Vertex partitions of (C_3, C_4, C_6)-free planar graphs

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Abstract

A graph is (k_1, k_2)-colorable if it admits a vertex partition into a graph with maximum degree at most k_1 and a graph with maximum degree at most k_2. We show that every (C_3, C_4, C_6)-free planar graph is $(0, 6)$-colorable. We also show that deciding whether a (C_3, C_4, C_6)-free planar graph is $(0, 3)$-colorable is NP-complete.

Keywords: Improper colorings, Planar graphs.

1. Introduction

A graph is (k_1, k_2)-colorable if it admits a vertex partition into a graph with maximum degree at most k_1 and a graph with maximum degree at most k_2. Choi, Liu, and Oum [2] have established that there exist exactly two minimal sets of forbidden cycle lengths such that every planar graph is $(0, k)$-colorable for some absolute constant k.

- Planar graphs without odd cycles are bipartite, that is, $(0, 0)$-colorable.
- Planar graphs without cycles of length 3, 4, and 6 are $(0, 45)$-colorable.

The aim of this paper is to improve this last result. Notice that forbidding cycles of length 3, 4, and 6 as subgraphs or as induced subgraphs result in the same graph class. For every $n \geqslant 3$, we denote by C_n the cycle on n vertices. So we are interested in the class $\mathcal{C}$ of (C_3, C_4, C_6)-free planar graphs.

We prove the following two theorems in the next two sections.

Theorem 1. *Every graph in $\mathcal{C}$ is $(0, 6)$-colorable.*

Theorem 2. *For every $k \geqslant 1$, either every graph in $\mathcal{C}$ is $(0, k)$-colorable, or deciding whether a graph in $\mathcal{C}$ is $(0, k)$-colorable is NP-complete.*

In addition, we construct a graph in $\mathcal{C}$ that is not $(0, 3)$-colorable in Section 4. This graph and Theorem 2 imply the following.

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Corollary 3. *Deciding whether a graph in $\mathcal{C}$ is $(0,3)$-colorable is NP-complete.*

The $(0,k)$-colorability of planar graphs with given girth has been investigated.

- Planar graphs with girth at least 11 are $(0,1)$-colorable [4].
- Planar graphs with girth at least 8 are $(0,2)$-colorable [1].
- Planar graphs with girth at least 7 are $(0,4)$-colorable [1].

These graph classes are subclasses of $\mathcal{C}$.

Since we deal with $(0,k)$-colorings for some $k \geqslant 2$, we denote by the letter 0 the color of the vertices that induce the independent set and we denote by the letter k the color of the vertices that induce the graph with maximum degree k.

2. Proof of Theorem 1

The proof uses the discharging method. For every plane graph G, we denote by $V(G)$ the set of vertices of G, by $E(G)$ the set of edges of G, and by $F(G)$ the set of faces of G.

For all d, let us call a vertex of a graph G of degree d, at most d, and at least d a *d-vertex*, a *d^--vertex*, and a *d^+-vertex*, respectively. For all vertex v, a *d-neighbor*, a *d^--neighbor*, and a *d^+-neighbor* of v is a neighbor of v that is a d-vertex, a d^--vertex, and a d^+-vertex, respectively. For all d, let us call a face of G of degree d, at most d, and at least d a *d-face*, a *d^--face*, and a *d^+-face*, respectively. For a set S of vertices, an *S-vertex* is a vertex that belongs to S, and an *S-neighbor* of a vertex v is a neighbor of v that belongs to S. For a set S of vertices, let $G[S]$ denote the graph induced by S, and $G - S = G[V(G) \setminus S]$. For convenience, we will denote $G - v$ for $G - \{v\}$.

Let us define the partial order $\preceq$. Let $n_3(G)$ be the number of 3^+-vertices in G. For any two graphs G_1 and G_2, we have $G_1 \prec G_2$ if and only if one of the following conditions holds:

- $|V(G_1)| < |V(G_2)|$ and $n_3(G_1) = n_3(G_2)$.
- $n_3(G_1) < n_3(G_2)$.

Note that the partial order $\preceq$ is well-defined and is a partial linear extension of the subgraph poset.

We suppose for contradiction that G is a graph in $\mathcal{C}$ that is not $(0,6)$-colorable and is minimal according to $\preceq$. Let n denote the number of vertices, m the number of edges, and f the number of faces of G. For every vertex v, the degree of v in G is denoted by $d(v)$. For every face α, the *degree* of α, denoted by $d(\alpha)$, is the length of a boundary walk of the face. More generally, when counting the number of edges of a certain type in a face, we will always count twice the edges that are only in this face.

Let us first prove some results on the structure of G, and then we will prove that G cannot exist, thus proving the theorem.

Lemma 4. *G is connected.*

Proof. If G is not connected, then every connected component of G is smaller than G and thus admits a $(0,6)$-coloring. The union of these $(0,6)$-colorings gives a $(0,6)$-coloring of G, a contradiction. $\square$

Lemma 5. *G has no 1-vertex.*

Proof. Let v be a 1-vertex and w be the neighbor of v. The graph $G-v$ admits a $(0,6)$-coloring since $G-v \prec G$. We get a $(0,6)$-coloring G by assigning to v the color distinct from the color of w, a contradiction. $\square$

Lemma 6. *Every 7^--vertex of G has an 8^+-neighbor.*

Proof. Let v be a 7^--vertex with no 8^+-neighbors. The graph $G-v$ admits a $(0,6)$-coloring since $G-v \prec G$. If there is a neighbor w of v with no neighbor colored 0, then we color w with 0. Thus, we can assume that every neighbor of v that is colored k has a neighbor colored 0 in $G-v$, and thus it has at most 5 neighbors colored k in $G-v$. Also, we can assume that v has at least one neighbor colored 0, since otherwise v can be colored 0. Thus, v has at most six neighbors colored k and v can be colored k, a contradiction. $\square$

Lemma 7. *Every vertex with degree at least 3 and at most 7 has two 8^+-neighbors.*

Proof. Suppose for contradiction that G contains a d-vertex v such that $3 \leqslant d \leqslant 7$ and v has at most one 8^+-neighbor. By Lemma 6, v has exactly one 8^+-neighbor w. Let $w_1, \dots, w_{d-1}$ be the other neighbors of v. Let H be the graph obtained from $G-v$ by adding $d-1$ 2-vertices $v_1, \dots, v_{d-1}$, such that for every $i \in \{1, \dots, d-1\}$, v_i is adjacent to w and w_i.

Notice that $H \prec G$ since $n_3(H) = n_3(G)-1$. Moreover, every cycle of length ℓ in H is associated to a cycle of length ℓ or $\ell-2$ in G. Therefore $H \in \mathcal{C}$, so H has a $(0,6)$-coloring.

If w is colored 0, then every v_i is colored k, so coloring v with k leads to a $(0,6)$-coloring of G, a contradiction. Therefore w is colored k.

If at least one of the v_i's is colored k, then w has at most five neighbors colored k in $G-v$. So we assign k to v and 0 to every w_i that is not adjacent to a vertex colored 0. This leads to a $(0,6)$-coloring of H. Otherwise, every v_i is colored 0, every w_i is colored k, and w is colored k. Thus we assign 0 to v to obtain a $(0,6)$-coloring of G, a contradiction. $\square$

Lemma 8. *No 3-vertex is adjacent to a 2-vertex.*

Proof. Let w be a 3-vertex adjacent to a 2-vertex v, let x_1 and x_2 be the other two neighbors of w, and let u be the other neighbor of v. Let H be the graph obtained from $G-\{v,w\}$ by adding five 2-vertices v_1, v_2, w_1, w_2, and x which form the 8-cycle $uv_1w_1x_1xx_2w_2v_2$. It is easy to check that H is in $\mathcal{C}$. By Lemmas 6 and 7, u, x_1, and x_2 are 8^+-vertices in G and thus are 9^+-vertices in

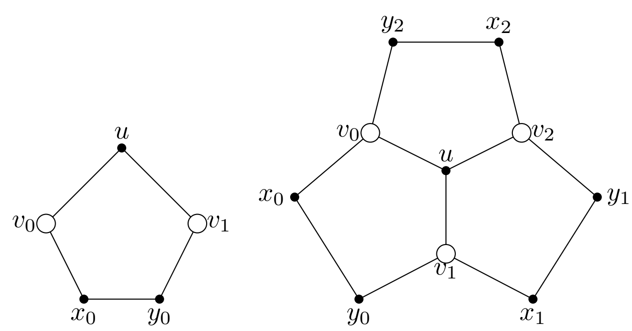

Figure 1: A special face (left) and a special configuration (right).

H. Since w is in G but not in H, $n_3(H) = n_3(G) - 1$, so $H \prec G$. Therefore H has a $(0, 6)$-coloring.

Suppose that v_1 and v_2 are both colored 0. Then w_1, w_2, and u are colored k. We color v with 0 and w with k. The number of neighbors of x_1 (resp. x_2) colored k in G is at most the number of neighbors of x_1 (resp. x_2) colored k in H. Thus we have a $(0, 6)$-coloring of G, a contradiction. Now we assume without loss of generality that v_1 is colored k. We color w with the color of x and we color v with k. The number of neighbors of u (resp. x_1, x_2) colored k in G is at most the number of neighbors of u (resp. x_1, x_2) colored k in H. Thus we have a $(0, 6)$-coloring of G, a contradiction. □

A *special face* is a 5-face with three 2-vertices and two non-adjacent 8^+-vertices. See Figure 1, left. A *special configuration* is three 5-faces sharing a common 3-vertex adjacent to three 8^+-vertices, such that all the other vertices of these faces are 2-vertices. See Figure 1, right. We say *special structure* to refer indifferently to a special face or to a special configuration.

Let us define a hypergraph $\widehat{G}$ whose vertices are the 8^+-vertices of G and whose hyperedges correspond to the sets of 8^+-vertices contained in the same special structure. For every vertex v of $\widehat{G}$, let $\hat{d}(v)$ denote the degree of v in $\widehat{G}$, that is the number of hyperedges containing v.

Lemma 9. *Let α be a special structure, with the notation of Figure 1. Consider a $(0, 6)$-coloring of α. We can change the color of the x_i's, y_i's, and u such that the v_i's have no more neighbors colored k than before, and for all i, if v_i is colored k, then v_i has a neighbor colored 0 in α.*

Proof. If all of the v_i's are colored 0, then there is nothing to do. If they are all colored k, then we assign 0 to u. If one of the v_i's, say v_0, is colored 0 and another one, say v_1, is colored k, then u and x_0 are colored k and we assign 0 to y_0. Moreover, if α is a special configuration and v_2 is colored k, then y_2 is colored k and we assign 0 to x_2. □

Lemma 10. *For every vertex v in $\widehat{G}$, $d(v) - \hat{d}(v) \geqslant 7$.*

Proof. Let v be a vertex that does not satisfy the lemma, i.e. such that $d(v) - \hat{d}(v) \leqslant 6$. Since v is an 8^+-vertex in G, $\hat{d}(v) \geqslant 2$. Let α be a special structure incident to v in $\widehat{G}$. We use the notation of Figure 1, with say $v = v_0$. The graph $G - x_0$ is smaller than G, thus it admits a $(0, 6)$-coloring. Since G does not admit a $(0, 6)$-coloring, v_0 is colored k and y_0 is colored 0. By Lemma 9, we can assume that v_0 has a neighbor colored 0 in each of its special structures distinct from α. Since y_0 is colored 0, v_1 is colored k. If α is a special face, or if v_2 is colored k, then we assign 0 to u. If α is a special configuration and v_2 is colored 0, then x_2 is colored k and we assign 0 to y_2. In both cases, $v = v_0$ has at least $\hat{d}(v)$ neighbors colored 0. Thus v has at most $d(v) - \hat{d}(v) \leqslant 6$ neighbors colored k and we can assign k to x_0, a contradiction. □

Lemma 11. *Every component of $\widehat{G}$ has at least one vertex v such that $d(v) - \hat{d}(v) \geqslant 8$.*

Proof. Suppose the lemma is false, and let C be a component of $\widehat{G}$ that does not satisfy the lemma. If C has only one vertex, then this vertex is an 8^+-vertex, which satisfies $d(v) - \hat{d}(v) \geqslant 8$. Therefore C has at least one hyperedge, which corresponds to a special structure α of G. By Lemma 10, every vertex of C satisfies $d(v) - \hat{d}(v) = 7$. We use the notation of Figure 1. The graph $G - \{x_0, y_0\}$ is smaller than G, thus it admits a $(0, 6)$-coloring. Since G admits no $(0, 6)$-coloring, v_0 and v_1 are colored k. If α is a special configuration and v_2 is colored 0, then x_2 and y_1 are colored k and we can color y_2 and x_1 with 0. Otherwise, we can color u with 0.

Note that v_0 and v_1 both have six neighbors colored k, otherwise we could color x_0 or y_0 with color k and the other one with color 0. By Lemma 9, we can assume that every v_i that is colored k has at least one neighbor colored 0 in each of its special structures besides α. Therefore if we can alter the coloring such that v_0 or v_1, say v_0, has two neighbors with color 0 in α, then it satisfies $d(v_0) - \hat{d}(v_0) \geqslant 8$, a contradiction. Therefore if α is a special configuration and v_2 is colored k, then as u is colored 0, x_1 and y_2 are colored k and cannot be recolored to 0, and thus x_2 and y_1 are colored 0 and cannot be recolored to k. Hence, in that case, v_2 has six neighbors in k, and thus $d(v_2) - \hat{d}(v_2) \geqslant 9$, a contradiction.

Thus, for every v_i, either v_i is colored 0 or v_i has no neighbor colored 0 outside of its special structures and at most one neighbor colored 0 in each special structure besides α.

We uncolor u and all the x_i's and y_i's, and let H be equal to G where u, the x_i's, and the y_i's are removed. By symmetry, we only consider the vertex v_0. The following procedure either assigns 0 to v_0 or ensures that v_0 has two neighbors colored 0 in one of its special structures:

- For each special structure β containing v_0 and completely contained in H, we use the notation of Figure 1, keeping the same vertex for v_0, but changing the other ones for the vertices in β, and do the following:

- By Lemma 9, we can assume that every v_i colored k has a neighbor colored 0 in each of its special structures that are completely contained in H.
- Suppose that one of the 8^+-vertices of β distinct from v_0, say v_1, has two neighbors colored 0 in a special structure distinct from β or a neighbor colored 0 outside of its special structures. Since $d(v_1) - \hat{d}(v_1) = 7$, v_1 has at most five neighbors colored k outside of β if β is a special face, and at most four neighbors colored k outside of β if β is a special configuration. We assign k to y_0 and 0 to x_0. If v_2 exists and is colored 0, then we assign 0 to y_2, and otherwise we assign 0 to u. Now v_0 has two neighbors colored 0 in β. We end the procedure.
- We uncolor the 7^--vertices of β and remove them from H.
- For every 8^+-vertex $w \neq v_0$ in β colored k, we apply the procedure with w instead of v_0. Now w is colored 0 or has two neighbors colored 0 in the same special structure.
- We add back to H the 7^--vertices of β. If v_0 is colored 0, then we give them color k if they are adjacent to a vertex colored 0 and we assign them 0 otherwise, and we end the procedure. If β is a special face and v_1 is colored k, or if β is a special configuration and v_1 and v_2 are colored k, then we color u and x_0 with 0, we color the other 2-vertices with k, and we end the procedure. Suppose β is a special configuration, either v_1 or v_2, say v_1, is colored k, and the other one is colored 0. We assign 0 to x_0, x_1, and y_2, and k to u, y_0, y_1, and x_1, and we end the procedure. Now all of the v_i's distinct from v_0 are colored 0. We color x_0 and y_2 (if it exists) with 0 and we color the other 7^- vertices in β with color k.

- Now in each special structure containing v_0 and completely contained in H, all of the 8^+-vertices distinct from v_0 are colored 0. We assign 0 to v_0 and k to all of the neighbors of v_0.

Let us prove that the previous procedure terminates. It always calls itself recursively on a graph with fewer vertices, thus the number of nested iterations is bounded by the order of the initial graph. Furthermore, each iteration of the procedure only does a bounded number of calls to the procedure (at most two). That proves that the procedure terminates.

In the end, if one of the v_i's is colored k, then it has at most five neighbors colored k outside of α if α is a special face, and at most four neighbors colored k outside of α if α is a special structure. If every v_i is colored k, then color u with color 0 and the other 7^--vertex of α with color k. Otherwise, assign k to u, and do the following:

- If every v_i is colored 0, then assign k to the x_i's and the y_i's.
- If α is a special face and one of the v_i's, say v_0, is colored 0 whereas the other one is colored k, then assign k to x_0 and 0 to y_0.

- If α is a special configuration, then assign k to the y_i's, and for all $i \in \{0,1,2\}$, if v_i is colored k, then assign 0 to x_i, and if v_i is colored 0 then assign k to x_i.

In all cases, we get a $(0,6)$-coloring of G, a contradiction. □

For each component C of $\widehat{G}$, we choose a vertex v in C such that $d(v)-\hat{d}(v) \geqslant 8$ as the *root* of C. We set that v ***sponsors*** all of its special structures. While there is a special structure α in C that has no sponsor, we choose a vertex in one such special structure that is also in a special structure that is already sponsored and we designate it to sponsor α. Since C is connected, each of its special structures gets a sponsor. Moreover, every vertex in C other than v is incident to at least one special structure that it does not sponsor.

Discharging procedure

For all d, we assign the weight $d-4$ to every d-vertex and d-face of G. Thus every face and every 4^+-vertex has non-negative initial weight.

We apply the following discharging procedure.

1. Every 8^+-vertex gives weight $\frac{1}{2}$ to each of its 7^--neighbors, to each special face it sponsors, and to the 3-vertex of each special configuration it sponsors. Additionally, for every edge vw where v and w are 8^+-vertices, v and w each give $\frac{1}{4}$ to each of the faces containing the edge vw, and $\frac{1}{4}$ more to the face containing vw if there is only one face containing vw.
2. For each 3^+-vertex v with degree at most 7 in G, v gives $\frac{1}{2}$ to each of its 2-neighbors. Moreover, v gives $\frac{1}{2}$ to every 5-face that is incident to v, to two 8^+-vertices adjacent to v, and to two 2-vertices.
3. For each face f, we consider a boundary walk of the face, and consider the vertices of this boundary walk in order. Each time a 3^+-vertex v with degree at most 7 appears in this boundary walk, right after or right before an 8^+-vertex, f gives $\frac{1}{4}$ to v (only once even if it is both right before an 8^+-vertex and right after an 8^+-vertex). Note that this means that f may give several times to the same vertex if it appears several times in the boundary walk.
4. Each 5-face gives $\frac{1}{4}$ to each of its 2-vertices with no 2-neighbor and $\frac{5}{8}$ to its 2-vertices with a 2-neighbor.
5. Each 7^+-face gives $\frac{3}{4}$ to each of its 2-vertices that belong to a 5-face and have no 2-neighbors, $\frac{7}{8}$ to each of its 2-vertices that belong to a 5-face and have a 2-neighbor, $\frac{1}{2}$ to each of its 2-vertices that do not belong to a 5-face and have no 2-neighbors, and $\frac{3}{4}$ to each of its 2-vertices that do not belong to a 5-face and have a 2-neighbor. If a 2-vertex is incident to only one face, then it receives twice the corresponding value from that face.

Let ω be the initial weight distribution, and let ω' be the final weight distribution, after the discharging procedure.

Lemma 12. *Every vertex v satisfies $\omega'(v) \geqslant 0$.*

Proof. Let v be a vertex of degree d. We have $\omega(v) = d - 4$.

- Suppose that $d \geqslant 8$. The vertex v gives $\frac{1}{2}$ to each of its 7^--neighbors and $\frac{1}{4}$ two times for each of its 8^+-neighbors in Step 1, for a total of $\frac{d}{2}$. As $d \geqslant 8$, we have $\omega(v) = d - 4 \geqslant \frac{d}{2}$, therefore if v sponsors no special structure, then $\omega'(v) = d - 4 - \frac{d}{2} \geqslant 0$.

  Suppose v sponsors a special structure. If v sponsors all of its special structures, then v is the root of its component in $\widehat{G}$, thus $d - \hat{d}(v) \geqslant 8$, and thus $\omega'(v) = d - 4 - \frac{\hat{d}(v)}{2} - \frac{d}{2} = d - \hat{d}(v) - 4 - \frac{d-\hat{d}(v)}{2} \geqslant 0$. If v does not sponsor all of its special structures, then $d - \hat{d}(v) \geqslant 7$, and $\omega'(v) = d - 4 - \frac{\hat{d}(v)-1}{2} - \frac{d}{2} = d - \hat{d}(v) - \frac{7}{2} - \frac{d-\hat{d}(v)}{2} \geqslant 0$.

- Suppose that $4 \leqslant d \leqslant 7$. By Lemma 7, v has at least two 8^+-neighbors. The vertex v only gives weight in Step 2. Moreover, it gives $\frac{1}{2}$ to each of its 2-neighbors plus $\frac{1}{2}$ for each pair of consecutive 8^+-vertices in Step 2. If v has only 8^+-neighbors, then it receives $\frac{d}{2}$ in Step 1, and gives at most $\frac{d}{2}$ in Step 2, so $\omega'(v) \geqslant \omega(v) = d - 4 \geqslant 0$. Suppose v has at least one 7^--neighbor. Let $d' \geqslant 2$ be the number of 8^+-neighbors of v. The vertex v receives $\frac{d'}{2}$ in Step 1. It gives at most $\frac{d-d'}{2}$ to the 2-vertices and at most $\frac{d'-1}{2}$ to the faces for a total of at most $\frac{d-d'}{2} + \frac{d'-1}{2} = \frac{d}{2} - \frac{1}{2}$ in Step 2. It receives at least $\frac{d'}{4}$ in Step 3. We have $\omega'(v) \geqslant d - 4 - \frac{d}{2} + 3\frac{d'}{4} + \frac{1}{2} \geqslant 0$, since $d' \geqslant 2$ and $d \geqslant 4$.

- Suppose that $d = 3$. By Lemma 7, v has at least two 8^+-neighbors, and by Lemma 8, v has no 2-neighbors. If v has exactly two 8^+-neighbors, then it receives 1 in Step 1, gives $\frac{1}{2}$ in Step 2, and receives $\frac{3}{4}$ in Step 3, therefore $\omega'(v) \geqslant \frac{1}{4} > 0$. If v has three 8^+-neighbors, then v receives $\frac{3}{2}$ in Step 1 and an additional $\frac{3}{4}$ in Step 3, and it gives at most 1 in Step 2 unless it is in a special configuration, in which case it gives at most $\frac{3}{2}$ in Step 2 and receives 2 in Step 1. Therefore if v has three 8^+-neighbors, then $\omega'(v) \geqslant \frac{1}{4} > 0$.

- Suppose that $d = 2$. Notice that v cannot be in two 5-faces, since otherwise G would contain C_6 or C_4.

  - If v is in a 5-face and adjacent to another 2-vertex, then it receives $\frac{1}{2}$ from its 8^+-neighbor in Step 1, $\frac{5}{8}$ from its 5-face in Step 4, and $\frac{7}{8}$ from its other face in Step 5.
  - If v is in a 5-face and adjacent to no other 2-vertex, then it receives 1 from its 3^+-neighbors in Steps 1 and 2, $\frac{1}{4}$ from its 5-face in Step 4, and $\frac{3}{4}$ from its other face in Step 5.
  - If v is not in a 5-face and is adjacent to another 2-vertex, then it receives $\frac{1}{2}$ from its 8^+-neighbor in Step 1, and $\frac{3}{2}$ from its faces in Step 5.

– If v is not in a 5-face and adjacent to no other 2-vertex, then it receives 1 from its 3^+-neighbors in Steps 1 and 2, and 1 from its faces in Step 5.

In all cases, v receives 2 over the procedure, and thus $\omega'(v) = 2-4+2 = 0$.

□

Lemma 13. *Every face α satisfies $\omega'(\alpha) \geqslant 0$.*

Proof. Let α be a face of degree d. We have $\omega(\alpha) = d - 4$.

- Suppose $d = 5$. If α is a special face, then it receives $\frac{1}{2}$ in Step 1 and gives $\frac{1}{4} + 2 \cdot \frac{5}{8} = \frac{3}{2}$ in Step 4.

  If α has no two consecutive 2-vertices, then it gives at most $\frac{1}{4}$ to its 7^--neighbors at Steps 3 and 4, and does not actually give anything unless one of its vertices is an 8^+-vertex, and thus gives at most 1 overall.

  If α has two consecutive 2-vertices and its three other vertices are 8^+-vertices, then it receives 1 in Step 1 and gives at most $2 \cdot \frac{5}{8} = \frac{5}{4} \leqslant 2$ overall.

  The only remaining case is when α has, in this consecutive order, two 2-vertices, an 8^+-vertex, a 3^+-vertex with degree at most 7, and another 8^+-vertex. In this case, α receives $\frac{1}{2}$ in Step 2, and gives $2 \cdot \frac{5}{8} + \frac{1}{4} = \frac{3}{2}$ over Steps 3 and 4.

  In all cases, $\omega'(\alpha) \geqslant 1 - 1 = 0$.

- Suppose $d = 7$. Note that if there are two adjacent 2-vertices in α, then these two vertices are not in a 5-face, otherwise there would be a cycle of length 6 in G. The face α has an initial charge of 3, gives at most $\frac{3}{4}$ to its 7^--vertices that are adjacent to an 8^+-vertex in α, and nothing to its other vertices. There can be at most four of these vertices. Therefore $\omega'(\alpha) \geqslant 3 - 4 \cdot \frac{3}{4} = 0$.

- Suppose $d = 8$. Note that at most one pair of adjacent 2-vertices is in a 5-face, otherwise there would be a cycle of length 6 in G. The face α has an initial charge of 4, gives at most $\frac{7}{8}$ to its 7^--vertices that are adjacent to an 8^+-vertex in α, and nothing to its other vertices. There can be at most five of these vertices, and at most two are given $\frac{7}{8}$, the other being given at most $\frac{3}{4}$. Therefore $\omega'(\alpha) \geqslant 4 - 2 \cdot \frac{7}{8} - 3 \cdot \frac{3}{4} = 0$.

- Suppose $d = 9$. Note that at most two pairs of adjacent 2-vertices are in a 5-face, otherwise there would be a cycle of length 6 in G. The face α has an initial charge of 5, gives at most $\frac{7}{8}$ to its 7^--vertices that are adjacent to an 8^+-vertex in α, and nothing to its other vertices. There can be at most six of these vertices, at most four are given $\frac{7}{8}$, and the others are given at most $\frac{3}{4}$. Therefore $\omega'(\alpha) \geqslant 5 - 4 \cdot \frac{7}{8} - 2 \cdot \frac{3}{4} = 0$.

- Suppose $d \geqslant 10$. The face α has an initial charge of $d-4$, gives at most $\frac{7}{8}$ to its 7^--vertices that are adjacent to an 8^+-vertex in α, and nothing to its other vertices. There can be at most $d-4$ of these vertices, therefore $\omega'(\alpha) \geqslant d-4-(d-4)\cdot\frac{7}{8} > 0$.

□

By Euler's formula, since G is connected by Lemma 4 and has at least one vertex, $n+f-m=2$. The initial weight of the graph is $\sum_{v\in V(G)}\omega(v)+\sum_{\alpha\in F(G)}\omega(\alpha)=\sum_{v\in V(G)}(d(v)-4)+\sum_{\alpha\in F(G)}(d(\alpha)-4)=\sum_{v\in V(G)}d(v)+\sum_{\alpha\in F(G)}d(\alpha)-4n-4f=4m-4n-4f=-8<0$. Therefore the initial weight of the graph is negative, thus the final weight of the graph is negative. Since by Lemmas 12 and 13, the final weight of every face and every vertex is non-negative, we get a contradiction. This completes the proof of Theorem 1.

3. Proof of Theorem 2

Recall that we have to prove that for every $k \geqslant 1$, either every graph in $\mathcal{C}$ is $(0,k)$-colorable, or deciding whether a graph in $\mathcal{C}$ is $(0,k)$-colorable is NP-complete.

Claim 14. *We can assume that $k \geqslant 3$.*

Proof. Both $(0,1)$-colorability and $(0,2)$-colorability are NP-complete for $\mathcal{C}$ since $(0,1)$-colorability is NP-complete for planar graphs with girth 9 [3] and $(0,2)$-colorability is NP-complete for planar graphs with girth 7 [5]. □

Let $k \geqslant 3$ be a fixed integer. Suppose that there exists a graph in $\mathcal{C}$ that is not $(0,k)$-colorable. We consider such a graph H_k that is minimal according to $\preceq$.

Claim 15. *The graph H_k contains a 2-vertex.*

Proof. By adapting the proofs of Lemmas 4, 5, and 6, we obtain that the minimum degree of H_k is at least two and every $(k+1)^-$-vertex in H_k is adjacent to a $(k+2)^+$-vertex. In particular, since $k \geqslant 3$, every 3-vertex is adjacent to a 5^+-vertex. Suppose for contradiction that H_k contains no 2-vertex. We consider the discharging procedure such that the initial charge of every vertex is equal to its degree and every 5^+-vertex gives $\frac{1}{3}$ to every adjacent 3-vertex. Then the final charge of a 3-vertex is at least $3+\frac{1}{3}=\frac{10}{3}$, the final charge of a 4-vertex is $4>\frac{10}{3}$, and the final charge of a d-vertex with $d \geqslant 5$ is at least $d-d\times\frac{1}{3}=\frac{2d}{3}\geqslant\frac{10}{3}$. This implies that the maximum average degree of H_k is at least $\frac{10}{3}$, which is a contradiction since H_k is a planar graph with girth at least 5. Thus, H_k contains a 2-vertex. □

By Claim 15, H_k contains a 2-vertex v adjacent to the vertices u_1 and u_5. By minimality of H_k, H_k-v is $(0,k)$-colorable, every $(0,k)$-coloring of H_k-v

is such that u_1 and u_5 get distinct colors, and the vertex in $\{u_1, u_5\}$ that is colored k has exactly k neighbors that are colored k.

Consider the graph H'_k obtained from $H_k - v$ by adding three 2-vertices u_2, u_3, and u_4 which form a path $u_1u_2u_3u_4u_5$. Notice that H'_k is $(0, k)$-colorable and that every $(0, k)$-coloring of H'_k is such that u_3 is colored k and is adjacent to exactly one vertex colored k. It is easy to see that H'_k is in $\mathcal{C}$.

We are ready to prove that deciding whether a graph in $\mathcal{C}$ is $(0, k)$-colorable is NP-complete. The reduction is from the NP-complete problem of deciding whether a planar graph with girth at least 9 is $(0, 1)$-colorable [3]. Given an instance G of this problem, we construct a graph $G' \in \mathcal{C}$, as follows. For every vertex v in G, we add $k-1$ copies of H'_k and we add an edge between v and the vertex u_3 of each of these copies. Notice that G' is in $\mathcal{C}$ since G' is planar and every cycle of length at most 8 is contained in a copy of H'_k which is in $\mathcal{C}$. Notice that a $(0, 1)$-coloring of G can be extended to a $(0, k)$-coloring of G'. Conversely, a $(0, k)$-coloring of G' induces a $(0, 1)$-coloring of G. So G is $(0, 1)$-colorable if and only if G' is $(0, k)$-colorable.

4. A graph in $\mathcal{C}$ that is not $(0, 3)$-colorable

Consider the graph $F_{x,y}$ depicted in Figure 2. Suppose for contradiction that $F_{x,y}$ admits a $(0, 3)$-coloring such that all the neighbors of x and y are colored 0 (the white vertices in the picture). Then the neighbors of those white vertices are colored k. We consider the 8 big vertices. Each of them is colored k and is adjacent to two vertices colored k. For every pair of adjacent red vertices, at least one of them is colored k. Since there are 9 pairs of adjacent red vertices, at least 9 red vertices are colored k. Notice that every red vertex is adjacent to a big vertex. By the pigeon-hole principle, at least one of the 8 big vertices is adjacent to at least two red vertices colored k. This big vertex is thus adjacent to at least four vertices colored k, which is a contradiction.

In the graph depicted in Figure 3, every dashed line represents a copy of $F_{x,y}$ such that the extremities are x and y. Suppose for contradiction that this (C_3, C_4, C_6)-free planar graph admits a $(0, 3)$-coloring. Each of the two drawn edges has at least one extremity colored k. Thus, there exist two vertices u and v colored k that are linked by 7 copies of $F_{x,y}$. Since at most 3 neighbors of u and at most 3 neighbors of v can be colored k, one of these 7 copies of $F_{x,y}$ is such that all the neighbors of x and y are colored 0. This contradiction proves Theorem 2.

Following the proof above, we see that if we remove the green parts in Figures 2 and 3, we obtain a planar graph with girth 7 that is not $(0, 2)$-colorable. A graph with such properties is already known [5], but this new graph is smaller (184 vertices instead of 1304) and the proof of non-$(0, 2)$-colorability is simpler.

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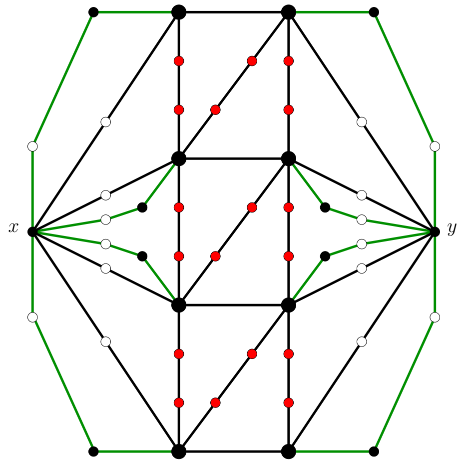

Figure 2: The forcing gadget $F_{x,y}$.

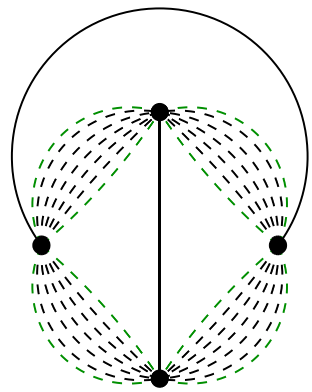

Figure 3: The non-$(0,3)$-colorable graph in $\mathcal{C}$.

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