Homomorphisms of planar (m, n)-colored-mixed graphs to planar targets

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Abstract

An (m, n)-colored-mixed graph $G = (V, A_1, A_2, \cdots, A_m, E_1, E_2, \cdots, E_n)$ is a graph having m colors of arcs and n colors of edges. We do not allow two arcs or edges to have the same endpoints. A homomorphism from an (m, n)-colored-mixed graph G to another (m, n)-colored-mixed graph H is a morphism $\varphi : V(G) \to V(H)$ such that each edge (resp. arc) of G is mapped to an edge (resp. arc) of H of the same color (and orientation). An (m, n)-colored-mixed graph T is said to be $P_g^{(m,n)}$-universal if every graph in $P_g^{(m,n)}$ (the planar (m, n)-colored-mixed graphs with girth at least g) admits a homomorphism to T.

We show that planar $P_g^{(m,n)}$-universal graphs do not exist for $2m + n \geqslant 3$ (and any value of g) and find a minimal (in the number vertices) planar $P_g^{(m,n)}$-universal graphs in the other cases.

1. Introduction

The concept of homomorphisms of (m, n)-colored-mixed graph was introduced by J. Nešetřil and A. Raspaud [1] in order to generalize homomorphisms of k-edge-colored graphs and oriented graphs.

An (m, n)*-colored-mixed graph* $G = (V, A_1, A_2, \cdots, A_m, E_1, E_2, \cdots, E_n)$ is a graph having m colors of arcs and n colors of edges. We do not allow two arcs or edges to have the same endpoints and we do not allow loops. The case $m = 0$ and $n = 1$ corresponds to simple graphs, $m = 1$ and $n = 0$ to oriented graphs and $m = 0$ and $n = k$ to k-edge-colored graphs. For the case $m = 0$ and $n = 2$ (2-edge-colored graphs) we refer to the two types of edges as *blue* and *red* edges.

A *homomorphism* from an (m, n)-colored-mixed graph G to another (m, n)-colored-mixed graph H is a mapping $\varphi : V(G) \to V(H)$ such that every edge (resp. arc) of G is mapped to an edge (resp. arc) of H of the same color (and orientation). If G admits a homomorphism to H, we say that G is H*-colorable* since this homomorphism can be seen as a coloring of the vertices of G using the vertices of H as colors. The edges and arcs of H (and their colors) give us the rules that this coloring must follow. Given a class of graphs $\mathcal{C}$, a graph is $\mathcal{C}$*-universal* if for every graph $G \in \mathcal{C}$ is H-colorable. The class $P_g^{(m,n)}$ contains every planar (m, n)-colored-mixed graph with girth at least g. Graph $\overrightarrow{C_6^2}$ is the graph with vertex set $\{0, 1, 2, 3, 4, 5\}$ such that uv is an arc if and only if $v - u \equiv 1 \pmod 6$ or $v - u \equiv 2 \pmod 6$.

In this paper, we consider some planar $P_g^{(m,n)}$-universal graphs with few vertices. They are depicted in Figures 1 and 2. The known results about this topic are as follows.

Theorem 1.

1. *K_4 is a planar $P_3^{(0,1)}$-universal graph. This is the four color theorem.*
2. *K_3 is a planar $P_4^{(0,1)}$-universal graph. This is Grötzsch's Theorem [2].*
3. *$\overrightarrow{C_6^2}$ is a planar $P_{16}^{(1,0)}$-universal graph [3].*

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Our first result shows that, in addition to the case of $(0,1)$-graphs covered by Theorems 1.1 and 1.2, our topic is actually restricted to the cases of oriented graphs (i.e., $(m,n)=(1,0)$) and 2-edge-colored graphs (i.e., $(m,n)=(0,2)$).

Theorem 2. *For every $g \geqslant 3$, there exists no planar $P_g^{(m,n)}$-universal graph if $2m+n \geqslant 3$.*

As Theorems 1.1 and 1.2 show for $(0,1)$-graphs, there might exist a trade-off between minimizing the girth g and the number of vertices of the universal graph, for a fixed pair (m,n). For oriented graphs, Theorem 1.3 tries to minimize the girth. For oriented graphs and 2-edge-colored graphs, we choose instead to minimize the number of vertices of the universal graph.

Theorem 3.

1. *$\overrightarrow{T_5}$ is a planar $P_{28}^{(1,0)}$-universal graph on 5 vertices.*
2. *T_6 is a planar $P_{22}^{(0,2)}$-universal graph on 6 vertices.*

The following results shows that Theorem 3 is optimal in terms of the number of vertices of the universal graph.

Theorem 4.

1. *For every $g \geqslant 3$, there exists an oriented bipartite cactus graph (i.e., K_4^- minor-free graph) with girth at least g and oriented chromatic number at least 5.*
2. *For every $g \geqslant 3$, there exists a 2-edge-colored bipartite outerplanar graph (i.e., $(K_4^-, K_{2,3})$ minor-free graph) with girth at least g that does not map to a planar graph with at most 5 vertices.*

Most probably, Theorem 3 is not optimal in terms of girth. The following constructions give lower bounds on the girth.

Theorem 5.

1. *There exists an oriented bipartite 2-outerplanar graph with girth 14 that does not map to $\overrightarrow{T_5}$.*
2. *There exists a 2-edge-colored planar graph with girth 11 that does not map to T_6.*
3. *There exists a 2-edge-colored bipartite planar graph with girth 10 that does not map to T_6.*

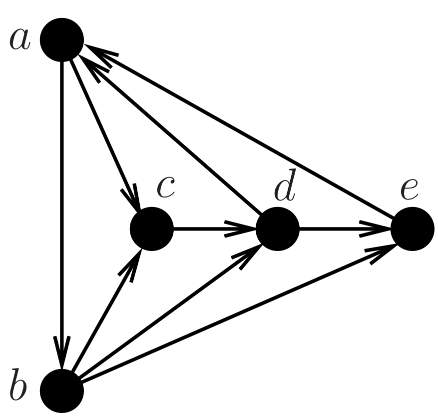

Figure 1: The $P_{28}^{(1,0)}$-universal graph $\overrightarrow{T_5}$.

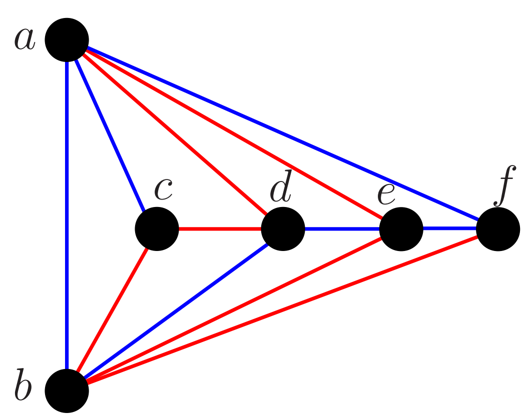

Figure 2: The $P_{22}^{(0,2)}$-universal graph T_6.

Next, we obtain the following complexity dichotomies:

Theorem 6.

1. *For any fixed girth $g \geqslant 3$, either every graph in $P_g^{(1,0)}$ maps to $\overrightarrow{T_5}$ or it is NP-complete to decide whether a graph in $P_g^{(1,0)}$ maps to $\overrightarrow{T_5}$. Either every bipartite graph in $P_g^{(1,0)}$ maps to $\overrightarrow{T_5}$ or it is NP-complete to decide whether a bipartite graph in $P_g^{(1,0)}$ maps to $\overrightarrow{T_5}$.*

2. *Either every graph in $P_g^{(0,2)}$ maps to T_6 or it is NP-complete to decide whether a graph in $P_g^{(1,0)}$ maps to T_6. Either every bipartite graph in $P_g^{(0,2)}$ maps to T_6 or it is NP-complete to decide whether a bipartite graph in $P_g^{(1,0)}$ maps to T_6.*

Finally, we can use Theorem 6 with the non-colorable graphs in Theorem 5.

Corollary 7.

1. *Deciding whether a bipartite graph in $P_{14}^{(1,0)}$ maps to $\overrightarrow{T_5}$ is NP-complete.*
2. *Deciding whether a graph in $P_{11}^{(0,2)}$ maps to T_6 is NP-complete.*
3. *Deciding whether a bipartite graph in $P_{10}^{(0,2)}$ maps to T_6 is NP-complete.*

A 2-edge-colored path or cycle is said to be *alternating* if any two adjacent edges have distinct colors.

Proposition 8 (folklore)**.**

- *Every planar simple graph on n vertices has at most $3n-6$ edges.*
- *Every planar simple graph satisfies $(\text{mad}(G)-2)\cdot(g(G)-2) < 4$.*

2. Proof of Theorem 3

We use the discharging method for both results in Theorem 3. The following lemma will handle the discharging part. We call a vertex of degree n an n-vertex and a vertex of degree at least n an n^+-vertex. If there is a path made only of 2-vertices linking two vertices u and v, we say that v is a weak-neighbor of u. If v is a neighbor of u, we also say that v is a weak-neighbor of u. We call a (weak-)neighbor of degree n an n-(weak-)neighbor.

Lemma 9. *Let k be a non-negative integer. Let G be a graph with minimum degree 2 such that every 3-vertex has at most k 2-weak-neighbors and every path contains at most $\frac{k+1}{2}$ consecutive 2-vertices. Then $\text{mad}(G) \geqslant 2 + \frac{2}{k+2}$. In particular, G cannot be a planar graph with girth at least $2k+6$.*

Proof. Let G be as stated. Every vertex has an initial charge equal to its degree. Every 3^+-vertex gives $\frac{1}{k+2}$ to each of its 2-weak-neighbors. Let us check that the final charge $ch(v)$ of every vertex v is at least $2 + \frac{2}{k+2}$.

- If $d(v) = 2$, then v receives $\frac{1}{k+2}$ from each of its 3-weak-neighbors. Thus $ch(v) = 2 + \frac{2}{k+2}$.
- If $d(v) = 3$, then v gives $\frac{1}{k+2}$ to each of its 2-weak-neighbors. Thus $ch(v) \geqslant 3 - \frac{k}{k+2} = 2 + \frac{2}{k+2}$.
- If $d(v) = d \geqslant 4$, then v has at most $\frac{k+1}{2}$ 2-weak-neighbors in each of the d incident paths. Thus $ch(v) \geqslant d - d\left(\frac{k+1}{2}\right)\left(\frac{1}{k+2}\right) = \frac{d}{2}\left(1 + \frac{1}{k+2}\right) \geqslant 2 + \frac{2}{k+2}$.

This implies that $mad(G) \geqslant 2 + \frac{2}{k+2}$. Finally, if G is planar, then the girth of G cannot be at least $2k+6$, since otherwise $(\text{mad}(G)-2)\cdot(g(G)-2) \geqslant \left(2 + \frac{2}{k+2} - 2\right)(2k+6-2) = \left(\frac{2}{k+2}\right)(2k+4) = 4$, which contradicts Proposition 8. $\square$

2.1. Proof of Theorem 3.1

We prove that the oriented planar graph $\overrightarrow{T_5}$ on 5 vertices from Figure 1 is $P_{28}^{(1,0)}$-universal by contradiction. Assume that G is an oriented planar graphs with girth at least 28 that does not admit a homomorphism to $\overrightarrow{T_5}$ and is minimal with respect to the number of vertices. By minimality, G cannot contain a vertex v with degree at most one since a $\overrightarrow{T_5}$-coloring of $G - v$ can be extended to G. Similarly, G does not contain the following configurations.

- A path with 6 consecutive 2-vertices.
- A 3-vertex with at least 12 2-weak-neighbors.

Suppose that G contains a path $u_0u_1u_2u_3u_4u_5u_6u_7$ such that the degree of u_i is two for $1 \leqslant i \leqslant 6$. By minimality of G, $G - u_1, u_2, u_3, u_4, u_5, u_6$ admits a $\overrightarrow{T_5}$-coloring φ. We checked on a computer that for any $\varphi(v_0)$ and $\varphi(v_6)$ in $V\left(\overrightarrow{T_5}\right)$ and every possible orientation of the 7 arcs u_iu_{i+1}, we can always extend φ into a $\overrightarrow{T_5}$-coloring of G, a contradiction.

Suppose that G contains a 3-vertex v with at least 12 2-weak-neighbors. Let u_1, u_2, u_3 be the 3^+-weak-neighbors of v and let l_i be the number of common 2-weak-neighbors of v and u_i, i.e., 2-vertices on the path between v and l_i. Without loss of generality and by the previous discussion, we have $5 \geqslant l_1 \geqslant l_2 \geqslant l_3$ and $l_1 + l_2 + l_3 \geqslant 12$. So we have to consider the following cases:

- **Case 1:** $l_1 = 5$, $l_2 = 5$, $l_3 = 2$.
- **Case 2:** $l_1 = 5$, $l_2 = 4$, $l_3 = 3$.
- **Case 3:** $l_1 = 4$, $l_2 = 4$, $l_3 = 4$.

By minimality, the graph G' obtained from G by removing v and its 2-weak-neighbors admits a $\overrightarrow{T_5}$-coloring φ. Let us show that in all three cases, we can extend φ into a $\overrightarrow{T_5}$-coloring of G to get a contradiction.

With an extensive search on a computer we found that if a vertex v is connected to a vertex u colored in $\varphi(u)$ by a path made of l 2-vertices ($0 \leqslant l \leqslant 5$) then v can be colored in:

- at least 1 color if $l = 0$,
- at least 2 colors if $l = 1$,
- at least 2 colors if $l = 2$ (the sets $\{c, d, e\}$ and $\{b, c, d\}$ are the only sets of size 3 that can be forbidden from v),
- at least 3 colors if $l = 3$,
- at least 4 colors if $l = 4$ and
- at least 4 colors if $l = 5$ (only the sets $\{b\}$, $\{c\}$, and $\{e\}$ can be forbidden from v).

In Case 1, u_3 forbids at most 3 colors from v since $l_3 = 2$. If it forbids less than 3 colors, we will be able to find a color for v since u_1 and u_2 forbid at most 1 color from v. The only sets of 3 colors that u_3 can forbid are $\{b, c, d\}$ and $\{c, d, e\}$. Since u_1 and u_2 can each only forbid b, c or e, we can always find a color for v.

In Case 2, u_1 and u_2 each forbid at most one color and u_3 forbids at most 2 colors so there remains at least one color for v.

In Case 3, u_1, u_2, and u_3 each forbid at most one color, so there remains at least two colors for v.

We can always extend φ into a $\overrightarrow{T_5}$-coloring of G, a contradiction.

So G contains at most 5 consecutive 2-vertices and every 3-vertex has at most 11 2-weak-neighbors. Using Lemma 9 with $k = 11$ contradicts the fact that the girth of G is at least 28.

2.2. Proof of Theorem 3.2

We prove that the 2-edge-colored planar graph T_6 on 6 vertices from Figure 2 is $P_{22}^{(0,2)}$-universal by contradiction. Assume that G is a 2-edge-colored planar graphs with girth at least 22 that does not admit a homomorphism to T_6 and is minimal with respect to the number of vertices. By minimality, G cannot contain a vertex v with degree at most one since a T_6-coloring of $G - v$ can be extended to G. Similarly, G does not contain the following configurations.

- A path with 5 consecutive 2-vertices.
- A 3-vertex with at least 9 2-weak-neighbors.

Suppose that G contains a path $u_0u_1u_2u_3u_4u_5u_6$ such that the degree of u_i is two for $1 \leqslant i \leqslant 5$. By minimality of G, $G - u_1, u_2, u_3, u_4, u_5$ admits a T_6-coloring φ. We checked on a computer that for any $\varphi(v_0)$ and $\varphi(v_6)$ in $V(T)$ and every possible colors of the 6 edges u_iu_{i+1}, we can always extend φ into a T_6-coloring of G, a contradiction.

Suppose that G contains a 3-vertex v with at least 9 2-weak-neighbors. Let u_1, u_2, u_3 be the 3^+-weak-neighbors of v and let l_i be the number of common 2-weak-neighbors of v and u_i, i.e., 2-vertices on the path between v and l_i. Without loss of generality and by the previous discussion, we have $4 \geqslant l_1 \geqslant l_2 \geqslant l_3$ and $l_1 + l_2 + l_3 \geqslant 9$. So we have to consider the following cases:

- **Case 1:** $l_1 = 3$, $l_2 = 3$, $l_3 = 3$.
- **Case 2:** $l_1 = 4$, $l_2 = 3$, $l_3 = 2$.
- **Case 3:** $l_1 = 4$, $l_2 = 4$, $l_3 = 1$.

By minimality of G, the graph G' obtained from G by removing v and its 2-weak-neighbors admits a T_6-coloring φ. Let us show that in all three cases, we can extend φ into a T_6-coloring of G to get a contradiction.

With an extensive search on a computer we found that if a vertex v is connected to a vertex u colored in $\varphi(u)$ by a path P made of l 2-vertices ($0 \leqslant l \leqslant 4$) then v can be colored in:

- at least 1 color if $l = 0$ (the sets a, c, d, e, f and b, c, d, e, f of colors are the only sets of size 5 that can be forbidden from v for some $\varphi(u) \in T$ and edge-colors on P),
- at least 2 colors if $l = 1$ (the sets a, b, c, f and b, c, e, f are the only sets of size 4 that can be forbidden from v),
- at least 3 colors if $l = 2$ (the sets b, c, f, c, e, f and d, e, f are the only sets of size 3 that can be forbidden from v),
- at least 4 colors if $l = 3$ (the set c, b is the only set of size 2 that can be forbidden from v), and
- at least 5 colors if $l = 4$ (the sets c and f are the only sets of size 1 that can be forbidden from v).

Suppose that we are in Case 1. Vertices u_1, u_2, and u_3 each forbid at most 2 colors from v since $l_1 = l_2 = l_3 = 3$. Suppose that u_1 forbids 2 colors. It has to forbid colors c and f (since it is the only pair of colors that can be forbidden by a path made of 3 2-vertices). If u_2 or u_3 also forbids 2 colors, they will forbid the exact same pair of colors. We can therefore assume that they each forbid 1 color from v. There are 6 available colors in T_6, so we can always find a color for v and extend φ to a T_6-coloring of G, a contradiction. We proceed similarly for the other two cases.

So G contains at most 4 consecutive 2-vertices and every 3-vertex has at most 8 2-weak-neighbors. Then Lemma 9 with $k = 8$ contradicts the fact that the girth of G is at least 22.

3. Proof of Theorem 4.1

We construct an oriented bipartite cactus graph with girth at least g and oriented chromatic number at least 5. Let g' be such that $g' \geqslant g$ and $g' \equiv 4 \pmod 6$. Consider a circuit $v_1, \cdots, v_{g'}$. Clearly, the oriented chromatic number of this circuit is 4 and the only tournament on 4 vertices it can map to is the tournament $\overrightarrow{T_4}$ induced by the vertices a, b, c, and d in $\overrightarrow{T_5}$. Now we consider the cycle $C = w_1, \cdots, w_{g'}$ containing the arcs $w_{2i-1}w_{2i}$ with $1 \leqslant i \leqslant g'/2$, $w_{2i+1}w_{2i}$ with $1 \leqslant i \leqslant g'/2 - 1$, and $w_{g'}w_1$.

Suppose for contradiction that C admits a homomorphism φ such that $\varphi(w_1) = d$. This implies that $\varphi(w_2) = a$, $\varphi(w_3) = d$, $\varphi(w_4) = a$, and so on until $\varphi(w_{g'}) = a$. Since $\varphi(w_{g'}) = a$ and $\varphi(w_1) = d$, $w_{g'}w_1$ should map to ad, which is not an arc of $\overrightarrow{T_4}$, a contradiction.

Our cactus graph is then obtain from the circuit $v_1, \cdots, v_{g'}$ and g' copies of C by identifying every vertex v_i with the vertex w_1 of a copy of C. This cactus graph does not map to $\overrightarrow{T_4}$ since one of the v_i would have to map to d and then the copy of C attached to v_i would not be $\overrightarrow{T_4}$-colorable.

4. Proof of Theorem 4.2

We construct a 2-edge-colored bipartite outerplanar graph with girth at least g that does not map to a 2-edge-colored planar graph with at most 5 vertices. Let g' be such that $g' \geqslant g$ and $g' \equiv 2 \pmod 4$. Consider an alternating cycle $C = v_0, \cdots, v_{g'-1}$. For every $0 \leqslant i \leqslant g' - 3$, we add $g' - 2$ 2-vertices $w_{i,1}, \cdots, w_{i,g'-2}$ that form the path $P_i = v_i w_{i,1} \cdots w_{i,g'-2} v_{i+1}$ such that the edges of P_i get the color distinct from the color of the edge $v_i v_{i+1}$. Let G be the obtained graph. The 2-edge-colored chromatic number of C is 5. So without loss of generality, we assume for contradiction that G admits a homomorphism φ to a 2-edge-colored planar graph H on 5 vertices. Let us define $\mathcal{E} = \bigcup_{i \texttt{ even}} \varphi(v_i)$ and $\mathcal{O} = \bigcup_{i \texttt{ odd}} \varphi(v_i)$. Since C is alternating, $\varphi(v_i) \neq \varphi(v_{i+2})$ (indices are modulo g'). Since $g' \equiv 2 \pmod 4$, there is an odd number of v_i with an even (resp. odd) index. Thus, $|\mathcal{E}| \geqslant 3$ and $|\mathcal{O}| \geqslant 3$. Therefore we must have $\mathcal{E} \cap \mathcal{O} \neq \emptyset$.

Notice that every two vertices v_i and v_j in G are joined by a blue path and a red path such that the lengths of these paths have the same parity as $i - j$. Thus, the blue (resp. red) edges of H must induce a connected spanning subgraph of H. Since $|V(H)| = 5$, H contains at least 4 blue (resp. red) edges. Since red and blue edges play symmetric roles in G and since $|E(H)| \leqslant 9$ by Proposition 8, we assume without loss of generality that H contains exactly 4 blue edges. Moreover, these 4 blue edges induce a tree. In particular, the blue edges induce a bipartite graph which partitions $V(H)$ into 2 parts. Thus, every v_i with even index is mapped into one part of $V(H)$ and every v_i with odd index is mapped into the other part of $V(H)$. So $\mathcal{E} \cap \mathcal{O} = \emptyset$, which is a contradiction.

5. Proof of Theorem 2

Let T be a $P_g^{(m,n)}$-universal planar graph for some g that is minimal with respect to the subgraph order.

By minimality of T, there exists a graph $G \in P_g^{(m,n)}$ such that every color in T has to be used at least once to color G. Without loss of generality, G is connected, since otherwise we can replace G by the connected graph obtained from G by choosing a vertex in each component of G and identifying them. We obtain a graph G' from G as follows:

For each edge or arc uv in G, we keep uv in G' and we add $4m + n$ paths starting at u and ending at v made of vertices of degree 2:

- For each type of edge, we add a path made of $g - 1$ edges of this type.
- For each type of arc, we add two paths made of $g - 1$ arcs of this type such that the paths alternate between forward and backward arcs. We make the paths such that u is the tail of the first arc of one path and the head of the first arc of the other path.
- Similarly, for each type of arc we add two paths made of g arcs of this type such that the paths alternate between forward and backward arcs. We make the paths such that u is the tail of the first arc of one path and the head of the first arc of the other path.

Notice that G' is in $P_g^{(m,n)}$ and thus admits a homomorphism φ to T. Since G is a connected subgraph of G' and every color in T has to be used at least once to color G, we can find for each pair of vertices (c_1, c_2) in T and each type of edge a path $(v_1, v_2, \cdots, v_l)$ in G' made only of edges of this type such that $\varphi(v_1) = c_1$ and $\varphi(v_l) = c_2$.

This implies that for every pair of vertices (c_1, c_2) in T and each type of edge, there exists a walk from c_1 to c_2 made of edges of this type. Therefore, for $1 \leqslant j \leqslant n$, the subgraph induced by $E_j(T)$ is connected and contains all the vertices of T. So $E_j(T)$ contains a spanning tree of T. Thus T contains at least $|V(T)| - 1$ edges of each type.

Similarly, we can find for each pair of vertices (c_1, c_2) in T and each type of arc a path of even length $(v_1, v_2, \cdots, v_{2l-1})$ in G' made only of arcs of this type, starting with a forward arc and alternating between forward and backward arcs such that $\varphi(v_1) = c_1$ and $\varphi(v_l) = c_2$. We can also find a path of the same kind with odd length.

This implies that for every pair of vertices (c_1, c_2) in T and each type of arc there exist a walk of odd length and a walk of even length from c_1 to c_2 made of arcs of this type, starting with a forward arc and alternating between forward and backward arcs. Let p be the maximum of the length of all these paths. Given one of these walks of length l, we can also find a walk of length $l+2$ that satisfies the same constraints by going through the last arc of the walk twice more. Therefore, for every $l \geqslant p$, every pair of vertices (c_1, c_2) in T, and every type of arc, it is possible to find a homomorphism from the path P of length l made of arcs of this type, starting with a forward arc and alternating between forward and backward arcs to T such that the first vertex is colored in c_1 and the last vertex is colored in c_2.

We now show that this implies that $|A_j(T)| \geqslant 2|V(T)| - 1$ for $1 \leqslant j \leqslant m$. Let P be a path $(v_1, v_2, \cdots, v_p, v_{p+1})$ of length p starting with a forward arc and alternating between forward and backward arcs of the same type. We color v_1 in some vertex c of T. Let C_i be the set of colors in which vertex v_i could be colored. We know that $C_1 = c$ and C_2 is the set of direct successors of c. Set C_3 is the set of direct predecessors of vertices in C_2 so $C_1 \subseteq C_3$ and, more generally, $C_i \subseteq C_i + 2$. Let uv be an arc in T. If $u \in C_i$ with i odd, then $v \in C_{i+1}$. If $v \in C_i$ with i even then $u \in C_{i+1}$. We can see that uv is capable of adding at most one vertex to a C_i (and every C_j with $j \equiv i \mod 2$ and $i \leqslant j$). We know that $C_{p+1} = V(T)$ hence T contains at least $2|V(T)| - 1$ arcs of each type.

Therefore, the underlying graph of T contains at least $m\,(2|V(T)| - 1) + n\,(|V(T)| - 1) = (2m+n)\,|V(T)| - m - n$ edges, which contradicts Proposition 8 for $2m + n \geqslant 3$.

6. Proof of Theorem 5.1

We construct an oriented bipartite 2-outerplanar graph with girth 14 that does not map to $\overrightarrow{T_5}$.

The oriented graph X is a cycle on 14 vertices $v_0, \cdots, v_{13}$ such that the tail of every arc is the vertex with even index, except for the arc $\overrightarrow{v_{13}v_0}$. Suppose for contradiction that X has a $\overrightarrow{T_5}$-coloring h such that no vertex with even index maps to b. The directed path $v_{12}v_{13}v_0$ implies that $h(v_{12}) \neq h(v_0)$. If $h(v_0) = a$, then $h(v_1) \in \{b, c\}$ and $h(v_2) = a$ since $h(v_2) \neq b$. By contagion, $h(v_0) = h(v_2) = \cdots = h(v_{12}) = a$, which is a contradiction. Thus $h(v_0) \neq a$. If $h(v_0) = c$, then $h(v_1) = d$ and $h(v_2) = c$ since $h(v_2) \neq b$. By contagion, $h(v_0) = h(v_2) = \cdots = h(v_{12}) = c$, which is a contradiction. Thus $h(v_0) \neq c$. So $h(v_0) \notin \{a, b, c\}$, that is, $h(v_0) \in \{d, e\}$. Similarly, $h(v_{12}) \in \{d, e\}$. Notice that $\overrightarrow{T_5}$ does not contain a directed path xyz such that x and z belong to $\{d, e\}$. So the path $v_{12}v_{13}v_0$ cannot be mapped to $\overrightarrow{T_5}$. Thus X does not have a $\overrightarrow{T_5}$-coloring h such that no vertex with even index maps to b.

Consider now the path P on 7 vertices $p_0, \cdots, p_6$ with the arcs $\overrightarrow{p_1p_0}$, $\overrightarrow{p_1p_2}$, $\overrightarrow{p_3p_2}$, $\overrightarrow{p_4p_3}$, $\overrightarrow{p_5p_4}$, $\overrightarrow{p_5p_6}$. It is easy to check that there exists no $\overrightarrow{T_5}$-coloring h of P such that $h(p_0) = h(p_6) = b$.

We construct the graph Y as follows: we take 8 copies of X called $X_{\mathtt{main}}$, X_0, X_2, X_4, $\cdots$, X_{12}. For every couple $(i, j) \in \{0, 2, 4, 6, 8, 10, 12\}^2$, we take a copy $P_{i,j}$ of P, we identify the vertex p_0 of $P_{i,j}$ with the vertex v_i of $X_{\mathtt{main}}$ and we identify the vertex p_6 of $P_{i,j}$ with the vertex v_j of H_i.

So Y is our oriented bipartite 2-outerplanar graph with girth 14. Suppose for contradiction that Y has a $\overrightarrow{T_5}$-coloring h. By previous discussion, there exists $i \in \{0, 2, 4, 6, 8, 10, 12\}$ such that the vertex v_i of $X_{\texttt{main}}$ maps to b. Also, there exists $j \in \{0, 2, 4, 6, 8, 10, 12\}$ such that the vertex v_j of X_i maps to b. So the corresponding path $P_{i,j}$ is such that $h(p_0) = h(p_6) = b$, a contradiction. Thus Y does not map to $\overrightarrow{T_5}$.

7. Proof of Theorem 5.2

We construct a 2-edge-colored 2-outerplanar graph with girth 11 that does not map to T_6. We take 12 copies $X_0, \cdots, X_{11}$ of a cycle of length 11 such that every edge is red. Let $v_{i,j}$ denote the j^{th} vertex of X_i. For every $0 \leqslant i \leqslant 10$ and $0 \leqslant j \leqslant 10$, we add a path consisting of 5 blue edges between $v_{i,11}$ and $v_{j,i}$.

Notice that in any T_6-coloring of a red odd cycle, one vertex must map to c. So we suppose without loss of generality that $v_{0,11}$ maps to c. We also suppose without loss of generality that $v_{0,0}$ maps to c. The blue path between $v_{0,11}$ and $v_{0,0}$ should map to a blue walk of length 5 from c to c in T_6. Since T_6 contains no such walk, our graph does not map to T_6.

8. Proof of Theorem 5.3

We construct a 2-edge-colored bipartite 2-outerplanar graph with girth 10 that does not map to T_6. By Theorem 4.2, there exists a bipartite outerplanar graph M with girth at least 10 such that for every T_6-coloring h of M, there exists a vertex v in M such that $h(v) = c$.

Let X be the graph obtained as follows. Take a main copy Y of M. For every vertex v of Y, take a copy Y_v of M. Since Y_v is bipartite, let A and B the two independent sets of Y_v. For every vertex w of A, we add a path consisting of 5 blue edges between v and w. For every vertex w of B, we add a path consisting of 4 edges colored (blue, blue, red, blue) between v and w.

Notice that X is indeed a bipartite 2-outerplanar graph with girth 10. We have seen in the previous proof that T_6 contains no blue walk of length 5 from c to c. We also check that T_6 contains no walk of length 4 colored (blue, blue, red, blue) from c to c. By the property of M, for every T_6-coloring h of X, there exist a vertex v in Y and a vertex w in Y_v such that $h(v) = h(w) = c$. Then h cannot be extended to the path of length 4 or 5 between v and w. So X does not map to T_6.

9. Proof of Theorem 6.1

Let g be the largest integer such that there exists a graph in $P_g^{(1,0)}$ that does not map to $\overrightarrow{T_5}$. Let $G \in P_g^{(1,0)}$ be a graph that does not map to $\overrightarrow{T_5}$ and such that the underlying graph of G is minimal with respect to the homomorphism order.

Let G' be obtained from G by removing an arbitrary arc v_0v_3 and adding two vertices v_1 and v_2 and the arcs v_0v_1, v_2v_1, v_2v_3. By minimality, G' admits a homomorphism φ to $\overrightarrow{T_5}$. Suppose for contradiction that $\varphi(v_2) = c$. This implies that $\varphi(v_1) = \varphi(v_3) = d$. Thus φ provides a $\overrightarrow{T_5}$-coloring of G, a contradiction. So $\varphi(v_2) \neq c$ and, similarly, $\varphi(v_2) \neq e$.

Given a set S of vertices of $\overrightarrow{T_5}$, we say that we force S if we specify a graph H and a vertex $v \in V(H)$ such that for every vertex $x \in V\left(\overrightarrow{T_5}\right)$, we have $x \in S$ if and only if there exists a $\overrightarrow{T_5}$-coloring φ of H such that $\varphi(v) = x$. Thus, with the graph G' and the vertex v_2, we force a non-empty set $\mathcal{S} \subset V\left(\overrightarrow{T_5}\right) \setminus \{c, e\} = \{a, b, d\}$.

We use a series of constructions in order to eventually force the set $\{a, b, c, d\}$ starting from $\mathcal{S}$. Recall that $\{a, b, c, d\}$ induces the tournament $\overrightarrow{T_4}$. We thus reduce $\overrightarrow{T_5}$-coloring to $\overrightarrow{T_4}$-coloring, which is NP-complete for subcubic bipartite planar graphs with any given girth [4].

These constructions are summarized in the tree depicted in Figure 3. The vertices of this forest contain the non-empty subsets of $\{a, b, d\}$ and a few other sets. In this tree, an arc from S_1 to

S_2 means that if we can force S_1, then we can force S_2. Every arc has a label indicating the construction that is performed. In every case, we suppose that S_1 is forced on the vertex v of a graph H_1 and we construct a graph H_2 that forces S_2 on the vertex w.

$$
\begin{tikzcd}
 & & & & bd \arrow[d, "\text{in}"] & & \\
 & & d \arrow[d, "\text{in}"] & & abc \arrow[d, "\text{out}"] & & \\
b \arrow[r, "\text{in}"] & a \arrow[r, "\text{out}"] & bc \arrow[r, "\text{in}"] & ab \arrow[r, "\text{out}"] & bcde \arrow[d, "\text{in}"] & ad \arrow[l, "\text{in}"'] & abd \arrow[l, "\text{Z}"'] \\
 & & & & abcd & &
\end{tikzcd}
$$

Figure 3: Forcing the set $\{a, b, c, d\}$.

- Arcs labelled "out": The set S_2 is the out-neighborhood of S_1 in $\overrightarrow{T_5}$. We construct H_2 from H_1 by adding a vertex w and the arc vw. Thus, S_2 is indeed forced on the vertex w of H_2.

- Arcs labelled "in": The set S_2 is the in-neighborhood of S_1 in $\overrightarrow{T_5}$. We construct H_2 from H_1 by adding a vertex w and the arc wv. Thus, S_2 is indeed forced on the vertex w of H_2.

- Arc labelled "Z": Let g' be the smallest integer such that $g' \geqslant g$ and $g' \equiv 4 \pmod 6$. We consider a circuit $v_1, \cdots, v_{g'}$. For $2 \leqslant i \leqslant g'$, we take a copy of H_1 and we identify its vertex v with v_i. We thus obtain the graph H_2 and we set $w = v_2$. Let φ be any T_6-coloring of H_2. By construction, $\{\varphi(v_2), \cdots, \varphi(v_{g'})\} \subset S_1 = \{a, b, d\}$. A circuit of length $\not\equiv 0 \pmod 3$ cannot map to the 3-circuit induced by $\{a, b, d\}$, so $\varphi(v_1) \in \{c, e\}$. If $\varphi(v_1) = c$ then $\varphi(v_2) = d$ and if $\varphi(v_1) = e$ then $\varphi(v_2) = a$. Thus $S_2 = \{ad\}$.

10. Proof of Theorem 6.2

Let g be the largest integer such that there exists a graph in $P_g^{(0,2)}$ that does not map to T_6. Let $G \in P_g^{(0,2)}$ be a graph that does not map to T_6 and such that the underlying graph of G is minimal with respect to the homomorphism order.

Let G' be obtained from G by subdividing an arbitrary edge v_0v_3 twice to create the path $v_0v_1v_2v_3$ such that the edges v_0v_1 and v_1v_2 are red and the edge v_2v_3 gets the color of the original edge v_0v_3. By minimality, G' admits a homomorphism φ to T_6. Suppose for contradiction that $\varphi(v_1) = f$. This implies that $\varphi(v_0) = \varphi(v_2) = b$. Thus φ provides a T_6-coloring of G, a contradiction.

Given a set S of vertices of T_6, we say that we force S if we specify a graph H and a vertex $v \in V(H)$ such that for every vertex $x \in V(T_6)$, we have $x \in S$ if and only if there exists T_6-coloring φ of H such that $\varphi(v) = x$. Thus, with the graph G' and the vertex v_1, we force a non-empty set $\mathcal{S} \subset V(T_6) \setminus \{f\} = \{a, b, c, d, e\}$.

Recall that the core of a graph is the smallest subgraph which is also a homomorphic image. We say that a subset S of $V(T_6)$ is *good* if the core of the subgraph induced by S is isomorphic to the graph T_4 which is a a clique on 4 vertices such that both the red and the blue edges induce a

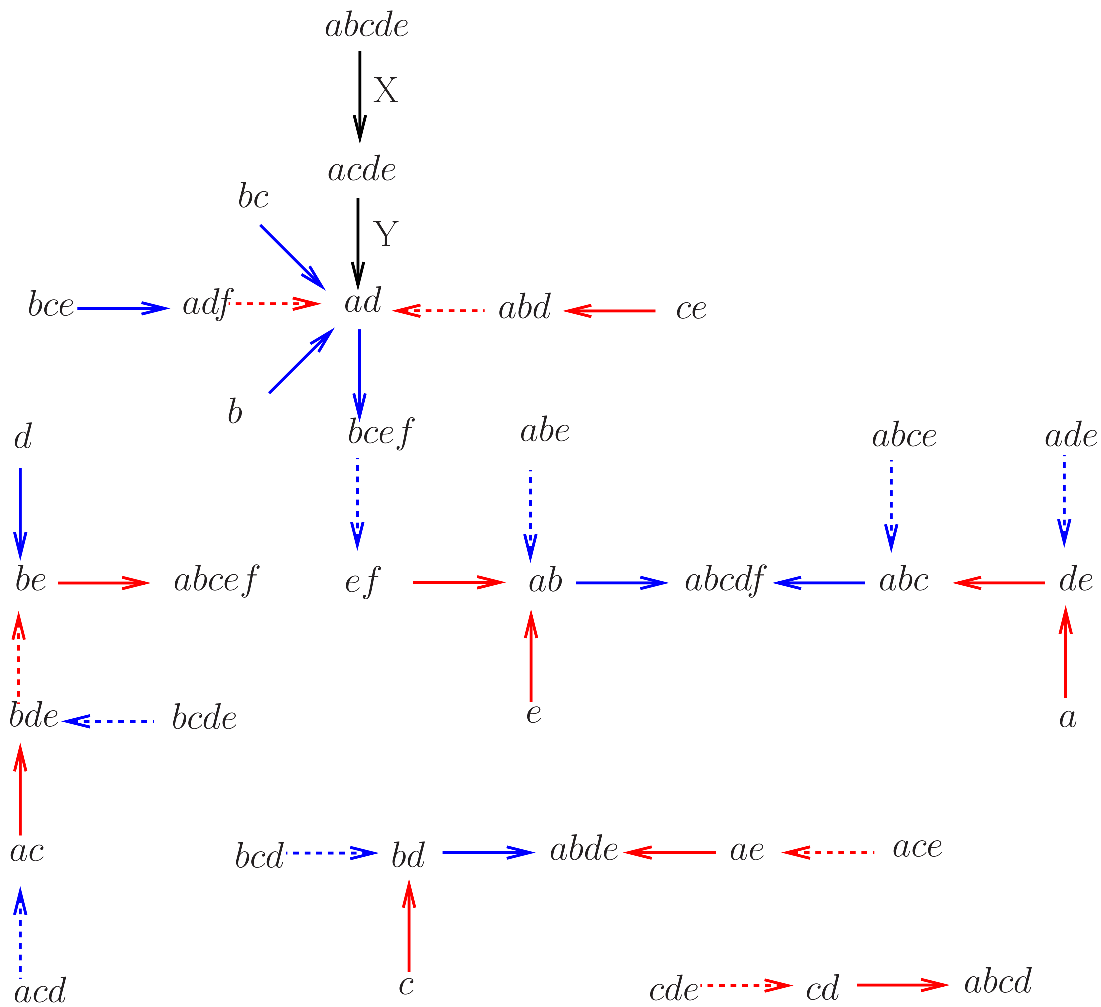

Figure 4: Forcing a good set.

path of length 3. We use a series of constructions in order to eventually force a good set starting from $\mathcal{S}$. We thus reduce T_6-coloring to T_4-coloring, which is NP-complete for subcubic bipartite planar graphs with any given girth [5].

These constructions are summarized in the forest depicted in Figure 4. The vertices of this forest are the non-empty subsets of $\{a, b, c, d, e\}$ together with a few auxiliary sets of vertices containing f. In this forest, an arc from S_1 to S_2 means that if we can force S_1, then we can force S_2. Every set with no outgoing arc is good. We detail below the construction that is performed for each arc. In every case, we suppose that S_1 is forced on the vertex v of a graph H_1 and we construct a graph H_2 that forces S_2 on the vertex w.

- Blue arcs: The set S_2 is the blue neighborhood of S_1 in T_6. We construct H_2 from H_1 by adding a vertex w adjacent to v such that vw is blue. Thus, S_2 is indeed forced on the vertex w of H_2.

- Red arcs: The set S_2 is the red neighborhood of S_1 in T_6. The construction is as above except that the edge vw is red.

- Dashed blue arcs: The set S_2 is the set of vertices incident to a blue edge contained in the subgraph induced by S_1 in T_6. We construct H_2 from two copies of H_1 by adding a blue edge between the vertex v of one copy and the vertex v of the other copy. Then w is one of the vertices v.

- Dashed red arcs: The set S_2 is the set of vertices incident to a red edge contained in the subgraph induced by S_1 in T_6. The construction is as above except that the added edge is red.

- Arc labelled "X": Let $g' = 2\lceil g/2 \rceil$. We consider an even cycle $v_1, \cdots, v_{g'}$ such that $v_1 v_{g'}$ is red and the other edges are blue. For every vertex v_i, we take a copy of H_1 and we identify its vertex v with v_i. We thus obtain the graph H_2 and we set $w = v_1$. Let φ be any T_6-coloring of H_2. In any T_6-coloring of H_2, the cycle $v_1, \cdots, v_{g'}$ maps to a 4-cycle with exactly one red edge contained in the subgraph of T_6 induced by $S_1 = \{a, b, c, d, e\}$. These 4-cycles are $aedb$ with red edge ae and $cdba$ with red edge cd. Since w is incident to the red edge in the cycle $v_1, \cdots, v_{g'}$, w can be mapped to a, e, c, or d but not to b. Thus $S_2 = \{a, c, d, e\}$.

- Arc labelled "Y": We consider an alternating cycle $v_0, \cdots, v_{8g-1}$. For every vertex v_i, we take a copy of H_1 and we identify its vertex v with v_i. We obtain the graph H_2 by adding the vertex x adjacent to v_0 and v_{4g+2} such that xv_0 and xv_{4g+2} are blue. We set $w = v_0$. In any T_6-coloring φ of H_2, the cycle $v_1, \cdots, v_{g'}$ maps to the alternating 4-cycle $acde$ contained in $S_1 = \{a, c, d, e\}$ such that $\varphi(v_i) = \varphi(v_{i+4 \pmod{8g}})$. So, a priori, either $\{\varphi(v_0), \varphi(v_{4g+2})\} = \{a, d\}$ or $\{\varphi(v_0), \varphi(v_{4g+2})\} = \{c, e\}$. In the former case, we can extend φ to H_2 by setting $\varphi(x) = b$. In the latter case, we cannot color x since c and e have no common blue neighbor in T_6. Thus, $\{\varphi(v_0), \varphi(v_{4g+2})\} = \{a, d\}$ and $S_2 = \{a, d\}$.

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