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On circle graphs with girth at least five

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ABSTRACT

Circle graphs with girth at least five are known to be 2-degenerate [A.A. Ageev, Every circle graph with girth at least 5 is 3-colourable, Discrete Math. 195 (1999) 229–233]. In this paper, we prove that circle graphs with girth at least $g \geq 5$ and minimum degree at least two contain a chain of $g-4$ vertices of degree two, which implies Ageev's result in the case $g = 5$. We then use this structural property to give an upper bound on the circular chromatic number of circle graphs with girth at least $g \geq 5$ as well as a precise estimate of their maximum average degree.

1. Introduction

A circle graph is the intersection graph of the chords of a circle. It is not difficult to show that the class of circle graphs contains all the complete bipartite graphs. This implies that triangle-free circle graphs have arbitrarily large minimum degree. However, Ageev [1] proved that circle graphs with girth (size of a shortest cycle) at least five have minimum degree at most two:

Theorem 1 (*Ageev [1]*). *Every circle graph with girth at least five contains a vertex with degree at most two.*

In this paper, we prove the following extension of Ageev's result:

Theorem 2. *Every circle graph with girth $g \geq 5$ and minimum degree at least two contains a chain of $(g-4)$ vertices of degree two.*

Ageev [1] uses his structural result to prove that circle graphs with girth at least five have chromatic number at most three. We can use Theorem 2 to obtain a refinement of this result for circle graphs with larger girth. Instead of considering the chromatic number of these graphs, we consider their circular chromatic number. For two integers $1 \leq q \leq p$, a (p, q)-coloring of a graph G is a coloring c of the vertices of G with colors $\{0, \dots, p-1\}$ such that for any pair of adjacent vertices x and y, we have $q \leq |c(x) - c(y)| \leq p - q$. The circular chromatic number of G is

$$\chi_c(G) = \inf\left(\frac{p}{q}\,\middle|\, \text{there exists a } (p,q)\text{-coloring of } G\right).$$

It is known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$, and so $\chi(G) = \lceil \chi_c(G) \rceil$. The chromatic number can thus be considered as an approximation of the circular chromatic number.

Using a well-known observation on circular coloring (see e.g. Corollary 2.2 in [2]), the existence of a chain of $(g-4)$ vertices of degree two implies the following result:

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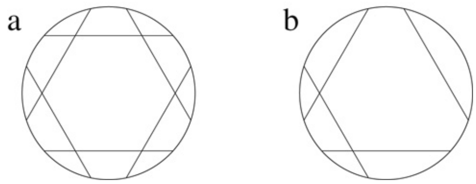

Fig. 1. (a) The unique circle representation $\mathcal{R}_6$ of C_6. (b) A sub-representation of $\mathcal{R}_6$.

Corollary 1. *Every circle graph G with girth $g \geq 5$ has circular chromatic number*

$$\chi_c(G) \leq 2 + \frac{1}{\left\lfloor \frac{g-3}{2} \right\rfloor}.$$

In Section 4, we study the *maximum average degree* of a graph G (see e.g. [5]) defined as

$$\text{mad}(G) = \max\left\{\text{ad}(H), H \subseteq G\right\}, \text{ where ad}(H) = \frac{2|E(H)|}{|V(H)|}.$$

For planar graphs, there is a simple relation between girth and maximum average degree: any planar graph G with girth g is such that $\text{mad}(G) < 2g/(g-2)$. On the other hand, there exists a family $(G_n)_{n\geq 0}$ of planar graphs with girth g, such that $\text{mad}(G_n) \to 2g/(g-2)$ when $n \to \infty$. We would like to obtain the same kind of link between the girth and the maximum average degree of circle graphs. The following corollary is a straightforward consequence of Theorem 2:

Corollary 2. *Any circle graph G with girth $g \geq 5$ is such that* $\text{mad}(G) < 2 + 2/(g-4)$.

Note that Corollary 2 has some implications on the circular choosability of circle graphs. Using Proposition 32(i) in Section 5.4 of [3], we can prove:

Corollary 3. *Every circle graph G with girth $g \geq 5$ has circular choice number $cch(G) \leq 2 + \frac{4}{g-2}$.*

To improve Corollary 2, we consider

$$\mu_g(\mathcal{F}) = \sup\left\{\text{mad}(G) \mid G \in \mathcal{F} \text{ and } G \text{ has girth at least } g\right\}.$$

For planar graphs, outerplanar graphs, and partial 2-trees this parameter is easy to compute and is known to be always a rational number. The following theorem shows that this is not the case for the class of circle graphs.

Theorem 3. *For every $g \geq 5$, $\mu_g(\text{Circle}) = 2\sqrt{\frac{g-2}{g-4}}$.*

In the next section, we give some notation and definitions. We prove Theorem 2 in Section 3 and Theorem 3 in Section 4.

2. Notation

Let C denote the unit circle, and let us take the clockwise orientation as the positive orientation of C. Let $\{x_0, \ldots, x_{k-1}\} \subset C$, we say that $(x_0, \ldots, x_{k-1})$ are in *cyclic order* if the minimum between the sum of the length of the arcs $\overrightarrow{x_i x_{i+1}}$, $0 \leq i \leq k-1$, and the sum of the length of the arcs $\overrightarrow{x_{i+1} x_i}$, $0 \leq i \leq k-1$, is equal to one, where i is taken modulo k. A pair $\{x, y\}$ of elements of C is called a *chord* of C with *endpoints* x and y. Two chords $\{x_1, y_1\}$ and $\{x_2, y_2\}$ *intersect* if $(x_1 x_2 y_1 y_2)$ are in cyclic order, otherwise they are said to be *parallel*.

All graphs considered in this paper are simple: they do not have any loop nor parallel edges. We call a k-vertex (resp. $^{\leq}k$-vertex, $^{\geq}k$-vertex) a vertex of degree k (resp. at most k, at least k).

By definition, every circle graph G with set of vertices $V(G) = \{v_1, \ldots, v_n\}$ admits a *representation* $\mathcal{C} = \{\{x_1, y_1\}, \ldots, \{x_n, y_n\}\}$ such that for all i, j, v_i and v_j are adjacent in G if and only if the chords $\{x_i, y_i\}$ and $\{x_j, y_j\}$ intersect in $\mathcal{C}$. We only consider representations in which endpoints and intersection points of chords are all distinct. A representation $\mathcal{C}'$ obtained from $\mathcal{C}$ only by removing chords is called a *sub-representation* of $\mathcal{C}$. Observe that if $\mathcal{C}$ is a representation of G, a sub-representation of $\mathcal{C}$ corresponds to an induced subgraph of G.

Remark that in general, circle graphs do not have a unique representation. However, cycles are uniquely representable; Fig. 1 depicts the unique representation of a cycle, and a sub-representation of this representation.

3. Proof of Theorem 2

Let $G = (V, E)$ be a circle graph with girth $g \geq 5$ and minimum degree at least two, and let $\mathcal{C} = \{\{x_1, x_1'\}, \ldots, \{x_n, x_n'\}\}$ be a circle representation of G. We first decompose the chords of $\mathcal{C}$ into two sets, using the following rule: color each chord $\{y, y'\}$

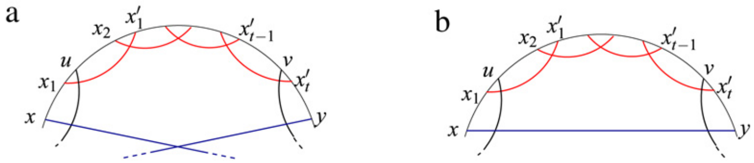

Fig. 2. A chain of $t \geq g-4$ vertices of degree two in G.

blue if there exists two chords $\{x, x'\}$ and $\{z, z'\}$, such that $(xyzz'y'x')$ are in order around the circle. Otherwise color the chord red. Let $\mathcal{C}^R$ (resp. $\mathcal{C}^B$) be the representation induced by the red (resp. blue) chords and G^R (resp. G^B) be the corresponding graph. We first prove the following lemma.

Lemma 1. *$\mathcal{C}^R$ is a sub-representation of the representation of a cycle.*

Proof. Assume that G^R contains a $^{\geq}3$-vertex v, adjacent to x, y, and z in G^R. Since $g \geq 5$, the graph G does not contain any triangle, and so $\{x, y, z\}$ is an independent set. This implies that the three corresponding red chords are parallel, which contradicts the definition of a red chord.

Hence, G^R has maximum degree two. Suppose now that G^R contains a cycle. Then if there exists a vertex which is not in the cycle, the corresponding chord, as well the chords corresponding to two non-adjacent vertices of the cycle, are parallel (recall that the cycle has length at least five, since $g \geq 5$). This contradicts the definition of a red chord. So G^R is either a cycle or a union of disjoint paths.

Suppose now that $\mathcal{C}^R$ is not a sub-representation of a cycle. Then G^R is necessarily a union of disjoint paths, and two of them are not in cyclic order in $\mathcal{C}^R$. This also contradicts the definition of a red chord, so $\mathcal{C}^R$ is a sub-representation of the representation of a cycle. □

Observe that each blue chord $\{x, y\}$ induces two arcs $\overrightarrow{xy}$ and $\overrightarrow{yx}$ on the circle. We denote by $\mathcal{A}_1$ the set of such arcs. Similarly, two intersecting blue chords $\{u, v\}$ and $\{x, y\}$ induce four consecutive arcs whose lengths add up to one, say without loss of generality $\overrightarrow{ux}$, $\overrightarrow{xv}$, $\overrightarrow{vy}$, and $\overrightarrow{yu}$. We denote by $\mathcal{A}_2$ the set of all such arcs.

For any arc $\overrightarrow{xy}$ of the circle, we define $\rho(\overrightarrow{xy})$ as the number of red chords having both endpoints in $\overrightarrow{xy}$. We consider the integer $t = \min\{\rho(\overrightarrow{xy}), \overrightarrow{xy} \in \mathcal{A}_1 \cup \mathcal{A}_2, \rho(\overrightarrow{xy}) > 0\}$.

If there is no blue chord in our decomposition, then G is either a cycle or a union of paths, and thus contains a $^{\leq}1$-vertex or g adjacent 2-vertices. So we can assume from now on that G^B is non-empty. Observe that for any blue chord $\{x, y\}$, we have $\rho(\overrightarrow{xy}) > 0$ and $\rho(\overrightarrow{yx}) > 0$ since otherwise $\{x, y\}$ would be red. Hence, the integer t exists. We now consider two cases, depending on whether the minimum is reached by two intersecting chords or by a single chord.

Case 1: The minimum $t > 0$ is reached by two intersecting blue chords, say $\{x, x'\}$ and $\{y, y'\}$, and for every blue chord $\{u, v\}$, we have $\rho(\overrightarrow{uv}) \neq t$. Let us assume without loss of generality that $t = \rho(\overrightarrow{xy})$. According to the clockwise order, we denote by $\{x_1, x'_1\}, \ldots \{x_t, x'_t\}$ the red chords having both endpoints in $\overrightarrow{xy}$ (see Fig. 2(a)). Observe that every blue chord has at most one endpoint in $\overrightarrow{xy}$, since otherwise we would have a blue chord $\{u, v\}$ with $1 \leq \rho(\overrightarrow{uv}) \leq t$, which would contradict the hypothesis.

We first prove that the graph induced by the chords $\{x_i, x'_i\}$ $(1 \leq i \leq t)$ is a path. If this is not the case, then for some i the chords $\{x_i, x'_i\}$ and $\{x_{i+1}, x'_{i+1}\}$ do not intersect. Then either one of them corresponds to a $^{\leq}1$-vertex, or each of them intersects a blue chord. Such a blue chord also intersects $\{x, x'\}$ or $\{y, y'\}$, since it has only one endpoint in $\overrightarrow{xy}$. This contradicts the minimality of t.

We now prove that the arc $\overrightarrow{x_2x'_{t-1}}$ does not contain any endpoint of a blue chord. Observe that if the arc contains the endpoint u of a blue chord, then there exists $1 \leq i \leq t-2$ such that $u \in \overrightarrow{x'_ix_{i+2}}$, since otherwise this would create a triangle. If such an endpoint u exists, the related blue chord along with $\{x, x'\}$ or $\{y, y'\}$ contradicts the minimality of t.

Hence, the vertices corresponding to $\{x_i, x'_i\}$ $(2 \leq i \leq t-1)$ are a chain of $(t-2)$ 2-vertices in G. Since G does not contain any 1-vertex, the chord $\{x_1, x'_1\}$ intersects a chord $\{u, u'\}$ distinct from $\{x_2, x'_2\}$. Such a chord may be blue or red, but by the minimality of t it cannot intersect $\{y, y'\}$. So the chord $\{u, u'\}$ has to intersect $\{x, x'\}$ and since $g \geq 4$, exactly one such $\{u, u'\}$ exists. Similarly, $\{x_t, x'_t\}$ intersects exactly one chord distinct from $\{x_{t-1}, x'_{t-1}\}$, say $\{v, v'\}$, and $\{v, v'\}$ also intersects $\{y, y'\}$. Thus the vertices corresponding to $\{x_i, x'_i\}$ $(1 \leq i \leq t)$ form a chain of t 2-vertices in G. Since the chords $\{x, x'\}, \{u, u'\}, \{x_1, x'_1\}, \ldots, \{x_t, x'_t\}, \{v, v'\}, \{y, y'\}$ correspond to a cycle in G, we have $t \geq g-4$.

Case 2: The minimum $t > 0$ is reached by a blue chord $\{x, y\}$. The proof is the same as the previous one, except that we obtain a chain of $(g-3)$ 2-vertices instead of $(g-4)$ 2-vertices (see Fig. 2(a)).

4. Proof of Theorem 3

Let us first give a construction to prove the lower bound. For every $g \geq 5$, we construct a family $(Q_{g,t})_{t\geq 0}$ of circle graphs with girth g such that $Q_{g,0} = C_g$ (the cycle on g vertices) and $Q_{g,t+1}$ is obtained by adding chords to the representation of $Q_{g,t}$.

A *k-region* is a region inside the circle, which is incident to the circle and to exactly k chords. Note that in any $Q_{g,t}$, every k-region is either a 2- or a 3-region. The representation of $Q_{g,t+1}$ is obtained from the representation of $Q_{g,t}$ by adding $(g-4)$

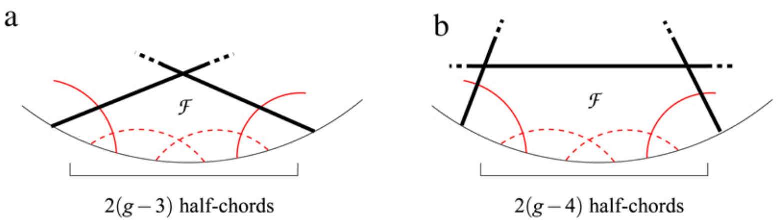

Fig. 3. From $Q_{g,t}$ to $Q_{g,t+1}$.

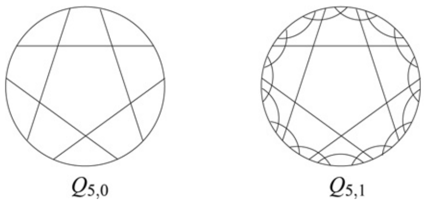

Fig. 4. Examples.

$$M_g = \begin{pmatrix} 1 & 0 & g-3 & g-4 \\ 0 & 1 & g-2 & g-3 \\ 0 & 0 & g-3 & g-4 \\ 0 & 0 & g-2 & g-3 \end{pmatrix} \qquad V = \begin{pmatrix} g-3+\sqrt{(g-2)(g-4)} \\ g-2+(g-3)\sqrt{(g-2)/(g-4)} \\ g-4+\sqrt{(g-2)(g-4)} \\ g-2+\sqrt{(g-2)(g-4)} \end{pmatrix}$$

Fig. 5. The matrix M_g and the eigenvector V.

chords and two half-chords in any 2-region, and $(g-5)$ chords and two half-chords in any 3-region (see Fig. 3, where thick chords correspond to chords of $Q_{g,t}$, and thin chords correspond to chords added to form $Q_{g,t+1}$).

Any 2-region in $Q_{g,t}$ produces in $Q_{g,t+1}$ a face $\mathcal{F}$ of size g, $(g-3)$ vertices ($2(g-3)$ half-chords), $(g-2)$ edges, $(g-3)$ 2-regions, and $(g-2)$ 3-regions. Any 3-region in $Q_{g,t}$ produces in $Q_{g,t+1}$ a face $\mathcal{F}$ of size g, $(g-4)$ vertices, $(g-3)$ edges, $(g-4)$ 2-regions, and $(g-3)$ 3-regions.

We now consider the vector $V_{g,t} = {}^t(n, m, R_2, R_3)$ whose components are respectively the number of vertices, edges, 2-regions, and 3-regions of $Q_{g,t}$. By construction, we have that $V_{g,t+1} = M_g V_{g,t}$, where M_g is the matrix given in Fig. 5.

The limit of the average degree $\text{ad}(Q_{g,t})$ of $Q_{g,t}$ when $t \to \infty$ can be obtained from the unique eigenvector V, given in Fig. 5, associated to the largest eigenvalue $g-3+\sqrt{(g-2)(g-4)}$ of M_g. We thus obtain:

$$\mu_g \geq \lim_{t\to\infty} \text{ad}(Q_{g,t}) = 2 \cdot \frac{g-2+(g-3)\sqrt{(g-2)/(g-4)}}{g-3+\sqrt{(g-2)(g-4)}} = 2\sqrt{\frac{g-2}{g-4}}.$$

Observe that the graphs $Q_{g,t}$ with $t \geq 1$ are circle graphs with girth $g \geq 5$ that contain neither ${}^{\leq}1$-vertices nor chains of $(g-3)$ 2-vertices (see Fig. 4 for an example with $g = 5$), which proves that Theorem 2 is optimal in a certain way. Remark also that for any $g \geq 5$, $Q_{g,t}$ contains K_{t+3} as a minor: $Q_{g,0}$ can be contracted into K_3, and if $Q_{g,t}$ contains a K_{t+3} minor, then the cycle we add to obtain $Q_{g,t+1}$ can be contracted into a dominating vertex, which gives a K_{t+4} minor.

We now prove the upper bound by contradiction. Since circle graphs of girth at least g are closed under taking induced subgraphs, it is sufficient to prove that every circle graph G with girth g at least five has average degree $\text{ad}(G) < 2\sqrt{\frac{g-2}{g-4}}$.

Let G be a circle graph and $\mathcal{C}$ be a circle representation of G. We denote by $R(\mathcal{C})$ the planar graph constructed as follows:

- the vertex set of $R(\mathcal{C})$ is the set of crossings of chords in $\mathcal{C}$,
- two distinct vertices are adjacent in $R(\mathcal{C})$ if and only if they correspond to consecutive crossings of a same chord in $\mathcal{C}$.

Observe that the construction above clearly gives a natural planar embedding of $R(\mathcal{C})$. In the following, we only consider this precise planar embedding. Note that $R(\mathcal{C})$ has maximum degree four.

Let us consider a fixed integer $g \geq 5$ and a circle graph G_1 with girth at least g, such that $\text{ad}(G_1) > 2\sqrt{\frac{g-2}{g-4}}$, and such that G_1 is minimal with this property. That is, for any circle graph H with girth at least g and such that $|V(H)| < |V(G_1)|$, we have

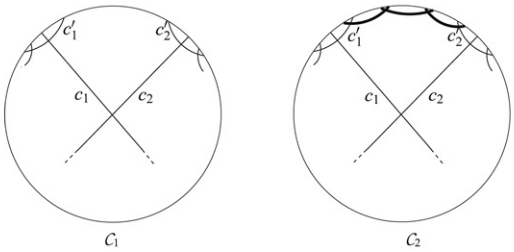

Fig. 6. From $\mathcal{C}_1$ to $\mathcal{C}_2$.

$\mathrm{ad}(H) < 2\sqrt{\frac{g-2}{g-4}}$. Observe that by minimality, G_1 does not contain any $^{\leq}1$-vertex, since otherwise by removing it we would obtain a smaller graph with larger average degree. Note also that G_1 is connected.

Let $\mathcal{C}_1$ be a representation of G_1. If the outerface of the planar embedding of $R(\mathcal{C}_1)$ contains a 4-vertex, we apply the following operation on $\mathcal{C}_1$, which gives a new representation $\mathcal{C}_2$ and a new circle graph G_2 with girth g. Let u denote a 4-vertex on the outerface of $R(\mathcal{C}_1)$. It corresponds to an edge between two vertices v_1 and v_2 of G_1, represented by two crossing chords c_1 and c_2 in $\mathcal{C}_1$. Since u is a 4-vertex in $R(\mathcal{C}_1)$, the chords c_1 and c_2 respectively cross two chords c_1' and c_2' as depicted in Fig. 6. Let v_1' and v_2' be the vertices of G_1 associated to c_1' and c_2'. Since u is on the outerface of $R(\mathcal{C}_1)$, v_1' and v_2' are not adjacent in G_1. Hence, we can add a path of $g-4$ chords between c_1' and c_2', as depicted in Fig. 6. Let $\mathcal{C}_2$ denote the new representation, and G_2 be the associated circle graph. The $g-4$ vertices added to G_1 to obtain G_2 form a cycle of length exactly g in G_2 containing v_1, v_2, v_1', and v_2'. Note that the number of 4-vertices on the outerface of the plane graph associated to the representation decreases by one after at most two iterations of this process.

Let n_1 and m_1 denote respectively the number of vertices and edges of G_1. By Corollary 2, we have that $\mathrm{ad}(G_1) < 2 \cdot \frac{g-3}{g-4}$. This implies that $\mathrm{ad}(G_2) = 2 \cdot \frac{m_1+g-3}{n_1+g-4} > 2 \cdot \frac{m_1}{n_1} = \mathrm{ad}(G_1)$. Thus the average degree increases during this operation.

We repeat this operation until we obtain a circle graph G with girth g having a representation $\mathcal{C}$ such that the outerface of the planar embedding of $R(\mathcal{C})$ does not contain any 4-vertex. The consequence of the previous observation is that $\mathrm{ad}(G) > \mathrm{ad}(G_1) > 2\sqrt{\frac{g-2}{g-4}}$. Let n and m be the number of vertices and edges of G. This implies in particular that:

$$\sqrt{\frac{g-2}{g-4}}n < m. \tag{1}$$

Let N, M, and F denote respectively the number of vertices, edges, and faces of $R(\mathcal{C})$. Since a crossing in $\mathcal{C}$ corresponds to both an edge in G and a vertex in $R(\mathcal{C})$, we have:

$$N = m. \tag{2}$$

We can write Euler's formula for the planar embedding of $R(\mathcal{C})$ as follows:

$$M + 2 = F + N. \tag{3}$$

Let N_d denote the number of d-vertices in $R(\mathcal{C})$. Since G_1 does not contain any $^{\leq}1$-vertex, and no new $^{\leq}1$-vertex is created during the transformation, the graph G does not contain any $^{\leq}1$-vertex either. This implies in particular that $R(\mathcal{C})$ does not contain $^{\leq}1$-vertices. Thus, the degree of a vertex in $R(\mathcal{C})$ is at least 2 and at most 4 and we have:

$$N = N_2 + N_3 + N_4. \tag{4}$$

The sum of vertex degrees is equal to twice the number of edges in $R(\mathcal{C})$:

$$2N_2 + 3N_3 + 4N_4 = 2M. \tag{5}$$

Any chord in a representation of G corresponding to some vertex $v \in G$ contains $(\deg(v) - 1)$ edges of $R(\mathcal{C})$. Since $\sum_{v \in G}(\deg(v) - 1) = 2m - n$, we have:

$$2m - n = M. \tag{6}$$

Note that the outerface of $R(\mathcal{C})$ contains every 2-vertex, every 3-vertex, and no 4-vertex of $R(\mathcal{C})$. Moreover, $R(\mathcal{C})$ cannot contain a face of degree strictly less than g, since otherwise G would contain a cycle of length strictly less than g. We thus obtain a lower bound on the sum of degrees of the faces of $R(\mathcal{C})$, which is equal to twice the number of edges in $R(\mathcal{C})$:

$$g(F-1) + N_2 + N_3 \leq 2M. \tag{7}$$

Let us decompose the chords of $\mathcal{C}$ into blue and red chords as done in the proof of Theorem 2. Using the previous notation, $\mathcal{C}^B$ is the sub-representation of $\mathcal{C}$ induced by the blue chords and G^B is the corresponding circle graph. Note that G^B is a proper induced subgraph of G_1 and G. We thus have:

$$\mathrm{ad}\left(G^B\right) = \frac{2(m - N_2 - N_3)}{n - N_2} < 2\sqrt{\frac{g-2}{g-4}} < \frac{2m}{n} = \mathrm{ad}\left(G\right).$$

This implies that $\frac{2(N_2+N_3)}{N_2} > \frac{2m}{n} > 2\sqrt{\frac{g-2}{g-4}}$, which gives:

$$\left(\sqrt{\frac{g-2}{g-4}} - 1\right) N_2 < N_3. \tag{8}$$

The combination $(g-4) \times (1) + (g-4)\left(2\sqrt{\frac{g-2}{g-4}} - 1\right) \times (2) + g \times (3) + 2(g-2)\left(1 - \sqrt{\frac{g-4}{g-2}}\right) \times (4) + \frac{1}{2}(g-2)\left(1 - \sqrt{\frac{g-4}{g-2}}\right) \times (5) + \sqrt{(g-2)(g-4)} \times (6) + (7) + \frac{1}{2}(g-4)\left(\sqrt{\frac{g-2}{g-4}} - 1\right) \times (8)$ gives $g < 0$, a contradiction.

The idea of the proof is to obtain a contradiction using Euler's Formula applied to the planar graph associated to a representation of the circle graph. Then we used a computer algebra system to ensure that the set of inequalities has no solution and also to find the positive weights appearing in the combination above.

5. Perspectives

In the present paper, we study the structure of sparse circle graphs. The opposite problem of studying the structure of dense circle graphs seems much harder. For example, the relation between the clique number of circle graphs and their chromatic number is not precisely established. Kostochka and Kratochvíl [4] proved that every circle graph with clique number ω has chromatic number at most $2^{\omega+6}$, but this is still far from the lower bound of $\Omega(\omega \log \omega)$.

Note that the upper bound of $2^{\omega+6}$ even holds for polygon-circle graphs, a superclass of circle graphs, defined as the intersection class of chords and convex polygons of the circle. The size of this class is known to be much larger, but we suspect that polygon-circle graphs with girth at least five behave like circle graphs with girth at least five. It would be interesting to see if the results of the present paper extend to the class of polygon-circle graphs.

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