A complexity dichotomy for the coloring of sparse graphs[*]

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Abstract

Gallucio, Goddyn and Hell proved in 2001 that in any minor-closed class of graphs, graphs with large enough girth have a homomorphism to any given odd cycle. In this paper, we study the computational aspects of this problem. Let $\mathcal{F}$ be a monotone class of graphs containing all planar graphs, and closed under clique-sum of order at most two. Examples of such class include minor-closed classes containing all planar graphs, and such that all minimal obstructions are 3-connected. We prove that for any k and g, either every graph of girth at least g in $\mathcal{F}$ has a homomorphism to C_{2k+1}, or deciding whether a graph of girth g in $\mathcal{F}$ has a homomorphism to C_{2k+1} is NP-complete.

We also show that the same dichotomy occurs when considering 3-colorability or acyclic 3-colorability of graphs under various notions of density that are related to a question of Havel (1969) and a conjecture of Steinberg (1976) about the 3-colorability of sparse planar graphs.

1   Introduction

Jaeger conjectured in 1988 [23] that for any $k \geq 1$, the edges of any $4k$-edge-connected graph can be oriented in the such way that for each vertex v, $d^+(v) \equiv d^-(v) \pmod{2k+1}$, where $d^-(v)$ and $d^+(v)$ denote the in- and out-degree of the vertex v. This conjecture is equivalent to Tutte's 3-flow conjecture [38] for $k = 1$ and implies Tutte's 5-flow conjecture [37] for $k = 2$. Restricted to planar graphs, Jaeger's conjecture is equivalent to the following statement:

Conjecture 1 (Jaeger, 1988) *For any $k \geq 1$, every planar graph of girth at least $4k$ has a homomorphism to C_{2k+1}.*

The case $k = 1$ is equivalent to the fact that triangle-free planar graphs are 3-colorable, proved by Grötzsch in 1959 [19], and the remaining cases are open. This result of Grötzsch led several researchers to investigate other (simple) sufficient conditions for a planar graph to be 3-colorable. The following question and conjecture are two different approaches in this direction.

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Problem 2 (Havel, 1969) *Is there a constant i such that every planar graph without triangles at distance less than i apart is 3-colorable?*

Conjecture 3 (Steinberg, 1976) *Every planar graph without cycles of length four and five is 3-colorable.*

While a positive answer to Havel's problem was recently announced [13] (with a very large constant), Steinberg's conjecture is still open. Erdős suggested the following relaxation of this conjecture: Does there exist a constant C such that the absence in a planar graph of cycles of length 4 to C guarantees its 3-colorability? Abbott and Zhou [1] proved in 1991 that such a C exists and $C \leq 11$. This result was then sequentially improved [3, 4, 34, 33], until Borodin *et al.* proved in 2005 that every planar graph without cycles of length 4 to 7 is 3-colorable [5].

All these problems have the same flavour: they suggest that for various types of coloring, every planar graph with sufficiently low density can be colored. Here, *density* should be understood as a broad notion, depending highly on the problem considered (and which does not necessarily coincide with the technical definition of density). For instance, in the case of Havel's problem, the density would be correlated with the minimum distance between two triangles: the larger the distance, the lower the density.

Consider the following problem: *does a graph G with maximum degree at most Δ admit a proper k-coloring?* If $k \leq \Delta - 1$, then the problem is NP-complete; if $k = \Delta$, then the problem can be solved in polynomial time using Brooks Theorem (but the answer is not always positive); and if $k \geq \Delta + 1$, then the answer is always positive. Our aim in this paper is to prove that in each of the problems mentioned above (Havel, Steinberg, Jaeger, and a couple others), this typically does not happen: the density threshold below which every planar graph becomes colorable is also a complexity threshold. We will show that in each of the questions we consider, by decreasing the density the decision problem drops directly from NP-complete to true always.

As was pointed out by a referee, this kind of complexity jump appears in different contexts. Let (k, s)-SAT denote the Boolean satisfiability problem restricted to instances with exactly k variables per clause, and at most s occurrences per variable. Tovey [36] proved that $(3, 4)$-SAT is NP-complete, while $(3, 3)$-SAT is trivial (every instance is satisfiable). This was generalized by Kratochvíl, Savický, and Tuza [25], who proved the existence of a function f (of exponential order) such that for any $k \geq 3$, $(k, f(k) + 1)$-SAT is NP-complete, while $(k, f(k))$-SAT is trivial. The reader is referred to [17] for a more detailed discussion about these results, together with the precise asymptotics for the function f (which is closely related to functions appearing in the Lovász Local Lemma).

Structure of the paper In Section 3, we consider the 3-Color Problem. Using the main result of [13], we prove that there exists an integer $d \geq 4$ such that every planar graph without triangles at distance less than d is 3-colorable, but deciding whether a planar graph without triangles at distance less than $d - 1$ is 3-colorable is NP-complete. We also show, using [5], that there exists an integer $i \in [5, 7]$ such that every planar graph without cycles of length 4 to i is 3-colorable, but deciding whether a planar graph without cycles of length 4 to $i - 1$ is 3-colorable is NP-complete (if Conjecture 3 is true, then i is precisely 5). The reductions in the proofs of these two results are fairly easy, and can be considered as an introduction to the reductions of the next sections.

In Section 4, we consider $(1,0)$-coloring of planar graphs. A graph is $(1,0)$-*colorable* if its vertex set can be partitioned into a stable set and a set inducing a graph with maximum degree at most one. Glebov and Zambalaeva proved that every planar graph with girth at least 16 is $(1,0)$-colorable [18]. The value 16 was later decreased to 14 by Borodin and Ivanova [6], and very recently to 12 by Borodin and Kostochka [9]. We prove that for any integers $d \geq 3$ and $g \geq 6$, either every planar graph of girth at least g and maximum degree at most d is $(1,0)$-colorable, or deciding whether a planar graph of girth at least g and maximum degree at most d is $(1,0)$-colorable is NP-complete. We then improve some known constructions of non-$(1,0)$-colorable sparse planar graphs.

In Section 5, we study the acyclic coloring of planar graphs. A graph is *acyclically k-colorable* if it admits a proper k-coloring in which every cycle contains at least three colors. A celebrated result of Borodin is that planar graphs are acyclically 5-colorable. He also proved that planar graphs with girth at least seven are 3-colorable [11]. Using this result, we show that there exists $g \in [5,7]$, such that every planar graph of girth at least g is acyclically 3-colorable, but deciding whether a planar graph of girth at least $g-1$ is acyclically 3-colorable is NP-complete.

In Section 6, we investigate homomophisms to odd cycles. A *homomorphism* from a graph G to a graph H is a mapping $h : V(G) \to V(H)$ that preserves the edges, i.e. if uv is an edge of G, then $h(u)h(v)$ is an edge of H. Using [10], we show that for every $k \geq 2$ there exists an integer $g = g(k)$, with $4k \leq g \leq \frac{1}{3}(20k - 2)$, such that every planar graph of girth at least g has a homomorphism to C_{2k+1}, but determining whether a planar graph of girth $g-1$ has a homomorphism to C_{2k+1} is NP-complete. If Conjecture 1 is true, then $g(k) = 4k$.

For the sake of clarity, all the results mentioned above are stated in the context of planar graphs. We will indeed prove that the dichotomy results still hold if instead of planar graphs we consider any monotone family $\mathcal{F}$ containing all planar graphs, and closed under small clique-sum. As will be proved in the next section, examples of such classes include K_n-minor free graphs ($n \geq 5$), graphs with no subdivision of K_n ($n \geq 5$), and graphs with Colin de Verdière parameter at most k, for some $k \geq 3$ (for instance, linklessly embeddable graphs).

2   Nice classes

For $k \geq 0$, a graph obtained from the disjoint union of two graphs G_1 and G_2 by identifying a k-clique of G_1 with a k-clique of G_2 is called a k-*clique-sum* of G_1 and G_2. A *small clique-sum* is a k-clique-sum with $0 \leq k \leq 2$.

A class of graphs containing all planar graphs is *nice* if it is closed under subgraphs and small clique-sums. The purpose of this section is to identify several important graph classes fitting this description. It is easy to remark that for any $k \geq 4$, the class of graphs with chromatic number at most k is nice (using the Four Color Theorem, such a class contains all planar graphs). Similarly, using a famous result of Borodin, it follows that for any $k \geq 5$, the class of graphs with acyclic chromatic number at most k is a nice class.

One of our main motivations is the result of Gallucio *et al.* [15] stating that in minor-closed families, high-girth graphs are almost bipartite. Hence, it is crucial to understand which minor-closed classes are nice.

Lemma 4 *A minor-closed class $\mathcal{F}$ containing all planar graphs is nice if and only if all minimal forbidden minors of $\mathcal{F}$ are 3-connected.*

PROOF.   Assume first that all minimal forbidden minors of $\mathcal{F}$ are 3-connected, and let $G_1, G_2 \in \mathcal{F}$ be such that a small clique-sum G of G_1 and G_2 is not in $\mathcal{F}$. Consider a minor-minimal minor H of G that is not in $\mathcal{F}$. Since $G_1, G_2 \in \mathcal{F}$, H is neither a minor of G_1, nor a minor of G_2. Hence, H is not 3-connected, a contradiction.

Assume now that $\mathcal{F}$ is nice. Let H be a minor-minimal graph such that $H \notin \mathcal{F}$. If H contains a clique-cutset of size at most two (in particular, H can be disconnected), then it is a small clique-sum of graphs from $\mathcal{F}$ (since H is minor-minimal), so H must be in $\mathcal{F}$, a contradiction. So H contains no clique-cutset of size at most two, which implies that H is 2-connected. Assume now that H contains two non-adjacent vertices u, v whose removal disconnects H, and let $H_1, \ldots, H_k$ be the graphs induced by each component together with u and v. Observe that $H + uv$ is the clique-sum of the graphs $(H_i + uv)_{1 \le i \le k}$ on a clique of size two, and all the H_i's are minors of H. Since H is minor-minimal, all the H_i's are in $\mathcal{F}$ and then $H + uv$ and H are also in $\mathcal{F}$, a contradiction. It follows that H is 3-connected. $\square$

A minor-monotone graph invariant, usually denoted by μ, was introduced by Colin de Verdière in 1990 [12]. It relates to the maximal multiplicity of the second largest eigenvalue of the adjacency matrix of a graph, in which the diagonal entries can take any positive value and the entries corresponding to edges can take any nonnegative values (a technical assumption, called the Strong Arnold Property, has to be added to avoid degenerate cases, but we omit the details). For a graph invariant f and an integer ℓ, let $\mathcal{F}(f, \ell)$ be the set of graphs G with $f(G) \le \ell$. It was proved by Colin de Verdière that $\mathcal{F}(\mu, 1)$ is the set of linear forests, $\mathcal{F}(\mu, 2)$ is the set of outerplanar graphs, and $\mathcal{F}(\mu, 3)$ is the set of planar graphs. It follows from [27] and [32] that $\mu(G) \le 4$ if and only if G is linklessly embeddable in $\mathbb{R}^3$. A direct consequence of a result of van der Holst et al. [22] is that for any $\ell \ge 3$, $\mathcal{F}(\mu, \ell)$ is closed under small clique-sums. Since $\mathcal{F}(\mu, 3)$ is the set of planar graphs, we have that for any $\ell \ge 3$, $\mathcal{F}(\mu, \ell)$ is a nice class.

Following the introduction of Colin de Verdière's parameter, van der Holst et al. [21] defined a new minor-monotone parameter, called λ, and proved that λ is preserved by the clique-sum operation. They also proved that $\mathcal{F}(\lambda, 1)$ is the set of forests, $\mathcal{F}(\lambda, 2)$ is the set of K_4-minor free graphs, and $\mathcal{F}(\lambda, 3)$ is the set of graphs that can be obtained from planar graphs by taking clique-sums and subgraphs. As previously, this shows that for any $\ell \ge 3$, the class $\mathcal{F}(\lambda, \ell)$ is a nice class. The definition of λ was then extended by Edmonds et al. [14] to a new minor-monotone graph parameter λ', satisfying $\lambda'(G) \ge \lambda(G)$ for any graph G, and having the same properties as the properties of λ mentioned above. It is not known whether there exists a graph G such that $\lambda'(G) < \lambda(G)$, so these two parameters might very well be equal. Anyway, again, we have that for any $\ell \ge 3$, the class $\mathcal{F}(\lambda', \ell)$ is a nice class.

Observe that the proof of Lemma 4 is still valid if *minor* is replaced by *topological minor* in the statement of the lemma. Hence, for any $n \ge 5$, the class of graphs with no subdivision of K_n is nice.

3   The 3-Color Problem

For two graphs G and $H(u, v)$, where u and v are two distinct vertices of $H(u, v)$, we denote by $G \oplus H(u, v)$ a graph constructed as follows: For every vertex x of G, we take $d_G(x) - 1$ copies $H_1^x(u, v), \cdots, H_{d_G(x)-1}^x(u, v)$ of $H(u, v)$ and identify the vertex v of $H_i^x(u, v)$ with the vertex u of H_{i+1}^x for $1 \le i \le d_G(x) - 2$. The vertices u of $H_i^x(u, v)$ for $1 \le i \le d_G(x) - 1$
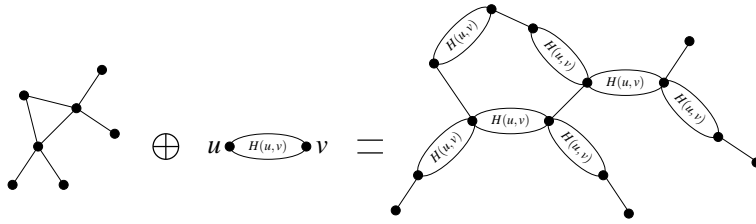
Figure 1: An example of construction of $G \oplus H(u, v)$.

together with the vertex v of $H^x_{d_G(x)-1}(u, v)$ are called the *duplicates* of x. For every edge xy in G, we add an edge between one duplicate of x and one duplicate of y. An example of construction of $G \oplus H(u, v)$ is depicted in Figure 1. By the definition of a nice class, if G is planar and $H(u, v) + uv$ is in a nice class $\mathcal{F}$, then we can choose $G \oplus H(u, v)$ to be also in $\mathcal{F}$.

For an integer $i \geq 4$, let $\mathcal{C}_i$ denote the class of graphs with no cycle of length 4 to i.

Theorem 5 *For every nice class $\mathcal{F}$ and integer $i \geq 4$, either every graph in $\mathcal{F} \cap \mathcal{C}_i$ is 3-colorable, or deciding whether a graph in $\mathcal{F} \cap \mathcal{C}_i$ is 3-colorable is NP-complete.*

PROOF. Suppose there exist non 3-colorable graphs in $\mathcal{F} \cap \mathcal{C}_i$, and consider such a graph H' that is minimal with respect to the number of edges. Let $H^*(u, v)$ be the graph obtained from H' by removing the edge uv. The graph $H^*(u, v)$ is thus 3-colorable. Moreover, every 3-coloring of $H^*(u, v)$ is such that u and v have the same color, otherwise H' would be 3-colorable.

We take $\ell = \lfloor i/3 \rfloor$ copies $(H^*_t(u, v))_{1 \leq t \leq \ell}$ of $H^*(u, v)$ and identify the vertex v of $H^*_t(u, v)$ with the vertex u of $H^*_{t+1}(u, v)$ for every $1 \leq t \leq \ell - 1$. We thus obtain a graph $H(u, v)$ having the same property as $H^*(u, v)$, except that u and v are now at distance at least $\lfloor i/3 \rfloor$ apart. Note that $H(u, v) + uv \in \mathcal{F}$, since this graph can be obtained from a cycle (of length $\ell + 1$) by replacing precisely ℓ edges by copies of $H^*(u, v)$, in other words we start with a planar graph, make ℓ clique-sums with H', and then remove ℓ edges, so the obtained graph is still in $\mathcal{F}$.

We prove the NP-completeness using a reduction from PLANAR 3-COLORABILITY, i.e., the problem of deciding whether a planar graph is 3-colorable, which is NP-complete [16]. Given an instance G of PLANAR 3-COLORABILITY, we construct a graph $G^* = G \oplus H(u, v) \in \mathcal{F} \cap \mathcal{C}_i$.

Notice that G^* can be chosen to be in $\mathcal{F}$, since $H(u, v) + uv \in \mathcal{F}$. Moreover, G^* has no cycle of length 4 to i: by the definition of $H(u, v)$, the vertices u and v are at distance at least $\lfloor i/3 \rfloor$ apart, so any cycle in G^* originating from a cycle in G must have length at least $3 \lfloor i/3 \rfloor + 3 \geq i + 1$. In a 3-coloring of G^*, all the duplicates of a vertex of G must get the same color, so G is 3-colorable if and only if G^* is 3-colorable. □

Recall that Borodin *et al.* [5] proved that every planar graph in $\mathcal{C}_7$ is 3-colorable. Moreover, there exist planar graphs in $\mathcal{C}_4$ that are not 3-colorable [35]. We can then deduce the following corollary:

Corollary 6 *There exists an integer $i \in [5, 7]$ such that every planar graph in $\mathcal{C}_i$ is 3-colorable, but deciding whether a planar graph in $\mathcal{C}_{i-1}$ is 3-colorable is NP-complete.*

Let i be an integer and let $\mathcal{T}_i$ denote the class of graphs with no triangles at distance less than i apart.

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Theorem 7 *For every nice class $\mathcal{F}$ and integer i, either every graph in $\mathcal{F} \cap \mathcal{T}_i$ is 3-colorable, or deciding whether a graph in $\mathcal{F} \cap \mathcal{T}_i$ is 3-colorable is NP-complete.*

PROOF.   We use the same construction and reduction as in the proof of Theorem 5. Suppose that H' is a non 3-colorable graph in $\mathcal{F} \cap \mathcal{T}_i$ that is minimal with respect to the number of edges. Let uv be an edge in H' that is contained in a triangle if H' does contain a triangle and any edge otherwise. The graph $H(u,v)$ is obtained from H' by removing the edge uv. Notice that u and v are at distance at least i from all the triangles in $H(u,v)$. Hence, the graph $G^* = G \oplus H(u,v) \in \mathcal{F}$ has no triangle at distance less than i apart. As previously, G^* is 3-colorable if and only if G is 3-colorable. $\qquad\square$

Using a construction of Aksionov and Mel'nikov [2] and the main result of [13], Theorem 7 has the following corollary.

Corollary 8 *There exists an integer $i \geq 4$ such that any planar graph in $\mathcal{T}_i$ is 3-colorable, but deciding whether a planar graph in $\mathcal{T}_{i-1}$ is 3-colorable is NP-complete.*

4 $(1,0)$-coloring of graphs

The *girth* of a graph G is the length of a shortest cycle of G. Let $\mathcal{C}_{g,d}$ denote the class of planar graphs with girth at least g and maximum degree at most d.

Borodin and Ivanova [6] proved that every planar graph with girth at least 14 is $(1,0)$-colorable. In this section we prove the following theorem:

Theorem 9 *Let $g \geq 6$ and $d \geq 3$ be integers. Either every graph in $\mathcal{C}_{g,d}$ is $(1,0)$-colorable, or deciding whether a graph in $\mathcal{C}_{g,d}$ is $(1,0)$-colorable is NP-complete.*

PROOF.   Consider a $(1,0)$-coloring c of a graph. For convenience, we will say that a vertex v has the color 1^0 if $c(v) = 1$ and none of its neighbors is colored with 1; a vertex has the color 1^1 if itself and one of its neighbors are colored with 1.

Suppose there exists a graph G with girth at least $g \geq 6$ that is not $(1,0)$-colorable, and take such a graph G with the minimum number of edges. One can easily observe that G has minimum degree at least 2. It is well-known that planar graphs with girth at least 6 are 2-degenerate, so G contains a vertex u of degree 2. Assume that u is adjacent to the vertices v and w. By minimality of G, $G \setminus u$ admits a $(1,0)$-coloring, and in any such coloring ϕ we have either $\phi(v) = 0$ and $\phi(w) = 1^1$ or the converse. Let $H(t)$ be the graph obtained from G as follows: we add a vertex t adjacent to u only and then we subdivide once the edges uv, uw and ut (see Figure 2). By the remark above $H(t)$ has a $(1,0)$-coloring and in any such coloring ϕ we have $\phi(u) = 1^1$ and $\phi(t) = 1^0$.

An instance of 3-SAT is said *planar* if its variable-clause graph is planar. We reduce our problem from PLANAR 3-SAT, which was proved to be NP-complete by Lichtenstein [26]. Consider an instance $\mathcal{I}$ of PLANAR 3-SAT. We construct a graph $H^{\mathcal{I}}$ and prove that it is $(1,0)$-colorable if and only if $\mathcal{I}$ is satisfiable. To each variable x of $\mathcal{I}$ we associate the graph H_x depicted in Figure 3, where each vertex drawn with a black square is identified with the vertex t of some copy of $H(t)$. Using the properties of $H(t)$ it follows that in any $(1,0)$-coloring of H_x, either all the black vertices labelled x are colored 0 and the black vertices labelled $\bar{x}$ have color 1^1, or vice-versa.
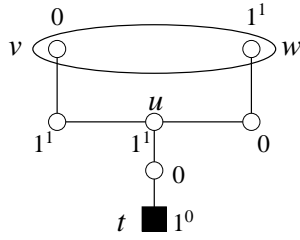
Figure 2: The graph $H(t)$ forcing color 1^0 on the vertex t of degree one.
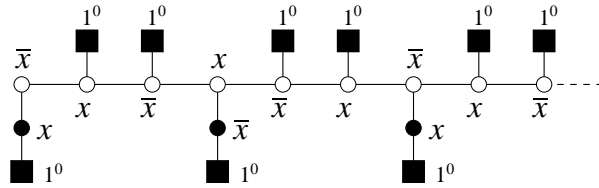

Figure 3: The graph H_x associated to a variable x.

For each clause $x' \vee y' \vee z'$ where x', y', z' are literals of the variables x, y, z, we consider a copy $T_{u,v,w}$ of the graph depicted in Figure 4, and identify u with a black vertex labelled x' in H_x, v with a black vertex labelled y' in H_y, and w with a black vertex labelled z' in H_z. This can be done in such way that the graph $H^{\mathcal{I}}$ obtained is planar. Since black vertices have degree two in each H_x, the maximum degree of $H^{\mathcal{I}}$ does not exceed that of G. Also, we can make sure that the girth of $H^{\mathcal{I}}$ is at least the girth of G by insisting that for any variable x, any two vertices of H_x that have been identified with an endpoint of some graph $T_{u,v,w}$ lie sufficiently far appart in H_x (say at distance at least g). Notice that the color of a black vertex is either 0 or 1^1. Notice also that the graph $T_{u,v,w}$ has the property that the only coloring of u, v, w with colors 0 and 1^1 that does not extend to a $(1,0)$-coloring of $T_{u,v,w}$ is the coloring where u, v, w are colored 1^1.

Suppose first that $\mathcal{I}$ is satisfiable. For each variable x of $\mathcal{I}$, we color the vertices of H_x as follows: if the value of x is true, we assign color 0 to each vertex of H_x labelled by x and color 1 to each vertex labelled $\overline{x}$. If the value of x is false, we assign color 1 to each vertex of H_x labelled by x, and color 0 to each vertex labelled $\overline{x}$. By the satisfiability of $\mathcal{I}$ and the remarks above, such a coloring extends to a $(1,0)$-coloring of $H^{\mathcal{I}}$.
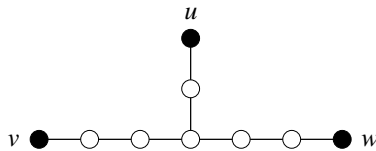

Figure 4: The graph $T_{u,v,w}$ has the property that all colorings of u, v, w except $\phi(u) = \phi(v) = \phi(w) = 1^1$ extend to a $(1,0)$-coloring of the whole graph.
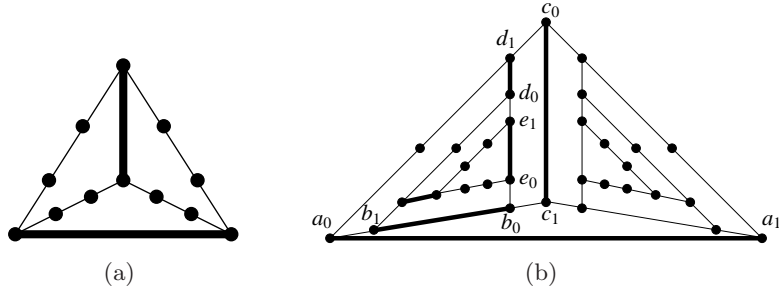
Figure 5: A non-$(1,0)$-colorable planar graph with girth 7 and maximum degree 3.

Conversely, suppose that $H^{\mathcal{I}}$ has a $(1,0)$-coloring ϕ. A variable x is assigned the value true if and only if a vertex labelled by x in H_x has color 0 in ϕ. By the previous remarks, this definition is consistent. Suppose now that there exists a clause c that is not satisfied. By definition, it means that there is a copy of $T_{u,v,w}$ in which $\phi(u) = \phi(v) = \phi(w) = 1^1$, a contradiction. Hence, $\mathcal{I}$ is satisfiable. $\square$

Corollary 10 *Deciding whether a planar graph of girth at least* 7 *and maximum degree at most* 3 *is* $(1,0)$-*colorable is NP-complete.*

PROOF. Borodin *et al.* [8] provided a non-$(1,0)$-colorable planar graph with girth 7 and maximum degree 7. To get our result, we construct a non-$(1,0)$-colorable planar graph with girth 7 and maximum degree 3 (see Figure 5). First check that in any $(1,0)$-coloring of the graph depicted in Figure 5(a), the endpoints of a thick edge cannot be both colored 1. Consider now a $(1,0)$-coloring of the graph depicted in Figure 5(b), where thick edges corresponding to copies of the graph of Figure 5(a) have been represented. Because of the thick edge a_0a_1, we may assume by symmetry that a_0 is colored 0. Then b_1 is colored 1, b_0 is colored 0 (because of the thick edge b_0b_1), c_1 is colored 1, c_0 is colored 0 (because of the thick edge c_0c_1), d_1 is colored 1, d_0 is colored 0 (because of the thick edge d_0d_1), e_1 is colored 1, e_0 is colored 0 (because of the thick edge e_0e_1). This is a contradiction since b_0 and e_0 cannot both get color 0. $\square$

We now present a planar graph G with girth 9 and maximum degree 4 that is not $(1,0)$-colorable. Figure 6 describes the construction of G; in this figure, a black vertex has no other neighbor than the ones already represented while a white vertex may have other neighbors.

Let us first consider the graph $H_{x,y}$ depicted in Figure 6(a). It is easy to check that if x and y are colored 1 and are adjacent to a vertex (not represented in Figure 6(a)) colored 1 (i.e. x and y are colored 1^1), then one cannot extend this partial coloring to a $(1,0)$-coloring of $H_{x,y}$.

Then consider the graph $J_{x,y}$ depicted in Figure 6(b). Since x and y are adjacent, they cannot be both colored with color 0. Suppose they are both colored with color 1. Since they are adjacent, they are colored 1^1. This implies that the vertices x_1 and y_1 are colored with color 1. It follows from the previous paragraph that one of z_1, z_3, say z_1, is colored 0. Thus z_2 is colored 1 and z_3 is colored 0. Therefore, x_2 and y_2' are colored 1. By symmetry, x_2' and y_2 are also colored 1. Thus x_2 and y_2 are colored 1^1 and then H_{x_2,y_2} is not $(1,0)$-colorable. Consequently, any $(1,0)$-coloring of $J_{x,y}$ forces x and y to have distinct colors.

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(a) The graph $H_{x,y}$ (b) The graph $J_{x,y}$ (c) The graph K_x (d) The graph G
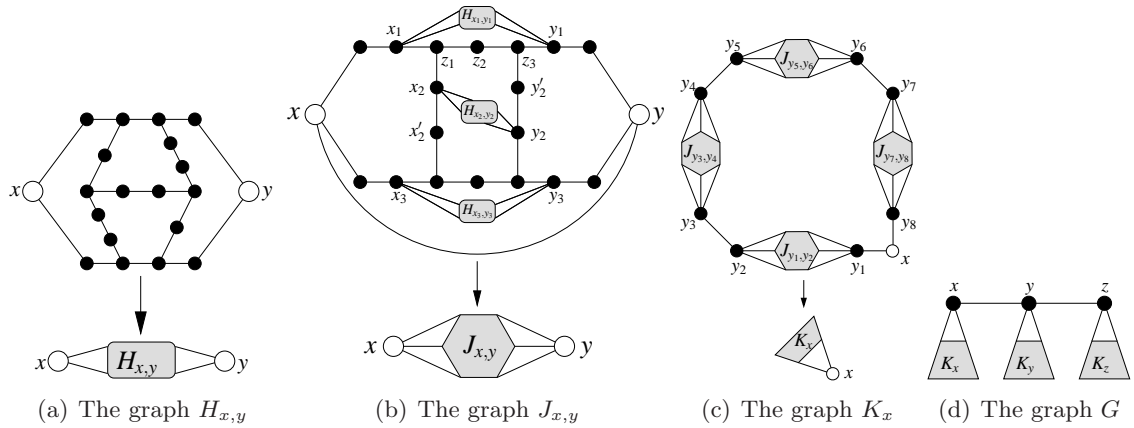
Figure 6: The construction of a non-$(1,0)$-colorable planar graph with girth 9 and maximum degree 4.

Consider now the graph K_x depicted in Figure 6(c). Suppose that x has color 0. Then each y_i is colored $i \pmod 2$, which is a contradiction since x and y_8 are adjacent. Thus x must be colored 1 in any $(1,0)$-coloring of K_x.

Finally, consider the graph G depicted in Figure 6(d). It is easy to check that G has maximum degree 4 and girth 9. It is clear that one of the vertices x, y, or z has to be colored with color 0, say x. This is a contradiction since x is colored 1 in any $(1,0)$-coloring of K_x.

Corollary 11 *Deciding whether a planar graph of girth at least* 9 *and maximum degree at most* 4 *is* $(1,0)$-*colorable is NP-complete.*

Note that unlike the other results of this paper, Theorem 9 cannot be safely generalized to nice classes of graphs. The reason is that the proof requires the 2-degeneracy of a minimal counterexample, which is not guaranteed in a nice class even if the girth is large enough.

5 Acyclic 3-coloring

For a graph G and a graph $H(u,v)$ with two specific vertices u and v, we define the graph $G \ominus H(u,v)$ as the graph obtained from G and $|E(G)|$ copies $(H_e(u,v))_{e \in E(G)}$ of $H(u,v)$ by doing the following: For every edge xy of G, remove xy, and identify x and y with the vertices u and v of $H_{xy}(u,v)$, respectively. In other words, $G \ominus H(u,v)$ is obtained by replacing every edge of G by a copy of $H(u,v)$. Note that if G is planar and $H(u,v) + uv$ is in a nice class $\mathcal{F}$, then the graph $G \ominus H(u,v)$ is also in $\mathcal{F}$. Remark also that given an orientation of the edges of G, the construction of $G \ominus H(u,v)$ defines a unique graph.

Borodin *et al.* [11] proved that planar graphs of girth at least 5 (resp. 7) are acyclically 4-colorable (resp. acyclically 3-colorable). Moreover, deciding whether a bipartite planar graph with maximum degree 4 is acyclically 3-colorable is NP-complete [30]. However, it is not known whether there exists a planar graph of girth at least 5 that is not acyclically 3-colorable. So the maximum acyclic chromatic number of a planar graph of girth 5 or 6 is not known, and is either 3 or 4. In this section, we prove the following:

Theorem 12 *Let $\mathcal{F}$ be a nice class and $g \geq 5$ be an integer. Either every graph in $\mathcal{F}$ of girth at least g is acyclically 3-colorable, or deciding whether a graph from $\mathcal{F}$ of girth at least g is acyclically 3-colorable is NP-complete.*

PROOF. Suppose there exist graphs in $\mathcal{F}$ of girth at least g that are not acyclically 3-colorable, and consider such a graph H' that is minimal with respect to the number of edges. Let $H(u,v)$ be the graph obtained from H' by removing the edge uv. Notice that the distance in $H(u,v)$ between u and v is at least $g-1$. Since $H(u,v)$ is acyclically 3-colorable, by minimality of H', every acyclic 3-coloring of $H(u,v)$ corresponds to one of the cases below (up to permutation of colors):

- coloring A_0: u and v are colored 0 and there is no 2-colored path between u and v.

- coloring A_1: u and v are colored 0 and there is a path colored 0, 1 between u and v, but no path colored 0, 2 between these two vertices.

- coloring A_2: u and v are colored 0 and there are a path colored 0, 1 and a path colored 0, 2 between u and v.

- coloring B: u is colored 0, v is colored 1, and there is a path colored 0, 1 between u and v.

Now $H(u,v)$ admits at least one of the acyclic 3-coloring above, so $H(u,v)$ is exactly of one of the following type:

- type A_0: $H(u,v)$ admits colorings A_0, possibly A_1, possibly A_2, but no coloring B.

- type A_1: $H(u,v)$ admits colorings A_1, possibly A_2, but no coloring A_0 or B.

- type A_2: $H(u,v)$ only admits colorings A_2.

- type B: $H(u,v)$ only admits colorings B.

- type A_0B: $H(u,v)$ admits colorings A_0 and B, possibly A_1, and possibly A_2.

- type A_1B: $H(u,v)$ admits colorings A_1 and B, possibly A_2, but no coloring A_0.

- type A_2B: $H(u,v)$ admits colorings A_2 and B, but no coloring A_0 or A_1.

For each of these types, we prove the NP-completeness:

- type A_0: The reduction is from PLANAR 3-COLORABILITY. Given an instance G of PLANAR 3-COLORABILITY, the graph $G \oplus H(u,v)$ is in $\mathcal{F}$ and has the same girth as $H(u,v)$ (recall that $\oplus$ was defined in Section 3). Moreover, $G \oplus H(u,v)$ has an acyclic 3-coloring if and only if G has a 3-coloring.

- type A_1: We construct the graph $K(u,v)$ from two copies $H_1(u,v_1)$ and $H_2(u_2,v)$ of $H(u,v)$ by identifying v_1 with u_2 (we put the two copies in series). Any acyclic 3-coloring of K is such that u, v_1 and v get the same color, say 0. We can color $K(u,v)$ such that there is no path colored 0, 2 between u and v_1 in $H_1(u,v_1)$, and such that there is no path colored 0, 1 between u_2 and v in $H_2(u_2,v)$. This way, there is no 2-colored path between u and v in $K(u,v)$. The acyclic 3-coloring properties of $K(u,v)$ are thus of type A_0, we can thus use the same reduction as for type A_0 using $K(u,v)$ instead of $H(u,v)$. Note that $K(u,v) + uv$ is clearly in $\mathcal{F}$, so $G \oplus K(u,v) \in \mathcal{F}$.

- type A_2: The reduction is from PLANAR ACYCLIC 3-COLORABILITY. Given an instance G of PLANAR ACYCLIC 3-COLORABILITY, the graph $G \oplus H(u,v)$ is in $\mathcal{F}$ and has the same girth as $H(u,v)$. Moreover, $G \oplus H(u,v)$ has an acyclic 3-coloring if and only if G has an acyclic 3-coloring, since any alternating cycle in G would result in an alternating cycle in $G \oplus H(u,v)$.

- type B: The reduction is from PLANAR ACYCLIC 3-COLORABILITY. Given an instance G of PLANAR ACYCLIC 3-COLORABILITY, the graph $G \ominus H(u,v)$ is in $\mathcal{F}$ and has the same girth as $H(u,v)$. Moreover, $G \ominus H(u,v)$ has an acyclic 3-coloring if and only if G has an acyclic 3-coloring.

- type A_0B: We construct the graph $K(u,v)$ from two copies of $H(u,v)$ by identifying the vertex u (resp. v) of each copy (we put the two copies $H(u,v)$ in parallel). Consider an acyclic 3-coloring of $K(u,v)$, and suppose u and v get distinct colors, say respectively 0 and 1. Then both copies of $H(u,v)$ have a acyclic 3-coloring B. So both copies contain a path colored 0, 1 between u and v, which creates a 2-colored cycle in $K(u,v)$, a contradiction. So u and v must get the same color. Moreover, there exists an acyclic 3-coloring of $K(u,v)$ with no 2-colored path between u and v. Observe that $K(u,v)+uv \in \mathcal{F}$, since this graph is the clique-sum of two copies of $H(u,v)+uv$. Moreover, the coloring properties of $K(u,v)$ are of type A_0, we can thus use the same reduction as for type A_0 using $K(u,v)$ instead of $H(u,v)$.

- type A_1B: We use the same construction of $K(u,v)$ as in the previous case. Again, u and v must get the same color, say 0 (otherwise we find a 2-colored cycle). Moreover, each copy of $H(u,v)$ contains at least one 2-colored path between u and v. Thus $K(u,v)$ is acyclically 3-colorable if and only if there exists a path colored 0, 1 in one copy and a path colored 0, 2 in the other copy. The coloring properties of $K(u,v)$ are thus of type A_2, we can thus use the same reduction as for type A_2 using $K(u,v)$ instead of $H(u,v)$.

- type A_2B: We construct the graph $K(u,v)$ from three copies $H_1(u,v)$, $H_2(u_2,v_2)$, and $H_3(u_3,v_3)$ of $H(u,v)$ by identifying v with u_2, v_2 with u_3, and v_3 with u. Suppose $H_1(u,v)$ is colored A_2. Then u and v get the same color, say 0. If u_3 gets color 0, then we have cycles colored 0, 1 and cycles colored 0, 2 in $K(u,v)$. If u_3 gets a color distinct from 0, say 1, then we have a cycle colored 0, 1 in $K(u,v)$. So each copy of $H(u,v)$ must be colored B, which means that u, v, and u_3 get distinct colors. In this case $K(u,v)$ is indeed acyclically 3-colorable as we do not create any alternating cycle. The coloring properties of $K(u,v)$ are thus of type B, so we can use the same reduction as for type B using $K(u,v)$ instead of $H(u,v)$. It is clear that $K(u,v) + uv \in \mathcal{F}$, so it follows that $G \ominus K(u,v) \in \mathcal{F}$.

$\square$

We immediately have the following corollary:

Corollary 13 *There exists an integer $g \in [5,7]$ such that every planar graph of girth at least g is acyclically 3-colorable and* ACYCLIC 3-COLORABILITY *is NP-complete for planar graphs of girth $g-1$.*

Recall that $\mathcal{C}_i$ is the class of graphs with no cycle of length 4 to i.
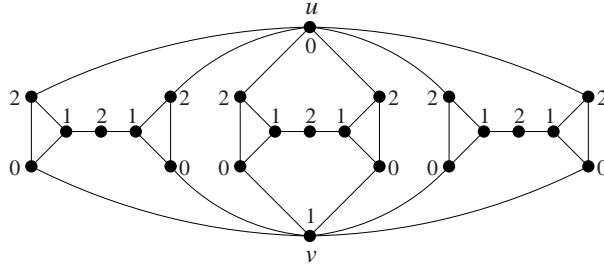
Figure 7: The edge-gadget $H(u, v)$ in Corollary 15.

Theorem 14 *Let $\mathcal{F}$ be a nice class and $i \geq 5$ be an integer. Either every graph in $\mathcal{F} \cap \mathcal{C}_i$ is acyclically 3-colorable, or deciding whether a graph in $\mathcal{F} \cap \mathcal{C}_i$ is acyclically 3-colorable is NP-complete.*

PROOF.   Suppose there exist graphs in $\mathcal{F} \cap \mathcal{C}_i$ that are not acyclically 3-colorable, and consider such a graph H' that is minimal with respect to the number of edges.

We first show that H' contains an edge that does not belong to a triangle. We use a result of Xu [39] that every maximal acyclically k-colorable graph with n vertices has exactly $(k-1)(n - \frac{k}{2})$ edges. So, every minimally non acyclically 3-colorable graph has most $2n - 2$ edges and is thus 3-degenerate. By minimality, H' is 2-connected, so H' contains a vertex x of degree 2 or 3. If H' contains a vertex x of degree 2, then by minimality x is not contained in a triangle, so the edges incident to x do not belong to a triangle. If H' contains a vertex x of degree 3, then the graph induced by the neighbors of x contains at most one edge since otherwise H' would contain a 4-cycle, x is thus incident to at least one edge that does not belong to a triangle.

Let $H(u, v)$ be the graph obtained from H' by removing an edge uv that is not contained in a triangle. Notice that the distance in $H(u, v)$ between u and v is at least i. This ensures that we can now use the same proof as for Theorem 12.   □

Corollary 15 *There exists an integer $i \in [6, 11]$ such that every planar graph in $\mathcal{C}_i$ is acyclically 3-colorable and ACYCLIC 3-COLORABILITY is NP-complete for planar graphs in $\mathcal{C}_{i-1}$.*

PROOF.   For the upper bound, Borodin and Ivanova [7] proved that graphs in $\mathcal{C}_{11}$ are acyclically 3-choosable, and thus acyclically 3-colorable. For the lower bound, it is easy to check that the graph $H(u, v)$ depicted in Figure 7 has no acyclic 3-coloring such that $c(u) = c(v)$. Then, the graph $H' = K_4 \ominus H(u, v)$ is clearly not acyclically 3-colorable.   □

6 Mapping graphs to odd cycles

A *homomorphism* from a graph G to a graph H is a mapping $h : V(G) \to V(H)$ that preserves the edges, i.e. if uv is an edge of G, then $h(u)h(v)$ is an edge of H. If such a mapping exists, then we say that G is H-*colorable*.

It was proved in [15] that for any proper minor-closed class $\mathcal{F}$, and any integer k there exists an integer g such that any graph $G \in \mathcal{F}$ with girth at least g has a homomorphism to C_{2k+1}. We prove the following:

Theorem 16 *For every nice class $\mathcal{F}$ and every integers $g \geq 3$ and $k \geq 2$, either every graph in $\mathcal{F}$ of girth at least g is C_{2k+1}-colorable, or deciding whether a graph in $\mathcal{F}$ of girth at least g is C_{2k+1}-colorable is NP-complete.*

PROOF. Assume that there exists a graph in $\mathcal{F}$ with girth at least g having no homomorphism to C_{2k+1}. To prove that it is NP-complete to determine whether a graph in $\mathcal{F}$ with girth at least g is C_{2k+1}-colorable, we use a reduction from PLANAR C_{2k+1}-COLORING, the problem of deciding whether a planar graph is C_{2k+1}-colorable, which is NP-complete [28].

Assume we are given a graph $H(u,v)$ such that $H(u,v) + uv \in \mathcal{F}$ (to be constructed later) of girth g, with two specific vertices u and v having the following property: $H(u,v)$ has a homomorphism to C_{2k+1}, and in any such homomorphism h, u and v have the same image, i.e., $h(u) = h(v)$. As in Section 3, observe that by putting sufficiently many copies of $H(u,v)$ in series, we can obtain a new graph playing the role of $H(u,v)$ such that the distance between u and v is at least $g/3$.

Now take an instance G of PLANAR C_{2k+1}-COLORING and consider $G^* = G \oplus H(u,v) \in \mathcal{F}$ (recall that $\oplus$ was defined in Section 3). We claim that G^* has girth at least g, and is C_{2k+1}-colorable if and only if G is C_{2k+1}-colorable.

First, observe that a cycle in G^* comes either from a cycle in $H(u,v)$ and then its length is at least g, or from a cycle in G and then its length is at least $3(g/3 + 1) > g$. Now assume that there is a homomorphism from G^* to C_{2k+1}. By the definition of $H(u,v)$, for any edge xy of G, all the duplicates of x (resp. y) have the same image, and the two images are adjacent on the cycle C_{2k+1}. Hence, G^* is C_{2k+1}-colorable if and only if G is C_{2k+1}-colorable.

It remains to prove that a graph $H(u,v)$ with the required properties exists assuming that there exists a graph in $\mathcal{F}$ of girth g with no homomorphism to C_{2k+1}. Let H' be such a graph with the minimum number of edges. Let $e = uv$ be any edge of H', and let $H^-(u,v)$ be the graph obtained from H' by removing e. The graph $H^-(u,v)$ is in $\mathcal{F}$ and has girth at least g, so by minimality of H' the graph $H^-(u,v)$ has a homomorphism to C_{2k+1}. For such a homomorphism h and any two vertices x,y of $H^-(u,v)$ we denote by $d_h(x,y)$ the distance between $h(x)$ and $h(y)$ in C_{2k+1}. We define the set $S = \{d_h(u,v) \mid h \text{ is a homomorphism from } H^-(u,v) \text{ to } C_{2k+1}\}$. Observe that S is non-empty and $S \subseteq \{0, \ldots, k\} \setminus \{1\}$. Indeed, S does not contain 1 since it would imply that H' admits a homomorphism to C_{2k+1}. We consider two cases, depending on the presence of an odd element in S.

If S contains an odd element, then let i be minimal such that $2i + 1 \in S$. Consider the graph $H_o(x,y)$ of Figure 8, left. It is obtained from a star with three leaves x, y, z and two copies of $H^-(u,v)$ by subdividing $2i - 1$ times the edge containing z, identifying x and z with the vertices u and v of the first copy of $H^-(u,v)$, respectively, and z and y with the vertices u and v of the second copy of $H^-(u,v)$. By the definition of S, any mapping from x and y to any vertex v of C_{2k+1}, and from z to a vertex at distance $2i + 1$ from v on the cycle can be extended to a homomorphism from $H_o(x,y)$ to C_{2k+1}, so $H_o(x,y)$ is C_{2k+1}-colorable. Now consider any homomorphism h from $H_o(x,y)$ to C_{2k+1}. Since there is a path of length $2i + 1$ between x and z in $H_o(x,y)$, the distance between $h(x)$ and $h(z)$ in C_{2k+1} is odd and at most $2i + 1$. Hence, by the minimality of i, the distance between $h(x)$ and $h(z)$ is precisely $2i + 1$. Similarly, the distance between $h(y)$ and $h(z)$ is $2i + 1$. Since there is a path of length
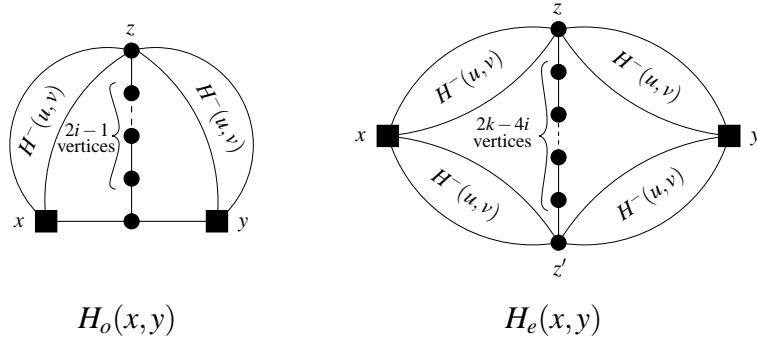
$$H_o(x,y) \qquad\qquad H_e(x,y)$$

Figure 8: One of these two graphs has a homomorphism to C_{2k+1} and in any such homomorphism h, we have $h(x) = h(y)$.

(number of edges) two between x and y and $4i + 2 \leq 2k$, we have $h(x) = h(y)$. Hence, we can use $H_o(x, y)$ to play the role of $H(u, v)$ in this case.

Otherwise all the elements of S are even. Let i be maximal such that $2i \in S$. Consider the graph $H_e(x, y)$ of Figure 8, right. It is obtained from $C_4 \ominus H^-(u, v)$ by adding a path of length $2k - 4i + 1$ between two opposite vertices of the original four cycle (recall that $\ominus$ was defined in Section 5). By the definition of S, any mapping from x and y to any vertex v of C_{2k+1}, and from z and z' to the two vertices at distance $2i$ from v on the cycle can be extended to a homomorphism from $H_e(x, y)$ to C_{2k+1}, so $H_e(x, y)$ is C_{2k+1}-colorable. Now consider any homomorphism h from $H_e(x, y)$ to C_{2k+1}. By the definition of S, the distances between $h(x)$ and $h(z), h(z')$ are both even, and at most $2i$. So any path between z and z' going through x is mapped to an even path of length at most $4i$. Since there is an odd path of length $2k - 4i + 1$ between z and z' in $H_e(x, y)$, the image of any cycle going through x, z, and z' is C_{2k+1} and by maximality of i, $h(z)$ and $h(z')$ are at distance precisely $2i$ from x and at distance $2k - 4i + 1$ from each other. By symmetry, we have $h(x) = h(y)$ and the graph $H_e(x, y)$ can be used to play the role of $H(u, v)$. This concludes the proof, since $H_o(x, y) + xy$ and $H_e(x, y) + xy$ are in $\mathcal{F}$ by the definition of a nice class. $\qquad\square$

It follows from the main result of [15] that Theorem 16 has the following corollary:

Corollary 17 *For every nice minor-closed class $\mathcal{F}$ and integer $k \geq 2$ there exists an integer $g = g(\mathcal{F}, k)$, such that every graph of $\mathcal{F}$ of girth at least g is C_{2k+1}-colorable, but determining whether a graph of $\mathcal{F}$ of girth $g - 1$ is C_{2k+1}-colorable is NP-complete.*

An unpublished construction of DeVos (see [10]) together with a result of Borodin *et al.* [10] imply that Theorem 16 also has the following corollary:

Corollary 18 *For every $k \geq 2$ there exists an integer $g = g(k)$, with $4k \leq g \leq \frac{1}{3}(20k - 2)$, such that every planar graph of girth at least g is C_{2k+1}-colorable, but determining whether a planar graph of girth $g - 1$ is C_{2k+1}-colorable is NP-complete.*

The *odd-girth* of a graph G is the size of a smallest odd cycle in G. In 2000, Klostermeyer and Zhang [24] proposed the following strengthening of Conjecture 1:

Conjecture 19 *For any $k \geq 1$, every planar graph of odd-girth at least $4k + 1$ is C_{2k+1}-colorable.*

We can prove the following theorem:

Theorem 20 *For every nice class $\mathcal{F}$ and every integers $g \geq 1$ and $k \geq 2$, either every graph in $\mathcal{F}$ of odd-girth at least $2g + 1$ is C_{2k+1}-colorable, or deciding whether a graph in $\mathcal{F}$ of odd-girth at least $2g + 1$ is C_{2k+1}-colorable is NP-complete.*

PROOF. The proof follows exactly the same arguments as the proof of Theorem 16. The only thing that has to be checked is that if G is planar and $H(u, v) \in \mathcal{F}$ with odd-girth $2g + 1$ is such that u and v are on the same face and at distance at least $\frac{1}{3}(2g + 1)$ apart in $H(u, v)$, then $G^* = G \oplus H(u, v)$ (which can be chosen to be in $\mathcal{F}$, as observed previously) has odd-girth at least $2g + 1$. Indeed, an odd cycle in G^* originates either from an odd cycle in $H(u, v)$, in which case its length is at least $2g + 1$, or from a cycle in G, in which case its length is at least $3 \times (2g + 1)/3 = 2g + 1$ in G^*. □

The construction of DeVos mentioned previously, together with a result of Zhu [41] imply that Theorem 20 has the following corollary:

Corollary 21 *For every $k \geq 2$ there exists an integer $g = g(k)$, with $2k \leq g \leq 4k - 2$, such that every planar graph of odd-girth at least $2g + 1$ is C_{2k+1}-colorable, but determining whether a planar graph of odd-girth $2g - 1$ is C_{2k+1}-colorable is NP-complete.*

Youngs [40] has constructed non 3-colorable projective planar graphs with arbitrarily large odd-girth, so Corollary 21 does not extend to nice classes containing all projective planar graphs.

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