Homomorphisms of planar graphs

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1 Homomorphisms

The graphs that we consider are simple graphs: no loop, no multiple edges, no orientation. A homomorphism from G to H is a mapping m from $V(G)$ to $V(H)$ such that if uv is an edge of G, then $m(u)m(v)$ is an edge of H. This notion generalizes the proper coloring problem with n colors: the chromatic number of a graph is at most n if and if it admits a homomorphism to the n-vertex clique K_n.

For every graph H, H-COLORING is the problem of deciding whether an input graph G admits a homomorphism to H.

Theorem 1. *[1] If H is bipartite then H-COLORING is polynomial. If H is not bipartite then H-COLORING is NP-complete.*

Notice that G has a homomorphism to a bipartite graph H if and only if G is bipartite itself, that is, G has a homomorphism to K_2. To generalize this remark, we need the notion of *core*. A graph H is a core if H admits no homomorphism to proper subgraph of G. For example, the octahedron $K_{2,2,2}$ is not a core since it contains a triangle and its chromatic number is 3. More generally, if a perfect graph H is a core, then H is a clique. It is not hard to see that if H is not a core, then it contains a (unique) core as an induced subraph, which is thus called *the core of H*. Then, a graph G admits a homomorphism to H if and and only if G admits a homomorphism to the core of H.

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Thus, in the study of H-COLORING, the graph H can be assumed to be a core. Now we can re-phrase Theorem 1.

Theorem 2. *[1] Let H be a core. H-COLORING is polynomial if H is K_1 or K_2. H-COLORING is NP-complete otherwise.*

2 Planar inputs

The stage is about PLANAR H-COLORING, which asks whether an input planar graph admits a homomorphism to H. Let us review the known results. Again, H is assumed to be a core. Obviously, PLANAR K_n-COLORING is polynomial for every $n \geqslant 4$.

Theorem 3. *[1] If H is an odd hole, or if H has girth 5 and maximum degree 3, then PLANAR H-COLORING is NP-complete.*

An interesting polynomial case is given by the Clebsch graph which is triangle-free.

Theorem 4. *[3] Every triangle-free planar graph has a homomorphism to the Clebsch graph.*

Thus, if H the Clebsch graph, then PLANAR H-COLORING is polynomial since it is equivalent to test if G contains a triangle.

3 The stage

The goal of the stage is to determine the complexity of PLANAR H-COLORING for as many cores H as possible. If a particular core is given, it is often not too difficult to determine the complexity of PLANAR H-COLORING. Thus, we are also interested in finding general techniques to prove NP-completeness for large families of cores.

References

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