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Oriented cliques and colorings of graphs with low maximum degree

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ABSTRACT

An oriented clique, or oclique, is an oriented graph G such that its oriented chromatic number $\chi_o(G)$ equals its order $|V(G)|$. We disprove a conjecture of Duffy, MacGillivray, and Sopena [Oriented colourings of graphs with maximum degree three and four, Duffy et al. (2019) by showing that for maximum degree 4, the maximum order of an oclique is equal to 12. For maximum degree 5, we prove that the maximum order of an oclique is between 16 and 18. In the same paper, Duffy et al. also proved that the oriented chromatic number of *connected* oriented graphs with maximum degree 3 and 4 is at most 9 and 69, respectively. We improve these results by showing that the oriented chromatic number of *non-necessarily connected* oriented graphs with maximum degree 3 (resp. 4) is at most 9 (resp. 26). The bound of 26 actually follows from a general result which determines properties for a target graph to be universal for graphs of bounded maximum degree. This generalization also allows us to get the upper bound of 90 (resp. 306, 1322) for the oriented chromatic number of graphs with maximum degree 5 (resp. 6, 7).

1. Introduction

Oriented graphs are directed graphs with neither loops nor opposite arcs. Unless otherwise specified, the term *graph* refers to *oriented graph* in the sequel.

For a graph G, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. For two adjacent vertices u and v, we denote by $\overrightarrow{uv}$ the arc from u to v, or simply uv whenever its orientation is not relevant (therefore, $uv = \overrightarrow{uv}$ or $uv = \overrightarrow{vu}$).

Given two graphs G and H, a *homomorphism* from G to H is a mapping $\varphi : V(G) \to V(H)$ that preserves the arcs, that is, $\overrightarrow{\varphi(x)\varphi(y)} \in A(H)$ whenever $\overrightarrow{xy} \in A(G)$.

An *oriented k-coloring* of G can be defined as a homomorphism from G to H, where H is a graph with k vertices. The existence of such a homomorphism from G to H is denoted by $G \to H$. The vertices of H are called *colors*, and we say that G is H-colorable. The *oriented chromatic number* of a graph G, denoted by $\chi_o(G)$, is defined as the smallest number of vertices of a graph H such that $G \to H$. If $\mathcal{F}$ is a family of oriented graphs, then $\chi_o(\mathcal{F})$ denotes the maximum of $\chi_o(G)$ over all $G \in \mathcal{F}$.

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The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors and the problem of bounding the oriented chromatic number has been investigated for various graph classes: outerplanar graphs (with given minimum girth) [18,20], 2-outerplanar graphs [9,16], planar graphs (with given minimum girth) [1–4,14,16,17,19], graphs with bounded maximum average degree [3,4], graphs with bounded degree [7,11,23], graphs with bounded treewidth [15,20,21], Halin graphs [8], and graph subdivisions [25]. A survey on the study of oriented colorings has been done by Sopena in 2001 and recently updated [22].

For bounded degree graphs, Kostochka et al. [11] proved as a general bound that graphs with maximum degree Δ have oriented chromatic number at most $2\Delta^2 2^\Delta$. They also showed that, for every Δ, there exists graphs with maximum degree Δ and oriented chromatic number at least $2^{\Delta/2}$. For low maximum degrees, specific results are only known for graphs with maximum degree 3 and 4. Sopena [20] proved that graphs with maximum degree 3 have an oriented chromatic number at most 16 and conjectured that any such connected graphs have an oriented chromatic number at most 7. The upper bound was later improved by Sopena and Vignal [23] to 11. Recently, Duffy et al. [7] proved that 9 colors are enough for *connected* graphs with maximum degree 3. They proved in the same paper that *connected* graphs with maximum degree 4 have oriented chromatic number at most 69. Lower bounds are given by Duffy et al. [7] who exhibit a graph with maximum degree 3 (resp. 4) and oriented chromatic number 7 (resp. 11). Note that the above-mentioned conjecture of Sopena is thus best possible. In each of the above cases, the lower bound is achieved by presenting an oclique. An *oclique*, or oriented clique, is an oriented graph G such that $\chi_o(G) = |V(G)|$.

Theorem 1 (*[10]*). *An oriented graph is an oclique if and only if any two vertices are connected by a directed path of length* 1 *or* 2.

In their paper, Duffy et al. proved the following upper bound on maximum size of an oclique with maximum degree Δ.

Theorem 2 (*[7]*). *Every oclique with maximum degree Δ has at most $\left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor$ vertices.*

The theorem gives the upper bound 8 (resp. 13, 18) for $\Delta = 3$ (resp. $\Delta = 4$, $\Delta = 5$). They improved the above general result for $\Delta = 3$ by showing that the largest number of vertices in a subcubic oclique is 7. They also prove that there exists an oclique of size 11 with maximum degree $\Delta = 4$. Moreover they conjectured that the maximum order of an oclique with maximum degree $\Delta = 4$ is 11.

In this paper, we first improve the known upper bounds for graphs with low maximum degree. Second, we consider ocliques of maximum degree 4 and 5, and disprove the above-mentioned conjecture of Duffy et al. [7].

We prove in Section 4 that the oriented chromatic number of graphs with maximum degree 3 is at most 9 ($\chi_o(\mathcal{G}_3) \leqslant 9$), that is, we remove the condition of connectivity; see Theorem 8. In Section 5, we prove a general result which determines properties of a target graph to be universal for (non-necessarily connected) graphs of maximum degree $\Delta \geqslant 4$; see Theorem 11. As a consequence of this general result, we obtain that the oriented chromatic number of graphs with maximum degree 4 is at most 26 ($\chi_o(\mathcal{G}_4) \leqslant 26$), substantially decreasing the bound of 69 due to Duffy et al. [7]. We also get that $\chi_o(\mathcal{G}_5) \leqslant 90$, $\chi_o(\mathcal{G}_6) \leqslant 306$, and $\chi_o(\mathcal{G}_7) \leqslant 1322$.

In Section 6, we disprove the conjecture of Duffy et al. [7] by showing that the maximum order of an oclique maximum degree 4 equals 12 ($\chi_o(\mathcal{G}_4) \geqslant 12$). More precisely, we exhibit an oclique of order 12 and maximum degree 4, and show that there is no such oclique of order at least 13. Similarly in Section 7, we exhibit an oclique of order 16 and maximum degree 5 ($\chi_o(\mathcal{G}_5) \geqslant 16$).

The next two sections will be devoted to define the notation and to present the properties of the target graphs we use to prove our upper bounds.

2. Notation

In the remainder of this paper, we use the following notions. For a vertex v of a graph G, we denote by $N_G^+(v)$ the set of outgoing neighbors of v, by $N_G^-(v)$ the set of incoming neighbors of v and by $N_G(v) = N_G^+(v) \cup N_G^-(v)$ the set of neighbors of v (subscripts are omitted when the considered graph is clearly identified from the context). The *degree* of a vertex v (resp. *in-degree*, *out-degree*), denoted by $d(v)$ (resp. $d^-(v)$, $d^+(v)$), is the number of its neighbors $|N(v)|$ (resp. incoming neighbors $|N^-(v)|$, outgoing neighbors $|N^+(v)|$). Let $\mathcal{G}_\Delta$ denote the family of oriented graphs with maximum degree Δ. If two graphs G and H are isomorphic, we denote this by $G \cong H$.

3. Paley tournaments and Tromp digraphs

In this section, we describe the general construction of graphs that will be used to prove Theorems 8 and 11, and present some of their useful properties.

For a prime power $p \equiv 3 \pmod 4$, the *Paley tournament* QR_p is defined as the graph whose vertices are the elements of the field $\mathbb{F}_p$ and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of $\mathbb{F}_p$. Clearly QR_p is vertex- and arc-transitive.

An *orientation n-vector* is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{-1, 1\}^n$ of n elements. An *ordered n-clique* is a sequence $S = (v_1, v_2, \ldots, v_n)$ of n vertices that induce an n-clique of a graph G. The vertex u is said to be an *α-successor of S* if for

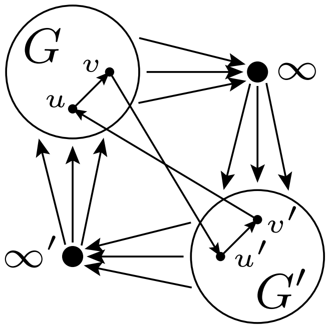


Fig. 1. The Tromp graph $Tr(G)$.

any i, $1 \leqslant i \leqslant n$, we have $\overrightarrow{uv_i} \in A(G)$ whenever $\alpha_i = 1$ and $\overrightarrow{v_iu} \in A(G)$ otherwise. We say that a graph G has *Property* $P_{n,k}$ if, for every ordered n-clique S of G and any orientation n-vector α, there exist k distinct α-successors of S. Such Properties $P_{n,k}$ have been extensively used in many papers dealing with oriented coloring.

Proposition 3. *The Paley tournament QR_p has Properties $P_{1,\frac{p-1}{2}}$ and $P_{2,\frac{p-3}{4}}$.*

Proof. By the vertex-transitivity of QR_p, the in-degree of every vertex is equal to its out-degree. This implies that QR_p has Property $P_{1,\frac{p-1}{2}}$.

Let us prove that QR_p has Property $P_{2,\frac{p-3}{4}}$. To do so, by arc-transitivity of QR_p, we just have to show that there exist at least $\frac{p-3}{4}$ α-successors of the sequence $S = (0, 1)$ for any of the four orientation vector $\alpha \in \{-1, 1\}^2$.

We first need to count the transitive triangles with arcs $\overrightarrow{xy}$, $\overrightarrow{yz}$, and $\overrightarrow{xz}$ in QR_p. There are p choices for the source vertex x of a transitive triangle. The number of transitive triangles such that $x = 0$ is equal to the number of arcs in $N^+(0)$, that is, $\binom{(p-1)/2}{2} = \frac{(p-1)(p-3)}{8}$. Thus, QR_p contains $\frac{p(p-1)(p-3)}{8}$ transitive triangles.

Considering $\alpha = (+1, +1)$, we can notice that $|N^+(0) \cap N^+(1)|$ is the number of transitive triangles such that $\overrightarrow{xy} = \overrightarrow{01}$. Since QR_p is arc-transitive and contains $\frac{p(p-1)}{2}$ arcs, $|N^+(0) \cap N^+(1)| = \frac{p(p-1)(p-3)/8}{p(p-1)/2} = \frac{p-3}{4}$. Similarly for $\alpha = (-1, -1)$ and $\alpha = (+1, -1)$, considering $\overrightarrow{yz} = \overrightarrow{01}$ gives $|N^-(0) \cap N^-(1)| = \frac{p-3}{4}$ and considering $\overrightarrow{xz} = \overrightarrow{01}$ gives $|N^+(0) \cap N^-(1)| = \frac{p-3}{4}$. Finally for $\alpha = (-1, +1)$, we have $N^-(0) \cap N^+(1) = V\left(QR_p\right) \setminus \{0, 1, N^+(0) \cap N^+(1), N^-(0) \cap N^-(1), N^+(0) \cap N^-(1)\}$, so that $|N^-(0) \cap N^+(1)| = p - 2 - \frac{3(p-3)}{4} = \frac{p+1}{4} > \frac{p-3}{4}$. This proves $P_{2,\frac{p-3}{4}}$. $\square$

Paley tournaments will be used as basic brick to build new graphs as explained below. Tromp (unpublished manuscript) proposed the following construction. Let G be a graph and let $G' \cong G$. The Tromp graph $Tr(G)$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$
- $\forall u \in V(G) : \overrightarrow{u\infty}, \overrightarrow{\infty u'}, \overrightarrow{u'\infty'}, \overrightarrow{\infty' u} \in A(Tr(G))$
- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G) : \overrightarrow{uv}, \overrightarrow{u'v'}, \overrightarrow{vu'}, \overrightarrow{v'u} \in A(Tr(G))$

Fig. 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between u and u'. Such pairs of vertices will be called *anti-twin vertices*, and we denote by $\mathrm{at}(u) = u'$ the anti-twin vertex of u.

In the following, we apply Tromp's construction to Paley tournaments QR_p which produces graphs with interesting structural properties. First of all, Marshall [12] proved that any $Tr(QR_p)$ is *vertex-transitive* and *arc-transitive*. He also prove that any $Tr(QR_p)$ is *triangle-transitive*, meaning that, given two triangles $u_1u_2u_3$ and $v_1v_2v_3$ of $Tr(QR_p)$ with the same orientation, there exists an automorphism that maps u_i to v_i. Second, it is possible to derive Properties $P_{n,k}$ for $Tr(QR_p)$ knowing those of QR_p (see Proposition 4). The authors already studied properties of $Tr(QR_{19})$ (see [16, Proposition 5]) and their results can be easily generalized to $Tr(QR_p)$:

Proposition 4 *([16]). If QR_p has Property $P_{n-1,k}$, then $Tr(QR_p)$ has Property $P_{n,k}$.*

Let us now introduce another type of properties. We say that a graph G has *Property* $C_{n,k}$ if for every ordered n-clique $v_1, v_2, \ldots, v_n$ of G, we have $|\bigcup_{1\leqslant i\leqslant n} N^+(v_i)| \geqslant k$ and $|\bigcup_{1\leqslant i\leqslant n} N^-(v_i)| \geqslant k$.

Remark 5. Given two integers n and k, a graph having Property $C_{n,k}$ has Property $C_{n',k'}$ for any n' and k' such that $n' \geqslant n$ and $k' \leqslant k$.

Proposition 6. *The graph $Tr(QR_p)$ has Properties $C_{2,\frac{3p+1}{2}}$ and $C_{3,\frac{7p+3}{4}}$.*

Proof. Recall that $Tr(QR_p)$ is built from two copies of QR_p, that will be named QR_p and QR'_p in the following (see Fig. 1). In this proof, the in- and out-neighborhood of a vertex v of $Tr(QR_p)$ will be denoted by $N^-(v)$ and $N^+(v)$. The in- and out-neighborhood of a vertex v in a subgraph H of $Tr(QR_p)$ will be denoted by $N^-_H(v)$ and $N^+_H(v)$.

Given an ordered n-clique $v_1, v_2, \ldots, v_n$ of $Tr(QR_p)$, if $z \in \bigcup_{1\leqslant i\leqslant n} N^+(v_i)$, then $\mathrm{at}(z) = z' \in \bigcup_{1\leqslant i\leqslant n} N^-(v_i)$. Thus $|\bigcup_{1\leqslant i\leqslant n} N^+(v_i)| = |\bigcup_{1\leqslant i\leqslant n} N^-(v_i)|$.

- Let us first consider Property $C_{2,\frac{3p+1}{2}}$. We have to prove that given two adjacent vertices x and y of $Tr(QR_p)$, we have $|N^+(x) \cup N^+(y)| \geqslant \frac{3p+1}{2}$.
  Since x and y are adjacent, w.l.o.g. $x = 0$ and $y = \infty$ by arc-transitivity of $Tr(QR_p)$. Then $N^+(0) \cup N^+(\infty)$ contains:
  - $N^+(\infty) = \left\{0', 1', \ldots, (p-1)'\right\}$ (p vertices);
  - Note that $N^+(0) = N^+_{QR_p}(0) \uplus N^+_{QR'_p}(0) \uplus \{\infty\}$. Since $N^+_{QR'_p}(0) \subset N^+(\infty)$ is already counted in the previous point, we just consider $N^+_{QR_p}(0)$ (at least $\frac{p-1}{2}$ vertices by Proposition 3) and ∞ (1 vertex);

  So that $N^+(\infty) \cup N^+(0)$ contains at least $p + \frac{p-1}{2} + 1 = \frac{3p+1}{2}$ vertices and thus $Tr(QR_p)$ has Property $C_{2,\frac{3p+1}{2}}$.
- Let us now consider Property $C_{3,\frac{7p+3}{4}}$. We have to prove that given three vertices x, y, and z of $Tr(QR_p)$ that form a triangle, we have $|N^+(x) \cup N^+(y) \cup N^+(z)| \geqslant \frac{7p+3}{4}$.
  We have to consider two cases depending on whether x, y, z form a transitive triangle or x, y, z form a directed triangle. By triangle-transitivity of $Tr(QR_p)$, it suffices to consider the cases $x, y, z = 0, 1, \infty$ (transitive triangle) and $x, y, z = 0, 1', \infty$ (directed triangle).

  Case $x, y, z = 0, 1, \infty$**:** Let $A = N^+(1) \setminus \left\{N^+(0) \cup N^+(\infty)\right\}$. We clearly have $|N^+(0) \cup N^+(1) \cup N^+(\infty)| = |N^+(0) \cup N^+(\infty)| + |A|$. Since we already know that $|N^+(0) \cup N^+(\infty)| = \frac{3p+1}{2}$ (see the previous point), let us focus on the set A. We have $N^+(1) = N^+_{QR_p}(1) \uplus N^+_{QR'_p}(1) \uplus \{\infty\}$. Since $N^+_{QR'_p}(1) \subset N^+(\infty)$ and $\{\infty\} \subset N^+(0)$, we have $A = N^+_{QR_p}(1) \setminus \left\{N^+(0) \cup N^+(\infty)\right\}$. Since vertex ∞ has no out-neighbor in QR_p, we have $A = N^+_{QR_p}(1) \setminus N^+(0)$, that corresponds to the set of out-neighbors of 1 in QR_p which are not out-neighbors of 0. Since QR_p is a tournament, the vertices which are not out-neighbors of a given vertex are the in-neighbors of this vertex. Therefore, the set A corresponds to the set of out-neighbors of 1 in QR_p which are in-neighbors of 0 and thus $A = N^+_{QR_p}(1) \cap N^-(0)$. This set has already been considered in the proof of Proposition 3 where we showed that $|A| = \frac{p+1}{4}$.
  Therefore, the set $N^+(0) \cup N^+(1) \cup N^+(\infty)$ contains $|N^+(0) \cup N^+(\infty)| + |A| \geqslant \frac{3p+1}{2} + \frac{p+1}{4} = \frac{7p+3}{4}$ vertices.

  Case $x, y, z = 0, 1', \infty$**:** Note that $N^+(1') = N^-(1)$. Thus $N^+(0) \cup N^+(1') \cup N^+(\infty) = N^+(0) \cup N^-(1) \cup N^+(\infty)$. Using the same kind of arguments as previous case, we get that $|N^+(0) \cup N^-(1) \cup N^+(\infty)| \geqslant \frac{7p+3}{4}$. □

4. Upper bound of the oriented chromatic number of graphs with maximum degree 3

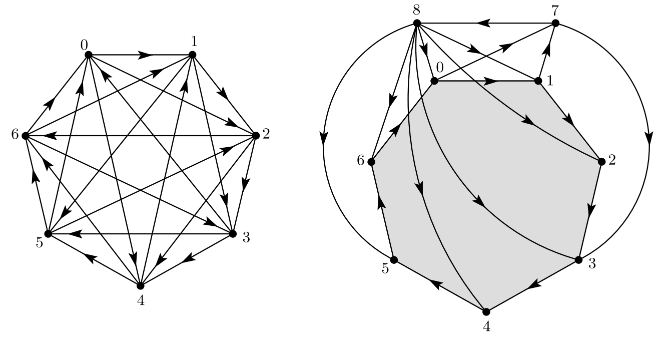
In this section, we consider graphs with maximum degree 3 and we prove that they all admit a homomorphism to the same target graph on nine vertices.

Duffy et al. [7] proved that every *connected* graph with maximum degree 3 has an oriented chromatic number at most 9. To achieve this bound, they use the Paley tournament QR_7 which has vertex set $V(QR_7) = \{0, 1, \ldots, 6\}$ and $\overrightarrow{uv} \in A(QR_7)$ whenever $v - u \equiv r \pmod 7$ for $r \in \{1, 2, 4\}$ (see Fig. 2(a)). Here is a quick sketch of their proof. They first consider 2-degenerated graphs with maximum degree 3 (not necessarily connected) and prove the following:

Theorem 7 (*[7]*)**.** *Every 2-degenerate graph with maximum degree 3 which does not contain a 3-source adjacent to a 3-sink is QR_7-colorable.*

Then, given a *connected* graph G with maximum degree 3, they first consider the case where G contains a 3-source. By removing all 3-sources from G, we obtain a graph G' that is QR_7-colorable by Theorem 7. It is then easy to put back all the 3-sources and color them with a new color 7. Then, subsequently, they consider the case where G does not contain 3-sources. Removing any arc $\overrightarrow{uv}$ from G leads to a graph G' which admits a QR_7-coloring φ by Theorem 7. To extend φ to G, it suffices to recolor u and v with two new colors so that $\varphi(u) = 7$ and $\varphi(v) = 8$. This gives that G has an oriented chromatic number at most 9.

The condition of connectivity of G is a necessary condition in their proof. Indeed, given a graph G with maximum degree 3 which is not connected, we need to remove one arc $\overrightarrow{u_iv_i}$ from each 3-regular component C_i of G to get a 2-degenerate graph G' which is QR_7-colorable by Theorem 7. However, to extend the coloring to G using two new colors, say color 7 for the u_i's and color 8 for the v_i's, the colorings of each component must agree on the neighbors of each u_i's and on the neighbors of each v_i's, which is not necessarily the case. This potentially leads to different target graphs on



(a) QR_7. (b) T_9.

Fig. 2. The oriented graphs QR_7 and T_9.

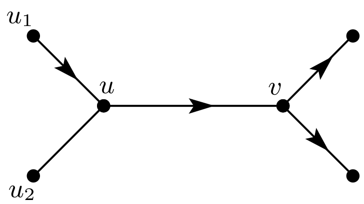


Fig. 3. Configuration of Theorem 8.

nine vertices for each component. Therefore, even if each component has an oriented chromatic number at most 9, the whole graph may have an oriented chromatic number strictly greater than 9.

In the following, we prove this is not the case by showing that it is possible to color each component with the same target graph T_9 on nine vertices whose construction is described below. This implies that the condition of connectivity is no more needed.

The oriented graph T_9 is obtained from QR_7 (see Fig. 2(a)) by adding two vertices labeled 7 and 8, and the arcs $\overrightarrow{07}$, $\overrightarrow{17}$, $\overrightarrow{73}$, $\overrightarrow{78}$, and $\overrightarrow{8i}$ for every $0 \leqslant i \leqslant 6$ (see Fig. 2(b) where the gray part stands for QR_7).

We prove the following:

Theorem 8. *Every graph with maximum degree* 3 *admits a* T_9*-coloring and thus* $\chi_o(\mathcal{G}_3) \leqslant 9$.

Proof. It is sufficient to show that every connected graph G with maximum degree 3 admits a T_9-coloring. We consider the following cases.

- We suppose that G is 2-degenerate or G contains a 3-source. Let G' be the oriented graph obtained from G by removing every 3-source. Since G' is 2-degenerate and contains no 3-source, G' admits a QR_7-coloring φ by Theorem 7. We extend φ to a T_9-coloring of G by setting $\varphi(u) = 8$ for every 3-source u of G (indeed, the vertex 8 of T_9 dominates all the vertices of QR_7).
- We suppose that G is 3-regular and contains no 3-source. Notice that G necessarily contains a vertex v of out-degree two. Let u denote the in-neighbor of v. Since u is not a 3-source, it has an in-neighbor u_1. Let u_2 denote the neighbor of u distinct from u_1 and v (see Fig. 3). We consider the graph G' obtained from G by removing the arc $\overrightarrow{uv}$. Since G' is 2-degenerate and contains no 3-source, G' admits a QR_7-coloring φ by Theorem 7.
  - If G (or equivalently G') contains the arc $\overrightarrow{uu_2}$, then we necessarily have $\varphi(u_1) \neq \varphi(u_2)$. If $\overrightarrow{\varphi(u_1)\varphi(u_2)} \in A(QR_7)$ (resp. $\overrightarrow{\varphi(u_2)\varphi(u_1)} \in A(QR_7)$), we recolor G' so that $\varphi(u_1) = 1$ (resp. $\varphi(u_1) = 0$) and $\varphi(u_2) = 3$ by the arc-transitivity of QR_7. It can be easily checked that we can extend φ to a T_9-coloring of G by setting $\varphi(u) = 7$ and $\varphi(v) = 8$.
  - If G contains $\overrightarrow{u_2u}$, then by the arc-transitivity of QR_7, we can assume that $\{\varphi(u_1), \varphi(u_2)\} \subseteq \{0, 1\}$. Again, φ can be extended to T_9-coloring of G by setting $\varphi(u) = 7$ and $\varphi(v) = 8$. $\square$

5. Upper bound of the oriented chromatic number of graphs with maximum degree at least 4

In this section, we consider graphs with maximum degree at least 4.

Duffy et al. [7] recently proved that every *connected* graph with maximum degree 4 has an oriented chromatic number at most 69. To prove their result, they first consider the case of 3-degenerate graphs with maximum degree 4 and prove that they admit a homomorphism to the Paley tournament QR_{67} on 67 vertices. Then they show how to extend such a 67-coloring to *connected* graphs with maximum degree 4 using two more colors, leading to a 69-coloring.

We propose a general result which determines properties of a target graph to be universal for graphs of maximum degree $\Delta \geqslant 4$. As for graphs with maximum degree 3 (see Section 4), the condition of connectivity is not needed. In particular, our general result substantially decreases the bound of 69 colors for graphs with maximum 4 due to Duffy et al. [7] to 26 colors.

Lemma 9. *Let $Tr(QR_p)$ be a graph with Property $P_{n,k}$. Let φ be a $Tr(QR_p)$-coloring of a graph G. If $u \in V(G)$ has degree at most n, then there exists at least k possible colors for u that leave unchanged $\varphi(v)$ for $v \neq u$. Moreover, these k colors induce a clique of $Tr(QR_p)$.*

Proof. Let us show that the proof can be reduced to the case where $\varphi(N(u))$ induces a clique of $Tr(QR_p)$. Let u_i denote the ith neighbor of u. If $\varphi(u_i) = \varphi(u_j)$ for some $i < j$, then G contains either $\overrightarrow{uu_i}$ and $\overrightarrow{uu_j}$ or $\overrightarrow{u_iu}$ and $\overrightarrow{u_ju}$. Thus, the coloring constraints on u due to the arcs uu_i and uu_j are the same. Similarly, if $\varphi(u_i) = \mathrm{at}(\varphi(u_j))$, then w.l.o.g. G contains $\overrightarrow{u_iu}$ and $\overrightarrow{uu_j}$. Thus, the coloring constraints on u due to the arcs $\overrightarrow{u_iu}$ and $\overrightarrow{uu_j}$ are the same. In both cases, the coloring constraints on u are unchanged by removing the arc uu_j.

So we can assume that $\varphi(u_i) \notin \{\varphi(u_j), \mathrm{at}(\varphi(u_j))\}$ for every $1 \leqslant i < j \leqslant d(u)$. Thus, $\varphi(N(u))$ induces a clique of $Tr(QR_p)$ of size $d(u)$ and the result holds by definition of the Property $P_{n,k}$. $\square$

Theorem 10. *Let $\Delta \geqslant 3$. If $Tr(QR_p)$ has Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,n-\frac{n-1}{\Delta-1}}$, where $n = |Tr(QR_p)| = 2p + 2$, then every $(\Delta - 1)$-degenerate graph with maximum degree Δ admits a $Tr(QR_p)$-coloring.*

Proof. Let G be a minimal counter-example to Theorem 10. By definition, G contains a k-vertex u with $k \leqslant \Delta - 1$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of u.

Suppose first that the neighborhood of u contains an arc and assume w.l.o.g. that this arc is $\overrightarrow{v_1v_2}$. The graph G' obtained from G by removing the arc uv_1 admits a $Tr(QR_p)$-coloring φ by minimality of G. The degree of v_1 is at most $\Delta - 1$ in G', so by Lemma 9 and Property $P_{\Delta-1,\Delta-2}$, there exists a set S of $\Delta-2$ available colors for v_1. Since S is a clique, S cannot contain both a color c and $\mathrm{at}(c)$. Thus, we can assign to v_1 a color in S that is distinct from $\varphi(v_i)$ and $\mathrm{at}(\varphi(v_i))$ for every $3 \leqslant i \leqslant k$ because there are at most $\Delta - 3$ values of i and $|S| \geqslant \Delta - 2$. Note that we necessarily have $\{\varphi(v_2), \mathrm{at}(\varphi(v_2))\} \cap S = \emptyset$ since $\overrightarrow{v_1v_2}$ is an arc of G'.

Now φ is such that $\varphi(v_1) \notin \{\varphi(v_i), \mathrm{at}(\varphi(v_i))\}$ for every $2 \leqslant i \leqslant k$. Using arguments along the lines of the proof of Lemma 9, φ can be extended to G by recoloring u using Property $P_{\Delta-1,\Delta-2}$.

Assume now that the neighborhood of u contains no arc. The graph G' obtained from G by removing the vertex u admits a $Tr(QR_p)$-coloring φ by minimality of G. The degree of v_1 in G' is at most $\Delta - 1$. By Lemma 9 and Property $P_{\Delta-1,\Delta-2}$, we have $\Delta - 2$ possible colors for v_1 and these $\Delta - 2$ colors induce a clique. By Property $C_{\Delta-2,n-\frac{n-1}{\Delta-1}}$, given these $\Delta - 2$ possible colors for v_1, there are at least $n - \frac{n-1}{\Delta-1}$ choices of colors for u. Thus v_1 forbids at most $\frac{n-1}{\Delta-1}$ colors for u.

Since the neighborhood of u contains no arc, the previous arguments hold for each v_i independently. That is, every v_i forbids at most $\frac{n-1}{\Delta-1}$ colors for u. Therefore, at most $k\frac{n-1}{\Delta-1} \leqslant n - 1$ colors are forbidden. So there exists at least one available color for u and φ can be extended to a $Tr(QR_p)$-coloring of G, a contradiction. $\square$

To achieve our bounds on oriented chromatic number, we construct the graph $Tr^*(QR_p)$ on $2p + 4$ vertices by adding two vertices t_0 and t_1 such that t_0 is a twin vertex of vertex 0 (*i.e.* a vertex with the same neighborhood as vertex 0) and t_1 is a twin vertex of vertex 1. We finally add the arc $\overrightarrow{t_1t_0}$.

Theorem 11. *Every graph with maximum degree $\Delta \geqslant 4$ admits a $Tr^*(QR_p)$-coloring where $Tr(QR_p)$ is a Tromp graph with Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,2p+2-\frac{2p+1}{\Delta-1}}$.*

Proof. Since $Tr(QR_p)$ is a subgraph of $Tr^*(QR_p)$, it remains to prove that every connected Δ-regular graph H admits a $Tr^*(QR_p)$-coloring.

Let $H' = H \setminus \{\overrightarrow{uv}\}$ where $\overrightarrow{uv}$ is any arc of H. By Theorem 10, H' admits a $Tr(QR_p)$-coloring φ. By vertex-transitivity of $Tr(QR_p)$, we may assume that $\varphi(v) = 0$. By Property $P_{\Delta-1,\Delta-2}$ of $Tr(QR_p)$, we have $\Delta - 2$ available colors $c_1, c_2, \ldots, c_{\Delta-2}$ for u. Note that, given $1 \leqslant i < j \leqslant \Delta - 2$, we necessarily have $c_i \neq \mathrm{at}(c_j)$. We thus recolor u with one of the c_i's so that $\varphi(u) \notin \{0, 0'\}$ since $\Delta \geqslant 4$. Therefore, $\varphi(u)\varphi(v)$ is an arc of $Tr(QR_p)$.

If $\overrightarrow{\varphi(u)\varphi(v)}$ is an arc of $Tr(QR_p)$, then φ is a $Tr(QR_p)$-coloring of H and thus a $Tr^*(QR_p)$-coloring of H. Therefore, $\overrightarrow{\varphi(v)\varphi(u)}$ is an arc of $Tr(QR_p)$. By arc-transitivity of $Tr(QR_p)$, we may assume that $\varphi(u) = 1$. We recolor u and v so that $\varphi(u) = t_1$ and $\varphi(v) = t_0$. Since each t_i is a twin vertex of vertex i in $Tr^*(QR_p)$ and $\overrightarrow{t_1t_0}$ is an arc $Tr^*(QR_p)$, it is easy to verify that φ is now a $Tr^*(QR_p)$-coloring of H. $\square$

The Properties $P_{n,k}$ of Paley tournaments QR_p can be expressed by a formula for $n \leqslant 2$ (see Proposition 3). For higher values of n, there is no known formula. The Properties $P_{n,k}$ of QR_p that we have computed for small values of p and

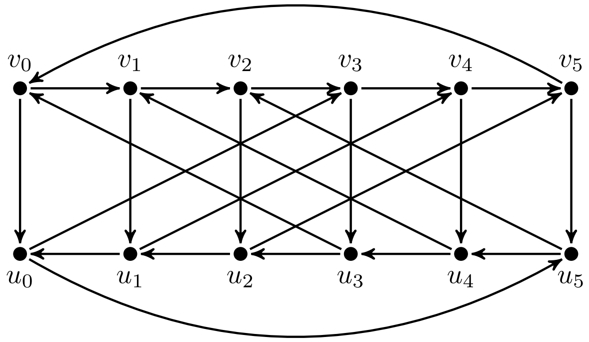


Fig. 4. 4-regular oclique on 12 vertices.

the program that we used can be found at http://www.lirmm.fr/~ochem/target/. For our purpose, we are interested in Properties of the form $P_{n,n}$.

Proposition 12. *The smallest Paley tournament with Property $P_{2,2}$ (resp. $P_{3,3}$, $P_{4,4}$, $P_{5,5}$) is QR_{11} (resp. QR_{43}, QR_{151}, QR_{659}).*

Determining more of such properties would be possible with more computing power and time. Note that Paley tournament QR_{151} has in fact Property $P_{5,6}$ and there exist no smaller Paley tournament with Property $P_{5,5}$.

As a corollary of Propositions 4 and 12, we get the following.

Corollary 13. *The Tromp graph $Tr(QR_{11})$ (resp. $Tr(QR_{43})$, $Tr(QR_{151})$, $Tr(QR_{659})$) has Property $P_{3,2}$ (resp. $P_{4,3}$, $P_{5,4}$, $P_{6,5}$).*

As corollaries of Remark 5, Proposition 6, Theorem 11 and Corollary 13, we obtain the following four results. We give a proof of the last one (Corollary 17), and the other three corollaries follow the same arguments.

Corollary 14. *Every graph $G \in \mathcal{G}_4$ admits a $Tr^*(QR_{11})$-coloring. Thus, $\chi_o(\mathcal{G}_4) \leqslant 26$.*

Corollary 15. *Every graph $G \in \mathcal{G}_5$ admits a $Tr^*(QR_{43})$-coloring. Thus, $\chi_o(\mathcal{G}_5) \leqslant 90$.*

Corollary 16. *Every graph $G \in \mathcal{G}_6$ admits a $Tr^*(QR_{151})$-coloring. Thus, $\chi_o(\mathcal{G}_6) \leqslant 306$.*

Corollary 17. *Every graph $G \in \mathcal{G}_7$ admits a $Tr^*(QR_{659})$-coloring. Thus, $\chi_o(\mathcal{G}_7) \leqslant 1322$.*

Proof. Let $\Delta = 7$. By Proposition 6, the graph $Tr(QR_{659})$ has Property $C_{3,\frac{7p+3}{4}} = C_{3,1154}$. By Remark 5, it thus has Property $C_{5,1101} = C_{\Delta-2,2p+2-\frac{2p+1}{\Delta-1}}$. By Corollary 13, it also has Property $P_{6,5} = P_{\Delta-1,\Delta-2}$. The graph $Tr(QR_{659})$ verifies the hypothesis of Theorem 11, and thus every graph with maximum degree 7 admits a $Tr^*(QR_{659})$-coloring. □

6. Oclique with maximum degree 4

Lemma 18. *There is an oclique of order 12 with maximum degree $\Delta = 4$.*

Proof. Consider the oriented graph presented on Fig. 4. The graph consists of two directed cycles: $(v_0, v_1, v_2, v_3, v_4, v_5)$ and $(u_5, u_4, u_3, u_2, u_1, u_0)$. These two cycles are connected by the arcs $\overrightarrow{v_iu_i}$ and $\overrightarrow{u_{(i+3) \bmod 6}v_i}$ for every $0 \leqslant i \leqslant 5$. The graph is vertex-transitive. By Theorem 1, it is enough to find a directed 2-path between v_0 and every vertex not adjacent to v_0: $v_0 \to v_1 \to v_2$, $v_0 \to u_0 \to v_3$, $v_4 \to v_5 \to v_0$, $v_0 \to v_1 \to u_1$, $u_2 \to v_5 \to v_0$, $u_4 \to u_3 \to v_0$, $v_0 \to u_0 \to u_5$. □

Lemma 19. *Suppose that G is an oclique of maximum degree $\Delta \geqslant 2$ and order*

$$n = \left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor.$$

Then:

- *Every two vertices are connected by only one directed path of length at most 2.*
- *For every vertex v:*
  - *either $deg_{in}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $deg_{out}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$,*
  - *or $deg_{out}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $deg_{in}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$.*

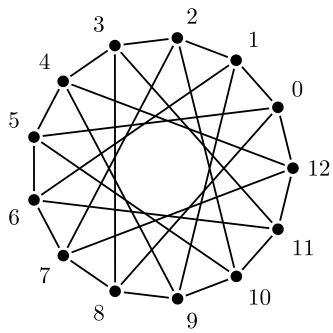


Fig. 5. Unique triangle-free 4-regular graph with 13 vertices and diameter 2.

Proof. The number of directed paths of length 1 is at most $\frac{\Delta \cdot n}{2}$. Each vertex $v \in V(G)$ is the midpoint of $deg_{in}(v) \cdot deg_{out}(v)$ directed paths of length 2, so we have

$$\sum_{v \in V(G)} deg_{in}(v) \cdot deg_{out}(v) \tag{$\star$}$$

directed paths of length 2 in the graph.

First, suppose that Δ is even. Then $n = \frac{(\Delta+1)^2+1}{2}$ and the sum ($\star$) achieves the maximum value of $\frac{\Delta^2 \cdot n}{4}$ if and only if $deg_{in}(v) = deg_{out}(v) = \frac{\Delta}{2}$ for every vertex v. Hence, we have at most $\frac{\Delta \cdot n}{2} + \frac{\Delta^2 \cdot n}{4}$ directed paths of length at most 2, and this maximal value is equal to $\binom{n}{2}$, that is the number of all pairs of vertices. Hence, if every pair of vertices is connected by a path of length 1 or two, then every vertex has in-degree and out-degree equal to $\frac{\Delta}{2}$, and every two vertices are connected by only one directed path of length at most 2.

If Δ is odd, then $n = \frac{(\Delta+1)^2}{2}$ and the sum ($\star$) achieves the maximum value of $\frac{(\Delta^2-1) \cdot n}{4}$ if and only if for every vertex v,

- either $deg_{in}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $deg_{out}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$,
- or $deg_{out}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $deg_{in}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$.

Hence, there are at most $\frac{\Delta \cdot n}{2} + \frac{(\Delta^2-1) \cdot n}{4}$ directed paths of length at most 2. Again, this maximal value is equal to $\binom{n}{2}$. □

Corollary 20. *The underlying graph of oclique on $n = \left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor$ vertices and maximum degree Δ is triangle free.*

Now, we shall prove that there is no oclique with maximum degree $\Delta = 4$ and order $n = \left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor = 13$. In [13] authors described a method to generate regular graphs of given girth (length of shortest cycle). Using this generator we can generate 4-regular graphs on 13 vertices with girth at least 4 (triangle-free).[2] Only one of these graphs has diameter 2, namely, the graph $G^{13} = (\mathbb{Z}_{13}, E)$, where $(u, v) \in E \iff (v - u) \in \{-1, 1, -5, 5\}$, see Fig. 5. All arithmetic in G^{13} is made modulo 13. Hence, only an orientation of G^{13} can be an oclique with $\Delta = 4$ and $n = 13$.

Lemma 21. *No orientation of G^{13} is an oclique.*

Proof. Suppose, for a contradiction, that there is an orientation of G^{13} which is an oclique. By Lemma 19, every vertex of this orientation has in-degree and out-degree equal to 2, and every two vertices are connected by only one directed path of length at most 2. Vertex 0 is connected to vertex 2 only by one path of length at most 2, namely 0–1–2. Without loss of generality assume that those edges are oriented: $0 \to 1 \to 2$. Furthermore, there is only one path going from 1 to 3, namely the one going through vertex 2. Hence, there is the arc $2 \to 3$ and so on, we show that there is the cycle $0 \to 1 \to 2 \to \cdots \to 12 \to 0$. Edges which are not oriented so far also form the cycle 0, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8.

Similarly as above also this cycle must be oriented in one direction. Hence, either every edge of the cycle goes from x to $x + 5$, or every edges of the cycle goes from $x + 5$ to x. However, these two cases are isomorphic via the mapping $x \to 8 \cdot x$, so we only consider the former orientation. Then we have two directed paths between 0 and 6, namely via 1 or via 5. A contradiction. □

Thus, we have proved the following theorem:

Theorem 22. *The maximum order of an oclique with maximum degree $\Delta = 4$ is 12.*

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[2] There are 31 such graphs which can be found at http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html.

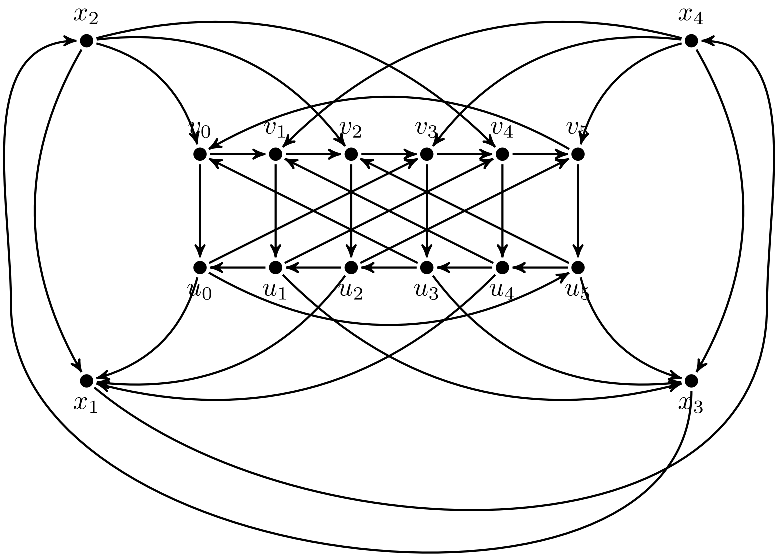

Fig. 6. 5-regular oclique on 16 vertices.

Corollary 23. $\chi_o(\mathcal{G}_4) \geq 12$.

7. Oclique with maximum degree 5

Lemma 24. *There is an oclique with maximum degree* 5 *and order* 16.

Proof. Consider the graph presented in Fig. 6. The graph is constructed from the oriented 4-regular oclique G on 12 vertices presented in Fig. 4 by adding four vertices x_1, x_2, x_3, x_4. These vertices form the cycle $x_1 \to x_4 \to x_3 \to x_2 \to x_1$ and there are arcs:

- $\overrightarrow{x_2v_0}$, $\overrightarrow{x_2v_2}$, $\overrightarrow{x_2v_4}$,
- $\overrightarrow{x_4v_1}$, $\overrightarrow{x_4v_3}$, $\overrightarrow{x_4v_5}$,
- $\overrightarrow{u_0x_1}$, $\overrightarrow{u_2x_1}$, $\overrightarrow{u_4x_1}$,
- $\overrightarrow{u_1x_3}$, $\overrightarrow{u_3x_3}$, $\overrightarrow{u_5x_3}$.

We will show that any two vertices in G are connected by a path of length at most 2. Since new vertices form the oriented cycle C_4, there are paths between these vertices. It is easy to find all other directed paths. For example, paths connecting x_2 and all other vertices are as follows:

- $x_2 \to v_i$ for $i \in \{0, 2, 4\}$,
- $x_2 \to v_i \to v_{i+1 \bmod 6}$ for $i \in \{1, 3, 5\}$,
- $x_2 \to v_i \to u_i$ for $i \in \{0, 2, 4\}$,
- $u_i \to x_3 \to x_2$ for $i \in \{1, 3, 5\}$. □

From Lemma 24 and Theorem 2 we have the following:

Corollary 25. *The maximum order of oclique with maximum degree* 5 *is between* 16 *and* 18.

Corollary 26. $\chi_o(\mathcal{G}_5) \geqslant 16$.

8. Discussion

Our attempts to construct an oclique based on the construction of the oriented 4-regular oclique on 12 vertices (see Fig. 4) lead to a linear estimation in terms of Δ. However, we can obtain a lower bound confirming that the maximum order of oclique is quadratic.

Lemma 27. *For every Δ, there exists an oclique of maximum degree Δ and order* $\frac{\Delta^2}{7} + O(\Delta)$.

Proof. The postage stamp problem[3] asks, given integers h and k, for a set of integers $V = \{v_1, v_2, \dots, v_k\}$ that maximizes the smallest integer $PSP(h, k)$ that cannot be written as the sum of at most h (not necessarily distinct) elements of V. In the context of ocliques, we are interested in the case of $h = 2$ stamps, which corresponds to directed paths of length at most 2. Using a set V corresponding to the value of $PSP(2, k)$, we construct the circulant oriented graph G_k on $n = 2PSP(2, k) - 1$ vertices $g_0, \dots, g_{n-1}$ and the arcs $\overrightarrow{g_i g_{i+v_j}}$ such that $0 \leqslant i < n$, $1 \leqslant j \leqslant k$, and indices are taken modulo n. By the properties of V, there exists a directed path of length at most 2 in G_k from g_0 to every vertex g_i with $1 \leqslant i \leqslant PSP(2, k) - 1$. Now since G_k is circular, G_k is an oclique. Also, G_k is Δ-regular with $\Delta = 2k$. We use the bound $PSP(2, k) \geqslant \frac{2}{7}k^2 + O(k)$ [24] to obtain $|V(G_k)| = 2PSP(2, k) - 1 \geqslant 2 \times \frac{2}{7}k^2 + O(k) = \frac{\Delta^2}{7} + O(\Delta)$. □

Finally, Duffy proves in a recent preprint [6] that the oriented chromatic number of a connected cubic graph is at most 8.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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3 https://en.wikipedia.org/wiki/Postage_stamp_problem.