \( k \), all have six neighbors colored \( k \), and by Lemma 9, we can assume that they all have at least one neighbor colored 0 in each of their special structures besides \( \alpha \).

If one of the \( v_i \)'s, say \( v_0 \), has an additional neighbor colored 0, it verifies \( d(v_0) - \hat{d}(v_0) \geq 8 \), a contradiction. Thus, for every \( v_i \), either \( v_i \) is colored 0 or \( v_i \) has no neighbor colored 0 outside of its special structures and at most one neighbor colored 0 in each special structure besides \( \alpha \).

We uncolor \( u \) and all the \( x_i \)'s and \( y_i \)'s, and let \( H \) equal to \( G \) where \( u \), the \( x_i \)'s and the \( y_i \)'s are removed. By symmetry, we only consider the vertex \( v_0 \). The following procedure either assigns 0 to \( v_0 \) or ensures that \( v_0 \) has two neighbors colored 0 in one of its special structures:

- For each special structure \( \beta \) containing \( v_0 \) and completely contained in \( H \), we use the notations of Figure 1, keeping the same vertex for \( v_0 \), but changing the other ones for the vertices in \( \beta \), and do the following:
  - By Lemma 9, we can assume that every \( v_i \) colored \( k \) has a neighbor colored 0 in each of its special structures that are completely contained in \( H \).
  - Suppose that one of the \( 8^+ \)-vertices of \( \beta \) distinct from \( v_0 \), say \( v_1 \), has two neighbors colored 0 in a special structure distinct from \( \beta \) or a neighbor colored 0 outside of its special structures. Since \( d(v_1) - \hat{d}(v_1) = 7 \), \( v_1 \) has at most five neighbors colored \( k \) outside of \( \beta \) if \( \beta \) is a special face, and at most four neighbors colored \( k \) outside of \( \beta \) if \( \beta \) is a special configuration. We assign \( k \) to \( y_0 \) and \( 0 \) to \( x_0 \). If \( v_2 \) exists and is colored 0, then we assign 0 to \( y_2 \), and otherwise we assign 0 to \( u \). We end the procedure.
  - We uncolor the \( 7^- \)-vertices of \( \beta \) and remove them from \( H \).
  - For every \( 8^+ \)-vertex \( w \neq v_0 \) in \( \beta \) colored \( k \), we apply the procedure with \( w \) instead of \( v_0 \). Now \( w \) is colored 0 or has two neighbors colored 0 in the same special structure.
– We add back to $H$ the $7^-$-vertices of $\beta$. If $v_0$ is colored 0, then we give them color $k$ if they are adjacent to a vertex colored 0 and we assign them 0 otherwise, and we end the procedure. If $\beta$ is a special face and $v_1$ is colored $k$, or if $\beta$ is a special configuration and $v_1$ and $v_2$ are colored $k$, then we color $u$ and $x_0$ with 0, we color the other 2-vertices with $k$, and we end the procedure. Suppose $\beta$ is a special configuration, either $v_1$ or $v_2$, say $v_1$, is colored $k$, and the other one is colored 0. We assign 0 to $x_0$, $x_1$, and $y_2$, and $k$ to $u$, $y_0$, $y_1$, and $x_1$, and we end the procedure.

Let us prove that the previous procedure terminates. It always calls itself iteratively on a graph with fewer vertices, thus the number of nested iterations is bounded by the order of the initial graph. Furthermore, each iteration of the procedure only does a bounded number of calls to the procedure (at most two). That proves that the procedure terminates.

In the end, if one of the $v_i$’s is colored $k$, then it has at most five neighbors colored $k$ outside of $\alpha$ if $\alpha$ is a special face, and at most four neighbors colored $k$ outside of $\alpha$ if $\alpha$ is a special structure. If every $v_i$ is colored $k$, then color $u$ with color 0 and the other $7^-$-vertex of $\alpha$ with color $k$. Otherwise, assign $k$ to $u$, and do the following:

- If every $v_i$ is colored 0, then assign $k$ to the $x_i$’s and the $y_i$’s.
- If $\alpha$ is a special face and one of the $v_i$’s, say $v_0$, is colored 0 while the other one is colored $k$, then assign $k$ to $x_0$ and 0 to $y_0$.
- If $\alpha$ is a special structure, then assign $k$ to the $y_i$’s, and for all $i \in \{0, 1, 2\}$, if $v_i$ is colored $k$, then assign 0 to $x_i$, and if $v_i$ is colored 0 then assign $k$ to $x_i$.

In all cases, we get a $(0, 6)$-coloring of $G$, a contradiction.

For each component $C$ of $\hat{G}$, we choose a vertex $v$ in $C$ such that $d(v) - \hat{d}(v) \geq 8$ as the root of $C$. We then choose an orientation of the edges of $\hat{G}$ such that the only vertices with no incoming edges are the roots (for example do a breadth first search from the root of each component). For each $8^+$-vertex $v$, $v$ is said to sponsor all of the special faces that correspond to its outgoing edges in $\hat{G}$.

We are now going to give some weight on the vertices and faces of the graph. Initially, for all $d$, every $d$-vertex has weight $d - 4$, and every $d$-face has weight $d - 4$. Thus every face and every $4^+$-vertex has non-negative initial weight.

We apply the following discharging procedure.
Lemma 12. Every vertex $v$ verifies $\omega'(v) \geq 0$.

Proof. Let $v$ be a vertex of degree $d$. We have $\omega(v) = d - 4$.

- Suppose first that $d \geq 8$. The vertex $v$ gives $\frac{1}{2}$ to each of its $7^-$-neighbors and two times $\frac{1}{4}$ to each of its $8^+$-neighbors in Step 1, for a total of $\frac{d}{2}$.

  As $d \geq 8$, we have $\omega(v) = d - 4 \geq \frac{d}{2}$, therefore if $v$ sponsors no special structure, then $\omega'(v) = d - 4 - \frac{d}{2} \geq 0$.

  Suppose $v$ sponsors a special structure. If $v$ sponsors all of its special structures, then $v$ is the root of its component in $\hat{G}$, thus $d - \hat{d}(v) \geq 8$, and thus $\omega'(v) = d - 4 - \frac{d(v)}{2} - \frac{d}{2} = d - \hat{d}(v) - 4 - \frac{d - d(v)}{2} \geq 0$. If $v$ does not sponsor all of its special structures, then $d - \hat{d}(v) \geq 7$, and $\omega'(v) = d - 4 - \frac{d(v) - 1}{2} - \frac{d}{2} = d - \hat{d}(v) - \frac{7}{2} - \frac{d - d(v)}{2} \geq 0$.

- Suppose now that $4 \leq d \leq 8$. By Lemma 7, $v$ has at least two $8^+$-neighbors. The vertex $v$ only gives weight in Step 2. Moreover, it gives at most $\frac{1}{2}$ to each of its 2-neighbors plus $\frac{1}{2}$ for each pair of consecutive $8^+$-vertices in Step 2. If $v$ has only $8^+$-neighbors, then it receives $\frac{d}{2}$ in Step 1, and gives at most $\frac{d}{2}$ in Step 2, so $\omega'(v) \geq \omega(v) = d - 4 \geq 0$.

  Suppose $v$ has at least one $7^-$-neighbor. Let $d' \geq 2$ be the number of $8^+$-neighbors of $v$. The vertex $v$ receives $\frac{d}{2}$ in Step 1. It gives at most $\frac{d-d'}{2}$ to the 2-vertices and at most $\frac{d'-1}{2}$ to the faces for a total of at most $\frac{d-d'}{2} + \frac{d'-1}{2} = \frac{d}{2} - \frac{d'}{2}$ in Step 2. It receives at least $\frac{d'}{4}$ in Step 3. We have $\omega'(v) \geq d - 4 - \frac{d}{2} + 3\frac{d'}{4} + \frac{1}{2} \geq 0$, since $d' \geq 2$ and $d \geq 4$. 

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• Suppose that \( d = 3 \). By Lemma 7, \( v \) has at least two \( 8^+ \)-neighbors, and by Lemma 8, \( v \) has no 2-neighbors. If \( v \) has exactly two \( 8^+ \)-neighbors, then it receives 1 in Step 1, gives \( \frac{1}{2} \) in Step 2, and receives \( \frac{3}{4} \) in Step 3, therefore \( \omega'(v) \geq \frac{1}{4} > 0 \). If \( v \) has three \( 8^+ \)-neighbors, then \( v \) receives \( \frac{3}{2} \) in Step 1 and an additional \( \frac{3}{2} \) in Step 3, and it gives at most 1 in Step 2 unless it is in a special configuration, in which case it gives at most \( \frac{3}{2} \) in Step 2 and receives 2 in Step 1. Therefore if \( v \) has three \( 8^+ \)-neighbors, then \( \omega'(v) \geq \frac{1}{4} > 0 \).

• Suppose that \( d = 2 \). Note that \( v \) cannot be in two 5-faces since \( G \in C \).
  - If \( v \) is in a 5-face and adjacent to another 2-vertex, then it receives \( \frac{1}{2} \) from its \( 8^+ \)-neighbor in Step 1, \( \frac{5}{8} \) from its 5-face in Step 4, and \( \frac{7}{8} \) from its other face in Step 5.
  - If \( v \) is in a 5-face and adjacent to no other 2-vertex, then it receives 1 from its \( 3^+ \)-neighbors in Steps 1 and 2, \( \frac{1}{4} \) from its 5-face in Step 4, and \( \frac{3}{4} \) from its other face in Step 5.
  - If \( v \) is not in a 5-face and is adjacent to another 2-vertex, then it receives \( \frac{1}{2} \) from its \( 8^+ \)-neighbor in Step 1, and \( \frac{1}{2} \) from its faces in Step 5.
  - If \( v \) is in a 5-face and adjacent to no other 2-vertex, then it receives 1 from its \( 3^+ \)-neighbors in Steps 1 and 2, and 1 from its faces in Step 5.

In all cases, \( v \) receives 2 over the procedure, and thus \( \omega'(v) = 2 - 4 + 2 = 0 \).

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Lemma 13. Every face \( \alpha \) satisfies \( \omega'(\alpha) \geq 0 \).

Proof. Let \( \alpha \) be a vertex of degree \( d \). We have \( \omega(\alpha) = d - 4 \).

• Suppose \( d = 5 \). If \( \alpha \) is a special face, then it receives \( \frac{1}{2} \) in Step 1 and gives \( \frac{1}{4} + 2 \cdot \frac{5}{8} = \frac{3}{2} \) in Step 4.

If \( \alpha \) has no two consecutive 2-vertices, then it gives at most \( \frac{1}{2} \) to its small vertices over Steps 3 and 4, and does not actually give anything unless one of its vertices is an \( 8^+ \)-vertex, and thus gives at most 1 overall.

If \( \alpha \) has two consecutive 2-vertices and its three other vertices are \( 8^+ \)-vertices, then it receives 1 in Step 1 and gives at most \( 2 \cdot \frac{5}{8} = \frac{5}{4} \leq 2 \) overall.

The only remaining case is when \( \alpha \) has, in this consecutive order, two 2-vertices, an \( 8^+ \)-vertex, a \( 3^+ \)-vertex with degree at most 7, and another \( 8^+ \)-vertex. In this case, \( \alpha \) receives \( \frac{1}{2} \) in Step 2, and gives \( 2 \cdot \frac{5}{8} + \frac{1}{4} = \frac{3}{2} \) over Steps 3 and 4.

In all cases, \( \omega'(\alpha) \geq 1 - 1 = 0 \).
• Suppose $d = 7$. Note that if there are two adjacent 2-vertices in $\alpha$, then these two vertices are not in a 5-face, otherwise there would be a cycle of length 6 in $G$. The face $\alpha$ has an initial charge of 3, gives at most $\frac{3}{4}$ to its $7^*$-vertices that are adjacent to an $8^*$-vertex in $\alpha$ and nothing to its other vertices. There can be at most four of these vertices. Therefore $\omega'\alpha \geq 3 - 4 \cdot \frac{3}{4} = 0$.

• Suppose $d = 8$. Note that at most one pair of adjacent 2-vertices is in a 5-face, otherwise there would be a cycle of length 6 in $G$. The face $\alpha$ has an initial charge of 4, gives at most $\frac{7}{8}$ to its $7^*$-vertices that are adjacent to an $8^*$-vertex in $\alpha$, and nothing to its other vertices. There can be at most five of these vertices, and at most two are given $\frac{7}{8}$, the other being given at most $\frac{3}{4}$. Therefore $\omega'\alpha \geq 4 - 2 \cdot \frac{7}{8} - 3 \cdot \frac{3}{4} = 0$.

• Suppose $d = 9$. Note that at most two pairs of adjacent 2-vertices are in a 5-face, otherwise there would be a cycle of length 6 in $G$. The face $\alpha$ has an initial charge of 5, gives at most $\frac{7}{8}$ to its $7^*$-vertices that are adjacent to an $8^*$-vertex in $\alpha$, and nothing to its other vertices. There can be at most six of these vertices, at most four are given $\frac{7}{8}$, and the others are given at most $\frac{3}{4}$. Therefore $\omega'\alpha \geq 5 - 4 \cdot \frac{7}{8} - 2 \cdot \frac{3}{4} = 0$.

• Suppose $d \geq 10$. The face $\alpha$ has an initial charge of $d - 4$, gives at most $\frac{7}{8}$ to its $7^*$-vertices that are adjacent to an $8^*$-vertex in $\alpha$, and nothing to its other vertices. There can be at most $d - 4$ of these vertices, therefore $\omega'\alpha \geq d - 4 - (d - 4) \cdot \frac{7}{8} > 0$.

By Euler’s formula, since $G$ is connected by Lemma 4 and has at least one vertex, $n + f - m = 2$. The initial weight of the graph is $\sum_{v \in V(G)} \omega(v) + \sum_{\alpha \in F(G)} \omega(\alpha) = \sum_{v \in V(G)} (d(v) - 4) + \sum_{\alpha \in F(G)} (d(\alpha) - 4) = \sum_{v \in V(G)} d(v) + \sum_{\alpha \in F(G)} d(\alpha) - 4n - 4f = 4m - 4n - 4f = -8 < 0$. Therefore the initial weight of the graph is negative, thus the final weight of the graph is negative. Since by Lemmas 12 and 13, the final weight of every face and every vertex is non-negative, we get a contradiction. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Let $k \geq 3$ be a fixed integer. Suppose that there exists a graph in $\mathcal{C}$ that is not $(0, k)$-colorable. We consider such a graph $H_k$ that is minimal according to $\preceq$. By adapting the proofs of Lemmas 4, 5, and 6, we obtain that the minimum degree of $H_k$ is at least two and every $(k + 1)^*$-vertex in $H_k$ is adjacent to a $(k + 2)^*$-vertex. Suppose for contradiction that $H_k$ contains no 2-vertex. We consider the discharging procedure such that the initial charge of every vertex is equal to its degree and every $5^*$-vertex gives $\frac{1}{3}$ to every adjacent $3^*$-vertex. Then the final charge of a $3^*$-vertex is at least $3 + \frac{1}{3} = \frac{10}{3}$, the final charge of a $d$-vertex with $d \geq k + 2$ is at least $d - d \cdot \frac{1}{3} \geq \frac{2d}{3} \geq \frac{2(k + 2)}{3} \geq \frac{10}{3}$, and the
final charge of every remaining vertex is at least $4 > \frac{10}{3}$. This implies that the maximum average degree of $H_k$ is at least $\frac{10}{3}$, which is a contradiction since $H_k$ is a planar graph with girth at least 5. Thus, $H_k$ contains a 2-vertex $v$ adjacent to the vertices $u_1$ and $u_5$.

By minimality of $H_k$, $H_k - v$ is $(0,k)$-colorable, every $(0,k)$-coloring of $H_k - v$ is such that $u_1$ and $u_5$ get distinct colors, and the vertex in $\{u_1, u_5\}$ that is colored $k$ has exactly $k$ neighbors that are colored $k$.

Consider the graph $H'_k$ obtained from $H_k - v$ by adding three 2-vertices $u_2$, $u_3$, and $u_4$ which form a path $u_1u_2u_3u_4u_5$. Notice that $H'_k$ is $(0,k)$-colorable and that every $(0,k)$-coloring of $H'_k$ is such that $u_3$ is colored $k$ and is adjacent to exactly one vertex colored $k$. It is easy to see that $H'_k$ is in $\mathcal{C}$.

We are ready to prove that deciding whether a graph in $\mathcal{C}$ is $(0,k)$-colorable is NP-complete. The reduction is from the NP-complete problem of deciding whether a planar graph with girth at least 9 is $(0,1)$-colorable [2]. Given an instance $G$ of this problem, we construct a graph $G' \in \mathcal{C}$, as follows. For every vertex $v$ in $G$, we add $k - 1$ copies of $H'_k$ and we add an edge between $v$ and the vertex $u_3$ of each these copies. Notice that $G'$ is in $\mathcal{C}$ since $G'$ is planar and every cycle of length at most 8 is contained in a copy of $H'_k$ which is in $\mathcal{C}$. Notice that a $(0,1)$-coloring of $G$ can be extended to a $(0,k)$-coloring of $G'$. Conversely, a $(0,k)$-coloring of $G'$ induces a $(0,1)$-coloring of $G$. So $G$ is $(0,1)$-colorable if and only if $G'$ is $(0,k)$-colorable.

4. A graph in $\mathcal{C}$ that is not $(0,3)$-colorable

Consider the graph $F_{x,y}$ depicted in Figure 2. Suppose for contradiction that $F_{x,y}$ admits a $(0,3)$-coloring such that all the neighbors of $x$ and $y$ are colored 0 (the white vertices in the picture). Then the neighbors of those white vertices are colored $k$. We consider the 8 big vertices. Each of them is colored $k$ and is adjacent to two vertices colored $k$. For every pair of adjacent red vertices, at least one of them is colored $k$. Notice that every red vertex is adjacent to a big vertex. Since there are 9 pairs of adjacent red vertices, there exists a big vertex that is adjacent to at least two red vertices colored $k$. This big vertex is thus adjacent to four vertices colored $k$, which is a contradiction.

In the graph depicted in Figure 3, every dashed line represents a copy of $F_{x,y}$ such that the extremities are $x$ and $y$. Suppose for contradiction that this $(C_3, C_4, C_6)$-free planar graph admits a $(0,3)$-coloring. Each of the two drawn edges has at least one extremity colored $k$. Thus, there exist two vertices $u$ and $v$ colored $k$ that are linked by 7 copies of $F_{x,y}$. Since at most 3 neighbors of $u$ and at most 3 neighbors of $v$ can be colored $k$, one of these 7 copies of $F_{x,y}$ is such that all the neighbors of $x$ and $y$ are colored 0. This contradiction proves Theorem 2.

Following the proof above, we see that if we remove the green parts in Figures 2 and 3, we obtain a planar graph with girth 7 that is not $(0,2)$-colorable. A graph with such properties is already known [3], but this new graph is smaller (184 vertices instead of 1304) and the proof of non-$(0,2)$-colorability is simpler.
References

