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On non-repetitive sequences of arithmetic progressions: The cases $k \in \{4, 5, 6, 7, 8\}$

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ABSTRACT

A *d-subsequence* of a sequence $\varphi = x_1 \dots x_n$ is a subsequence $x_i x_{i+d} x_{i+2d} \dots$, for any positive integer d and any i, $1 \leqslant i \leqslant n$. A *k-Thue sequence* is a sequence in which every d-subsequence, for $1 \leqslant d \leqslant k$, is non-repetitive, i.e. it contains no consecutive equal subsequences. In 2002, Grytczuk proposed a conjecture that for any k, $k+2$ symbols are enough to construct a k-Thue sequence of arbitrary lengths. So far, the conjecture has been confirmed for $k \in \{1, 2, 3, 5\}$. Here, we present two different proving techniques, and confirm it for all k, with $2 \leqslant k \leqslant 8$.

1. Introduction

A *repetition* in a sequence φ is a subsequence $\rho = x_1 \dots x_{2t}$ of consecutive terms of φ such that $x_i = x_{t+i}$ for every $i = 1, \dots, t$. The length of a repetition is hence always even and comprised of two identical *repetition blocks*, $\rho_1 = x_1 \dots x_t$ and $\rho_2 = x_{t+1} \dots x_{2t}$. A sequence is called *non-repetitive* or *Thue* if it does not contain any repetition. Surprisingly, as shown by Thue [14] (see [2] for a translation), having three distinct symbols suffices to construct non-repetitive sequences of arbitrary lengths. This result is a fundamental piece in the theory of combinatorics on words. After that, a number of other concepts related to repetitions has been presented (see e.g. [3] for more details).

In this paper, we continue dealing with the following generalization. A (possibly infinite) sequence φ is *k-Thue* (or *non-repetitive up to mod k*) if every d-subsequence of φ is Thue, for $1 \leqslant d \leqslant k$. By a *d-subsequence* of φ we mean an arithmetic subsequence $x_i x_{i+d} x_{i+2d} \dots$ of φ. Consider a sequence

$$a\ \underline{b}\ d\ \underline{c}\ b\ \underline{c},$$

which is Thue, but not 2-Thue, since the 2-subsequence $b\ c\ c$ is not Thue. On the other hand,

$$\underline{a}\ b\ c\ \underline{a}\ d\ b$$

is 2-Thue, but not 3-Thue, due to the repetition in the 3-subsequence $a\ a$.

This generalization was introduced by Currie and Simpson [7] and has been immediately followed by an intriguing conjecture due to Grytczuk [9].

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Conjecture 1 (*Grytczuk, 2002*). *For any positive integer k, $k+2$ distinct symbols suffice to construct a k-Thue sequence of any length.*

It is easy to show that having only $k+1$ symbols there is a repetition in any sequence of length at least $2k+2$, so the bound $k+2$ is tight.

Since 1-Thue sequences are simply Thue sequences, the above mentioned result establishes the conjecture for $k=1$. The conjecture has also been confirmed for $k=2$ in [7] and independently in [13], for $k=3$ in [7], and for $k=5$ in [5]. Although it has been considered also for the case $k=4$ by Currie and Pierce [6] using an application of the fixing block method, it remains open for all the cases except $k \in \{1,2,3,5\}$.

Several upper bounds have been established, first being $e^{33}k$ due to Grytczuk [9], and then substantially improved to $2k+O(\sqrt{k})$ in [10]. Currently the best known upper bound is due to Kranjc et al. [13].

Theorem 1 (*Kranjc et al. 2015*). *For any integer $k \geqslant 2$, $2k$ distinct symbols suffice to construct a k-Thue sequence of any length.*

The proof of the above is constructive and provides k-Thue sequences of given lengths.

The aim of this paper is two-fold. The main contribution is answering Conjecture 1 in affirmative for several additional values of k.

Theorem 2. *For any $k \in \{4,5,6,7,8\}$, $k+2$ distinct symbols suffice to construct a k-Thue sequence of any length.*

We present two different techniques of proving the above theorem. In the former, described in Section 3, we use exhaustive computer search to determine morphisms for each k, $k \in \{4,5,6,7,8\}$, from which we construct k-Thue sequences. In the latter, described in Section 5, we use concatenation of special blocks given by another morphism. The purpose of the latter one is to introduce its ability to deal with larger k's, therefore we only prove the cases $k=4$ and $k=6$. We believe, in the future, it could be used for proving Conjecture 1 for infinitely many values of k.

2. Preliminaries

In this section, we introduce additional terminology and notation used in the paper. Throughout the paper, i and t are used to determine positive integers, unless more details are given.

An $\mathbb{A}$-*sequence* (or simply a *sequence* when the alphabet is known from the context or not relevant) of length t is an ordered tuple of t symbols from some alphabet $\mathbb{A}$. Let $\varphi = x_1 \dots x_t$ be a sequence. A subsequence of φ of consecutive terms $x_i \dots x_j$, for some i,j, $1 \leqslant i \leqslant j \leqslant t$, is denoted by $\varphi(i,j)$. A *term* indicates an element of a sequence at a specified index. A *block* is a subsequence of consecutive terms of some sequence. When we refer to a term as a term of a block, by its index we mean the index of a term in the block. We denote the term at index i in a sequence φ (resp. a block β) by $\varphi(i)$ (resp. $\beta(i)$).

A *prefix* of a sequence $\varphi = x_1 \dots x_r$ is a sequence $\pi = x_1 \dots x_s$, for some integer $s \leqslant r$. A *suffix* is defined analogously. In a sequence φ consider a pair of sequences π and ε such that $\pi\varepsilon$ is a subsequence of φ, π has length at least 1, and ε is a prefix of $\pi\varepsilon$. The *exponent* of $\pi\varepsilon$ is

$$\exp(\pi\varepsilon) = \tfrac{|\pi\varepsilon|}{|\pi|}.$$

If a sequence has exponent p, we call it a *p-repetition*. A sequence is q^+-*free* if it contains no p-repetition such that $p > q$. For sequences over 3-letter alphabets, Dejean [8] proved the following.

Theorem 3 (*Dejean, 1972*). *Over 3-letter alphabets there exist $\frac{7}{4}^+$-free sequences of arbitrary lengths.*

A *morphism* is a mapping μ which assigns to each symbol of an alphabet a sequence. Applied to a sequence φ, $\mu(\varphi)$ is the sequence obtained from φ where every symbol is replaced by its image according to μ. We say that a morphism is *k-uniform* if it maps every symbol from the domain to some sequence of length k.

Given a sequence $\varphi = \beta_1 \dots \beta_t$ comprised of blocks β_i, for $1 \leqslant i \leqslant t$, the *covering subsequence* $\hat{\sigma}$ of a subsequence σ in φ is the subsequence $\varphi(i,j)$, where i is the index of the first term of the block containing the first term of σ, and j is the index of the last term of the block containing the last term of σ.

An *i-shift* of φ is the sequence $\varphi^i = x_{i+1} \dots x_\ell x_1 \dots x_i$, i.e. the sequence φ with the subsequence of the first i elements moved to the end. Let φ be a sequence of length ℓ. We define the *circular sequence* $\varphi_\zeta^{(\ell,t)}$ of order ℓ and length $\ell^2 \cdot t$ as

$$\varphi_\zeta^{(\ell,t)} = \underbrace{\varphi^0\varphi^1 \dots \varphi^{\ell-1} \ \dots \ \varphi^0\varphi^1 \dots \varphi^{\ell-1}}_{t}.$$

We call each subsequence φ^i of $\varphi_\zeta^{(\ell,t)}$ a ζ-*block*.

Apart from concatenation of sequences, we define another sequence combining operation. Let φ_1 and φ_2 be sequences of lengths $\ell \cdot t$. A *sequence wreathing of order* ℓ of φ_1 and φ_2, denoted by $\varphi_1 \wr_\ell \varphi_2$, is consecutive concatenation of ℓ subsequent elements of φ_1 and φ_2, i.e.

$$\varphi_1 \wr_\ell \varphi_2 = \varphi_1(1,\ell)\,\varphi_2(1,\ell) \dots \varphi_1((t-1)\ell+1, t\cdot\ell)\,\varphi_2((t-1)\ell+1, t\cdot\ell).$$

We call the sequences φ_1 and φ_2 the *base* and the *wrap* of sequence wrapping $\varphi_1 \wr_\ell \varphi_2$, respectively. Additionally, the blocks $\varphi_1(i\ell+1,(i+1)\ell)$ and $\varphi_2(i\ell+1,(i+1)\ell)$ are respectively called a *base-block* and a *wrap-block*.

We conclude this section with two lemmas we will use in the forthcoming sections. The former, due to Currie [4], states that insertion of non-repetitive subsequences (over distinct alphabets) into a non-repetitive sequence preserves non-repetitiveness.

Lemma 4 (*Currie, 1991*)**.** *Let $\varphi_0 = x_1 \dots x_t$ be a non-repetitive $\mathbb{A}$-sequence, and $\varphi_1, \dots \varphi_{t+1}$ be non-repetitive $\mathbb{B}$-sequences, where $\mathbb{A}$, $\mathbb{B}$ are disjoint alphabets. Additionally, the length of any φ_i, $1 \leqslant i \leqslant t+1$, may be 0. Then, the sequence $\varphi_1 x_1 \dots \varphi_t x_t \varphi_{t+1}$ is non-repetitive.*

Proving that a non-repetitive sequence φ is k-Thue for some integer $k > 1$, one needs to show that every ℓ-subsequence of φ is non-repetitive for every integer ℓ, $1 \leqslant \ell \leqslant k$. To prove that an ℓ-subsequence is non-repetitive, it suffices to have enough information about φ as we show in the next lemma. Let $\varphi = \beta_1 \dots \beta_t$ be a sequence comprised of blocks β_i, $1 \leqslant i \leqslant t$. We say that a block β_i is *uniquely determined by a subset of terms* if there is no block β_j, $\beta_i \neq \beta_j$, having the same terms at the same positions. E.g., from the construction of circular sequences, we have the following.

Observation 1. *A ζ-block φ^i, $0 \leqslant i \leqslant \ell - 1$, is uniquely determined by one term, i.e., given at least one term of a φ^i, one can determine i.*

We use the following lemma as a tool for proving that some d-subsequence of a Thue sequence does not contain a repetition.

Lemma 5. *Let σ be an ℓ-subsequence of a sequence $\varphi = \beta_1\beta_2 \dots \beta_t$, for some positive integers ℓ and t. Let $\rho_1\rho_2$ be a repetition in σ, and let, for some j, $\gamma_1 = \beta_{j+1} \dots \beta_{j+r}$, $\gamma_2 = \beta_{j+r+1} \dots \beta_{j+2r}$ be the covering sequences of ρ_1 and ρ_2, respectively. If it holds that*

- *the terms of ρ_1 uniquely determine the blocks β_i, for $i \in \{j+1, j+r\}$;*
- *the terms of ρ_2 uniquely determine the blocks β_i, for $i \in \{j+r+1, j+2r\}$;*
- *all the terms of ρ_1, ρ_2 appear in γ_1, γ_2 at the same indices within their blocks, respectively;*

then $\gamma_1\gamma_2$ is a repetition in φ.

Proof. Since all the blocks are uniquely determined and $r > 0$, it follows that $\beta_{j+i} = \beta_{j+r+i}$ for every j. $\square$

3. Technique #1: exhaustive search for morphisms

The aim of this section is to present a compact proof of Theorem 2. For completeness, in the proof, we provide constructions of k-Thue sequences also for $k \in \{2, 3\}$. We used an exhaustive computer search to determine appropriate morphisms which are then applied to appropriate sequences.

Proof of Theorem 2. Let $\mathbb{A}_3$ and $\mathbb{A}_{k+2}$ be alphabets on 3 and $k+2$ letters, respectively. For every k, $2 \leqslant k \leqslant 8$, let a morphism $\mu_k : \mathbb{A}_3^* \to \mathbb{A}_{k+2}^*$ be defined as given below:

- $k = 2$: a 7-uniform morphism

  $\mu_2(0) = \texttt{0310213}$
  $\mu_2(1) = \texttt{0230132}$
  $\mu_2(2) = \texttt{0120321}$

- $k = 3$: a 14-uniform morphism

  $\mu_3(0) = \texttt{10231402310243}$
  $\mu_3(1) = \texttt{01243024130243}$
  $\mu_3(2) = \texttt{01240312401234}$

- $k = 4$: a 12-uniform morphism

  $\mu_4(0) = \texttt{012350412534}$
  $\mu_4(1) = \texttt{012345103245}$
  $\mu_4(2) = \texttt{012340521345}$

- $k = 5$: a 27-uniform morphism

  $\mu_5(0) = \texttt{012345601235460235146023546}$
  $\mu_5(1) = \texttt{012345601234650134625013465}$
  $\mu_5(2) = \texttt{012345061234065123460152346}$

- $k = 6$: a 23-uniform morphism

$$\begin{aligned}
\mu_6(0) &= \texttt{01234560172436501243756}\\
\mu_6(1) &= \texttt{01234560127354061235476}\\
\mu_6(2) &= \texttt{01234560123746510324657}
\end{aligned}$$

- $k = 7$: a 36-uniform morphism

$$\begin{aligned}
\mu_7(0) &= \texttt{012345670812345608721345687201345678}\\
\mu_7(1) &= \texttt{012345670182345601872345618702345687}\\
\mu_7(2) &= \texttt{012345670128345670281345762801345768}
\end{aligned}$$

- $k = 8$: a 30-uniform morphism

$$\begin{aligned}
\mu_8(0) &= \texttt{012345678902315647890312645789}\\
\mu_8(1) &= \texttt{012345678902143675982014365789}\\
\mu_8(2) &= \texttt{012345678019324568079123548679}
\end{aligned}$$

In what follows, we show that $\mu_k(\varphi')$ is k-Thue for every $\left(\frac{7}{4}\right)^+$-free sequence $\varphi' \in \mathbb{A}_3^*$. Using a computer, we have verified the following.

Claim 1. *Let φ be any non-repetitive sequence over $\mathbb{A}_3$ of length at most 40. For each morphism μ_k, $\mu_k(\varphi)$ is k-Thue.*

Next, for every k and d such that $2 \leqslant k \leqslant 8$ and $1 \leqslant d \leqslant k$, we consider every sequence $\delta = x_1x_2x_3x_4$ of length 4 over $\mathbb{A}_3$ and every d-subsequence σ of $\mu_k(\delta)$ such that σ intersects both the prefix $\mu_k(x_1)$ and the suffix $\mu_k(x_4)$ of $\mu_k(\delta)$. We again used a computer to check that if such a d-subsequence appears in two sequences $\mu(\delta)$ and $\mu(\delta')$, where $\delta = x_1x_2x_3x_4$ and $\delta' = x_1'x_2'x_3'x_4'$, then $x_2x_3 = x_2'x_3'$.

Thus, long enough d-subsequences of $\mu_k(\varphi)$ allow to determine φ, except maybe for the first and the last term of φ. So, if a large repetition ρ occurs in some d-subsequence of $\mu_k(\varphi)$, then φ contains a factor uvu such that u is large and $|v| \leqslant 2$. For $|u| \geqslant 7$, such a factor uvu cannot appear in a $\left(\frac{7}{4}\right)^+$-free sequence. On the other hand, if $|u| \leqslant 6$, then the length of φ is at most 18 (including possible first and last term). For such sequences, $\mu_k(\varphi)$ are k-Thue by Claim 1. This completes the proof. □

4. Construction of Thue sequences using hexagonal morphism

Three decades after Thue, Aršon [1] presented another construction of infinite non-repetitive sequences using three symbols. The construction has later been subject of other studies; e.g. it has been shown that it is $\frac{7}{4}^+$-free [11].

Recently, in his master's thesis [12], Kočiško rediscovered Aršon's construction. Namely, they presented a uniform morphism κ, which maps a term x of a sequence to a block of three symbols regarding the mapping of the predecessor of x. In particular, instead of using an alphabet $\mathbb{A} = \{1, 2, 3\}$ an auxiliary alphabet

$$\overline{\mathbb{A}} = \left\{\overline{1}, \underline{1}, \overline{2}, \underline{2}, \overline{3}, \underline{3}\right\}$$

is used. The morphism κ is then defined as

$$\begin{aligned}
\kappa(\overline{1}) &= \overline{1}\ \underline{2}\ \overline{3}\,, &\quad \kappa(\overline{2}) &= \overline{2}\ \underline{3}\ \overline{1}\,, &\quad \kappa(\overline{3}) &= \overline{3}\ \underline{1}\ \overline{2}\,,\\
\kappa(\underline{1}) &= \underline{3}\ \overline{2}\ \underline{1}\,, & \kappa(\underline{2}) &= \underline{1}\ \overline{3}\ \underline{2}\,, & \kappa(\underline{3}) &= \underline{2}\ \overline{1}\ \underline{3}\,.
\end{aligned}$$

For a positive integer t, we recursively define the sequence

$$\overline{\varphi}_\kappa^t = \kappa(\overline{\varphi}_\kappa^{t-1})\,,$$

where $\overline{\varphi}_\kappa^0 = \overline{1}$. So, for example, we have

$$\begin{aligned}
\overline{\varphi}_\kappa^1 &= \kappa(\overline{1}) = \overline{1}\ \underline{2}\ \overline{3}\\
\overline{\varphi}_\kappa^2 &= \kappa(\overline{1}\ \underline{2}\ \overline{3}) = \overline{1}\ \underline{2}\ \overline{3}\ \ \underline{1}\ \overline{3}\ \underline{2}\ \ \overline{3}\ \underline{1}\ \overline{2}\,.
\end{aligned}$$

Notice that for every t, every symbol from $\overline{\mathbb{A}}$ is a neighbor of at most two symbols of $\overline{\mathbb{A}}$ (if $t > 3$, then precisely two); we say that neighboring symbols are *adjacent*. The adjacency is also preserved between the blocks of three symbols to which the symbols from $\overline{\mathbb{A}}$ are mapped by κ; we denote these blocks $\overline{\kappa}$*-triples*. Due to its structure, we refer to κ as the *hexagonal morphism*. In Fig. 1, the adjacencies between the symbols and the $\overline{\kappa}$-triples, and the mappings of κ are depicted.

Let $\pi\ :\ \overline{\mathbb{A}} \to \mathbb{A}$ be a projection of symbols from the auxiliary alphabet $\overline{\mathbb{A}}$ to $\mathbb{A}$ defined as $\pi(\overline{a}) = a$ and $\pi(\underline{a}) = a$, for every $a \in \{1, 2, 3\}$. By φ_κ^t, we denote the projected sequence $\overline{\varphi}_\kappa^t$, i.e. $\varphi_\kappa^t = \pi(\overline{\varphi}_\kappa^t)$; similarly a projected $\overline{\kappa}$-triple τ, $\pi(\tau)$, is referred to as a κ-block.

By the definition of $\varphi_\kappa^t = \{x_i\}_{i=1}^{3^t}$ and the mapping κ, one can easily derive the following basic properties:

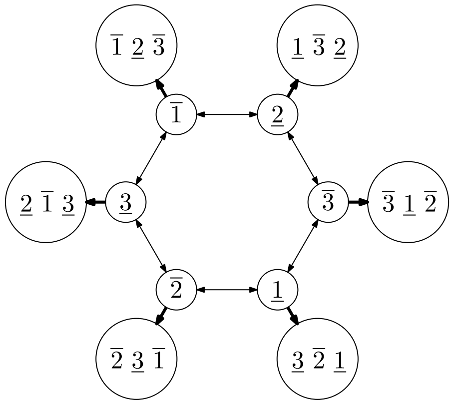

Fig. 1. The graph of adjacencies between the symbols of $\overline{\mathbb{A}}$ and $\overline{\kappa}$-triples, and the mappings defined by κ.

(K_1) For every pair of adjacent κ-blocks τ and σ, the sequence $\tau\sigma$ is Thue.
(K_2) The length of φ_κ^t is 3^t, and $x_{3i+1}x_{3i+2}x_{3i+3}$ is a κ-block for every i, $0 \leqslant i < 3^{t-1}$.
(K_3) $\{x_{3i+1}, x_{3i+2}, x_{3i+3}\} = \{1, 2, 3\}$ for every i, $0 \leqslant i < 3^{t-1}$.
(K_4) $x_{3i+2} \neq x_{3(i+1)+2}$ for every i, $0 \leqslant i < 3^{t-1} - 1$.
(K_5) Any three consecutive terms $x_{j+1}x_{j+2}x_{j+3}$ of φ_κ^t, which do not belong to the same κ-block, uniquely determine the two κ-blocks they belong to.
(K_6) For a pair τ_1, τ_2 of adjacent κ-blocks it holds that the first term of τ_1 is distinct from the third term of τ_2.
(K_7) If a pair of distinct κ-blocks has the same first or last term, then they are adjacent.
(K_8) A pair of adjacent κ-blocks is not adjacent to any other common κ-block.
(K_9) The middle term of the κ-block $\pi(\kappa(i))$, $i \in \overline{\mathbb{A}}$, equals $\pi(i) + 1$ (modulo 3).
(K_{10}) A pair of distinct κ-symbols x_1 and x_2, where x_1 and x_2 are the first (last) terms of adjacent κ-blocks τ_1 and τ_2, uniquely determines τ_1 and τ_2.
(K_{11}) A κ-block τ_1 and at least one term of a κ-block τ_2 adjacent to τ_1 uniquely determine τ_2.
(K_{12}) A pair of adjacent κ-blocks is in φ_κ^t always separated by an even number of κ-blocks, since the graph of adjacencies is bipartite.

We use (some of) the properties above, to prove the following theorem.

Theorem 6 (*Kočiško, 2013*). *The sequence φ_κ^t is Thue, for every non-negative integer t.*

For the sake of completeness, we present a short proof of Theorem 6 here also.

Proof. We prove the theorem by induction. Clearly, φ_κ^0 is Thue. Consider the sequence $\varphi_\kappa^t = \{x_i\}_{i=1}^{3^t}$ and suppose that φ_κ^j is Thue for every $j < t$. Suppose for a contradiction that there is a repetition in φ_κ^t and let $\rho_1\rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$ be a repetition with the minimum length (for later purposes we distinguish two repetition factors, although $\rho_1 = \rho_2$). By (K_1), we have that $r \geqslant 3$. We consider two subcases regarding the length r of $\rho_1 (= \rho_2)$.

Suppose first that r is divisible by 3. Then, as we show in the following claim, we may assume that the term y_1 is the first term of some κ-block.

Claim 2. *Let r be divisible by 3. If $y_1 = x_{3i+2}$ (resp. $y_1 = x_{3i+3}$) for some i, $0 \leqslant i < 3^{t-1}$, then $x_{3i+1}x_{3i+2} \dots x_{3i+2r}$ (resp. $x_{3(i+1)+1}x_{3(i+1)+2} \dots x_{3(i+1)+2r}$) is also a repetition.*

Proof. Suppose that $y_1 = x_{3i+2}$. By (K_3), every κ-block is uniquely determined by two symbols. So $x_{3i+1} = y_r$ and hence $x_{3i+1} \dots x_{3i+2r} = y_r y_1 \dots y_{2r-1}$ is a repetition. A proof for the case $y_1 = x_{3i+3}$ is analogous. ♦

Hence, we have that $\rho = \tau_1 \dots \tau_{\frac{r}{3}}$, where τ_j are κ-blocks for every j, $1 \leqslant j \leqslant \frac{r}{3}$. But in this case, there is a repetition already in φ_κ^{t-1}, contradicting the induction hypothesis.

Therefore, we may assume that r is not divisible by 3. This means that the first terms y_1 and y_{r+1} of the two repetition factors ρ_1 and ρ_2, respectively, are at different positions within the κ-blocks they belong to. For example, if $r = 3k + 1$, and y_1 is the first term of the κ-block $y_1y_2y_3$, then y_{r+1} is the second term of the κ-block $y_r y_{2r+1} y_{2r+2}$. There are hence six possible cases regarding the position of y_1 and y_{r+1} in their κ-blocks.

Suppose first that y_1 is the first term of the κ-block $x_1x_2x_3$. By (K_3), x_1, x_2, and x_3 are pairwise distinct. Since $\rho_1 = \rho_2$, we thus know three consecutive elements of two κ-blocks (the one of y_{r+1} and the subsequent one). By (K_5), we can determine both κ-blocks, which gives us information about the term y_4. Using (K_5) again, we can determine the κ-block $y_4y_5y_6$, namely $y_4y_5y_6 = x_2x_1x_3$ in the case when $r \equiv 1 \bmod 3$, and $y_4y_5y_6 = x_1x_3x_2$ in the case when $r \equiv 2 \bmod 3$. Using the information obtained by determining κ-blocks using (K_5), we infer that every κ-block of ρ_1 ends with x_3 in the former case, or starts with x_1 in the latter case. As ρ_1 and ρ_2 are concatenated, this leads us to contradiction on the existence of a repetition. With a similar argument, we obtain a contradiction in the case when y_{r+1} is the first term of its κ-block.

Suppose now that $r = 3k + 1$, for some positive integer k, and y_1 is the second term of its κ-block, say $x_3x_1x_2$. Then $y_{r+1} = x_1$ and $y_{r+2} = x_2$, where y_{r+1} and y_{r+2} belong to distinct κ-blocks. Notice that there are two possibilities for the value of y_{r+3}, namely x_1 and x_3. However, regardless the choice, after determining the κ-block $y_{r+2}y_{r+3}y_{r+4}$ by (K_5), and continue by alternately determining κ-blocks in ρ_1 and ρ_2, as described above, we infer that in both cases, every κ-block in ρ_1 ends with x_2, a contradiction. An analogous analysis may be performed in the last case, when $r = 3k + 2$ and y_1 being the third term of its κ-block. □

5. Technique #2: transposition & cyclic blocks

In this section, we present alternative proofs to answer Conjecture 1 in affirmative for the cases $k = 4$ and $k = 6$. For each of the two cases we present a special morphism and apply it on a non-repetitive sequence. Then, we use sequence wreathing to extend the sequence by circular blocks.

5.1. The case $k = 4$

In this part, to prove the case $k = 4$ in Theorem 2, we combine the sequence φ_κ^t obtained by the hexagonal morphism and the circular sequence $\varphi_\zeta^{(3,3^t)}$ by wreathing. For every positive integer t, we construct the sequence φ_κ^t over the alphabet $\{1, 2, 3\}$, and the sequence $\varphi_\zeta^{(3,3^{t-1})}$ with $\varphi = 4\ 5\ 6$. We then define

$$\varphi_4^t = \varphi_\kappa^{t+1} \wr_3 \varphi_\zeta^{(3,3^{t-1})}.$$

So, for example, we have

$$\begin{aligned}
\varphi_4^1 &= \varphi_\kappa^2 \wr_3 \varphi_\zeta^{(3,3^0)} = 1\,2\,3\ \ 1\,3\,2\ \ 3\,1\,2\ \ \wr_3\ \ 4\,5\,6\ \ 5\,6\,4\ \ 6\,4\,5 \\
&= 1\,2\,3\ \ 4\,5\,6\ \ 1\,3\,2\ \ 5\,6\,4\ \ 3\,1\,2\ \ 6\,4\,5 \\
\varphi_4^2 &= \varphi_\kappa^3 \wr_3 \varphi_\zeta^{(3,3^1)} = \\
&= 1\,2\,3\ \ 4\,5\,6\ \ 1\,3\,2\ \ 5\,6\,4\ \ 3\,1\,2\ \ 6\,4\,5 \\
&= 3\,2\,1\ \ 4\,5\,6\ \ 3\,1\,2\ \ 5\,6\,4\ \ 1\,3\,2\ \ 6\,4\,5 \\
&= 3\,1\,2\ \ 4\,5\,6\ \ 3\,2\,1\ \ 5\,6\,4\ \ 2\,3\,1\ \ 6\,4\,5
\end{aligned}$$

For clarity, we refer to the base-blocks of φ_4^t as *κ-blocks* (recall that the wrap-blocks are called *ζ-blocks*). Additionally, the terms from κ-blocks (resp. ζ-blocks) are called κ-terms (resp. ζ-terms).

Lemma 7. *The sequence φ_4^t is 4-Thue for every positive integer t.*

Proof. Since φ_κ^{t+1} is Thue by Theorem 6, φ_4^t is also Thue by Lemma 4. Thus, it remains to prove that every d-subsequence of φ_4^t is Thue, for every $d \in \{2, 3, 4\}$. Observe first that by (K_1), (K_6), and the definition of circular sequences, every five consecutive terms of φ_4^t are distinct. This in particular means that

(P_1) *there are no repetitions of length 2 or 4 in any d-subsequence of φ_4^t.*

Moreover,

(P_2) *in every d-subsequence of φ_4^t there are at most two consecutive κ-terms or ζ-terms;*
(P_3) *in every d-subsequence of φ_4^t any repetition contains κ-terms and ζ-terms;*
(P_4) *if a κ-term (resp. ζ-term) in a d-subsequence σ of φ_4^t, whose predecessor and successor in σ are ζ-terms (resp. κ-terms), is at index i within its κ-block (resp. ζ-block), then every κ-term (resp. ζ-term) in σ is at index i within its κ-block (resp. ζ-block).*

All the latter three properties are direct corollaries of (P_1) and the fact that every κ-block and ζ-block is of length 3.

Now, we prove that every d-subsequence of φ_4^t is non-repetitive, considering three cases with regard to d. In each case, we assume there is a repetition $\rho_1\rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$ in some d-subsequence σ and eventually reach a contradiction on its existence.

By (P_3), there is at least one ζ-term in ρ_1. Moreover, by the definition of circular sequences and φ_4^t, every three consecutive ζ-terms in ρ_1 (ignoring the κ-terms) are distinct, unless $d = 4$ and the ζ-terms of ρ_1 are at indices 1 and 3 in ζ-blocks. However, in such a case, by construction of circular sequences, without loss of generality, consecutive ζ-terms of ρ_1 are $4\ 4\ 5\ 5\ 6\ 6 \dots$, which means that r must be divisible by 6, to have the same sequence of ζ-terms in ρ_2. This implies that

(P_5) *the number of* ζ*-terms in* ρ_1 *is divisible by* 3,

and consequently, since in ζ-blocks the symbols repeat at the same indices in every third block:

(P_6) *the number of* ζ*-blocks to which the* ζ*-terms of* ρ_1 *belong to in* φ_4^t *is divisible by* 3.

Observe that, by the above properties,

(P_7) *the first terms of* ρ_1 *and* ρ_2 *are either both* κ*-terms or* ζ*-terms, and moreover, they appear at the same index within their blocks in* φ_4^t.

Now, we start the analysis regarding d:

- $d = 2$.
Suppose first that y_1 is the first term of some κ-block. Then, ρ_1 is comprised alternately of two κ-terms (the first and the third terms of a κ-block in φ_4^t) and one ζ-term (the second term of its ζ-block in φ_4^t). Consequently, y_{r+1} is the first term of a κ-block also, and the last term of ρ_1 must be a ζ-term. By (K_3), every κ-block is uniquely determined by two of its terms, hence one can determine all κ-blocks to which the κ-terms of ρ_1 and ρ_2 belong to in φ_4^t. Similarly, all the ζ-blocks, to which ζ-terms of ρ_1 and ρ_2 belong, are uniquely determined by Observation 1. Moreover, since the terms y_r and y_{r+1} belong to different blocks, we can apply Lemma 5 obtaining a contradiction on the existence of $\rho_1\rho_2$.
Suppose now that y_1 is the third term of some κ-block. A similar argument as in the paragraph above shows that y_r is the first term of some κ-block γ_r of φ_4^t, while y_{r+1} is the third term of β_r. Note that the terms y_2, y_3 and y_{r+2}, y_{r+3} uniquely determine the same block γ_2. Consider now the κ-block γ_1 to which y_1 belongs. By (K_7), it is one of the two possible κ-blocks that end with y_1, and since γ_1 and γ_r are adjacent to γ_2, by (K_8), we infer that $\gamma_1 = \gamma_r$. Thus, taking the first term z of γ_1, we have a repetition $zy_1 \dots y_{r-1}y_r \dots y_{2r-1}$ in σ, which satisfies the assumptions of Lemma 5. Hence, there is a repetition in φ_4^t, a contradiction.
Next, suppose that y_1 is the second term of some κ-block. Then, by (P_4), all the κ-terms in ρ_1 and ρ_2 are the second terms of κ-blocks in φ_4^t. By (K_9), we have that the second terms of κ-blocks in φ_4^t are exactly the terms of φ_κ^{t-1} shifted by 1, and thus form a non-repetitive sequence. Using Lemma 4, we infer that the sequence σ is also non-repetitive, a contradiction.
Finally, suppose that y_1 is a ζ-term. Let γ be the last ζ-block in φ_4^t to which some term of ρ_2 belongs. Since a ζ-block is uniquely determined by at least one of its terms, using (P_5), we infer that the ζ-block of φ_4^t following γ is equal to the ζ-block uniquely determined by y_1. Let $y \in \{y_2, y_3\}$ be the first κ-term of ρ_1. The observation above implies that there exists a repetition in σ starting with y and ending with the ζ-terms before y in ρ_1. Such a repetition cannot exist due to the analysis of the cases above.
- $d = 3$.
Suppose that y_1 is the first term of some κ-block. Clearly, the first term of ρ_2 is also a κ-term, and thus the number of κ-terms in ρ_1 is divisible by 3, by (P_5). Therefore, there are at least six κ-terms in $\rho_1\rho_2$, meaning there are two distinct consecutive κ-terms. Using (K_{10}) and (K_{11}), we can uniquely determine all κ-blocks to which the κ-terms of ρ_1 and ρ_2 belong. So, by Lemma 5, we obtain a contradiction.
If y_1 is the third term of some κ-block, we use the same argument as in the paragraph above.
The argument when y_1 is the second term of some κ-block is analogous to the subcase in the case $d = 2$, where y_1 is the second term of some κ-block.
In the case when y_1 is a ζ-term, we can again translate the analysis to the one of the above cases, since the ζ-triples have period 3.
- $d = 4$.
Suppose that y_1 is the first term of some κ-block. By (P_1), the length of ρ_1 is at least 3. Furthermore, y_2 and y_3 are ζ-terms (the second terms of some ζ-block) and a κ-term (the third term of some κ-block), respectively. By (P_4), all ζ-terms in ρ_1 are the second terms of ζ-blocks. Thus, by (P_5) and the fact that for every ζ-term in ρ_1 there are two ζ-blocks in $\hat{\rho}_1$, we have that the number of ζ-blocks in $\hat{\rho}_1$ is divisible by 6. By (P_7), y_{r+1} is also the first term of some κ-block in φ_4^t, meaning that the number of κ-blocks in $\hat{\rho}_1$ is also divisible by 6 and that the number of κ-blocks between the blocks of y_1 and y_{r+1} is odd. Hence, by (K_{12}), the κ-blocks of y_1 and y_{r+1} are the same. Analogously, all the blocks of the κ-terms y_i in ρ_1 are the same as the κ-blocks of y_{i+r} in ρ_2. Thus, there is a repetition in φ_4^t also, a contradiction.
Suppose now that y_1 is the second term of some κ-block. Similarly as in the case above, we notice that all κ-terms of ρ_1 are the second terms in their κ-blocks in $\hat{\rho}_1$, and that the number of κ-blocks in $\hat{\rho}_1$ is divisible by 6. Again, we deduce that for every two κ-terms y_i and y_j in σ, there are even number of κ-blocks between the κ-blocks of y_i and y_j in $\hat{\sigma}$. It follows that every pair of equal κ-symbols in σ belongs to the same κ-block, and hence $\widehat{(\rho_1)} = \hat{\rho}_2$, a contradiction.
The cases, when y_1 is the third term of some κ-block, or the second term of some ζ-block are analogous to the first case. The cases, when y_1 is the first or the third term of some ζ-block are analogous to the second case. □

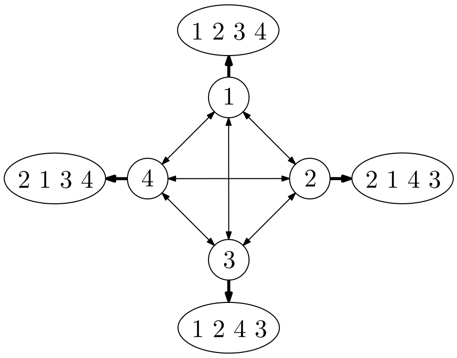

Fig. 2. The graph of adjacencies between the symbols of $\mathbb{B}$ and λ-blocks, and the mappings defined by λ.

5.2. The case $k = 6$

In this part, we present a construction of a 6-Thue sequence using 8 symbols, in a similar way as for the case $k = 4$. Again, we wreath a Thue sequence with a circular sequence, but now, the base sequence is formed by blocks of four symbols, where in each block we only permute symbols in fixed pairs.

Similarly as in Section 4, we start by constructing a Thue sequence over an alphabet

$$\mathbb{B} = \{1, 2, 3, 4\}$$

of 4 symbols. Let a morphism λ, mapping a symbol from the sequence to a block of four distinct symbols, be defined as

$$\lambda(1) = 1\ 2\ 3\ 4\,, \quad \lambda(2) = 2\ 1\ 4\ 3\,, \quad \lambda(3) = 1\ 2\ 4\ 3\,, \quad \lambda(4) = 2\ 1\ 3\ 4\,.$$

For a positive integer t, we recursively define the sequence

$$\varphi_\lambda^t = \lambda(\varphi_\lambda^{t-1})\,,$$

where $\varphi_\lambda^0 = 1$. Notice that for every positive integer t, every symbol from $\mathbb{B}$ is a neighbor of all symbols of $\mathbb{B}$. The blocks of four symbols to which the symbols from $\mathbb{B}$ are mapped by λ, are referred to as *λ-blocks*. In Fig. 2, the mappings of λ are depicted.

We first observe some basic properties of the sequence φ_λ^t, for any positive integer t.

- (L_1) For any pair of adjacent λ-blocks γ_1 and γ_2, the sequence $\gamma_1\gamma_2$ is Thue.
- (L_2) The length of φ_λ^t is 4^t, and $x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4}$ is a λ-block for every i, $0 \leqslant i \leqslant 4^{t-1} - 1$.
- (L_3) $\{x_{4i+1}, x_{4i+2}\} = \{1, 2\}$ and $\{x_{4i+3}, x_{4i+4}\} = \{3, 4\}$ for every i, $0 \leqslant i \leqslant 4^{t-1} - 1$. Consequently, by knowing at least one term at index 1 or 2, and at least one term at index 3 or 4, the λ-block is uniquely determined.
- (L_4) For every i, $0 \leqslant i \leqslant 4^{t-1} - 3$, it holds: $x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4} \neq x_{4i+9}x_{4i+10}x_{4i+11}x_{4i+12}$ (this is in fact a consequence of (L_3)).
- (L_5) Two consecutive λ-blocks with the same first two terms are mapped from $\{1, 3\}$ or $\{2, 4\}$. Similarly, two consecutive λ-blocks with the same last two terms are mapped from $\{1, 4\}$ or $\{2, 3\}$.
- (L_6) Let γ_1 and γ_2 be distinct λ-blocks with equal terms at indices 1 and 2 or at indices 3 and 4. For λ-blocks γ_3, γ_4, and γ_5, in φ_λ^t, there is at most one of the subsequences $\gamma_1\gamma_3\gamma_5$ and $\gamma_2\gamma_4\gamma_5$, since otherwise the property (L_3) would be violated in φ_λ^{t-1}.
- (L_7) If for two λ-blocks $\gamma_1 = x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4}$ and $\gamma_2 = x_{4j+1}x_{4j+2}x_{4j+3}x_{4j+4}$ there is such $\ell \in \{1, 2, 3, 4\}$ that $x_{4i+\ell} = x_{4j+\ell}$ and 4 divides $|j - i|$, then $\gamma_1 = \gamma_2$. On the other hand, if 4 does not divide $|j - i|$, but $|j - i|$ is even, then $\gamma_1 \neq \gamma_2$.
- (L_8) If for a λ-block γ one term is known, then it is one of two possible λ-blocks. In particular, if the known term is at index 1 or 2 in γ, then either $\lambda^{-1}(\gamma) \in \{1, 3\}$ or $\lambda^{-1}(\gamma) \in \{2, 4\}$. If the known term is at index 3 or 4 in γ, then either $\lambda^{-1}(\gamma) \in \{1, 4\}$ or $\lambda^{-1}(\gamma) \in \{2, 3\}$.

We leave the above properties to the reader to verify and proceed by proving that φ_λ^t is Thue.

Lemma 8. *The sequence φ_λ^t is Thue for every non-negative integer t.*

Proof. Suppose the contrary, and let t be the minimum such that there is a repetition in φ_λ^t. Denote the ith term of φ_λ^t by x_i. Let $\rho_1\rho_2 = y_1 \dots y_r y_{r+1} \dots y_{2r}$ be a repetition of minimum length. We first show that $r > 4$. The cases with $r \leqslant 3$ are trivial, so suppose $r = 4$. By (L_1), we have that y_1 is not at index $4i + 1$ in φ_λ^t (for any i, $0 \leqslant i \leqslant 4^{t-1} - 1$), and by (L_3), it is not at index $4i+2$ nor $4i+4$. Hence, assume y_1 is at index $4i+3$. Denote the λ-block $x_{4i+5}x_{4i+6}x_{4i+7}x_{4i+8}(= y_3y_4y_5y_6)$ by γ_1. By (L_1), we have that $\gamma_0 = x_{4i+1}x_{4i+2}x_{4i+3}x_{4i+4} = y_4y_3y_5y_6$ and similarly, $\gamma_2 = x_{4i+9}x_{4i+10}x_{4i+11}x_{4i+12} = y_3y_4y_6y_5$. By (L_5), this means that if $\gamma_1 \in \{1,2\}$, then $\gamma_0, \gamma_2 \in \{3,4\}$, and analogously, if $\gamma_1 \in \{3,4\}$, then $\gamma_0, \gamma_2 \in \{1,2\}$, a contradiction to (L_3). Hence, $r > 4$.

Let j be the index of y_1 in φ_λ^t, i.e. $y_1 = x_j$. If j is odd, then by (L_3), either $x_jx_{j+1} = \{1,2\}$ or $x_jx_{j+1} = \{3,4\}$, and without loss of generality, we may assume the former. Thus, also $x_{j+r}x_{j+r+1} = \{1,2\}$, which implies that r must be even. In the case when j is even, (L_3) similarly implies that $x_j \in \{1,2\}$ and $x_{j+1} \in \{3,4\}$, and hence $x_{j+r} \in \{1,2\}$ and $x_{j+r+1} \in \{3,4\}$. Consequently, r is again even. Finally observe that by (L_2), from r being even and $x_j = x_{j+r}$ it follows that r is divisible by 4.

Suppose now that $j = 4i+1$, for some i. Then, since r is divisible by 4, ρ_1 and ρ_2 are comprised of $\frac{r}{4}$ λ-blocks each, the first starting with x_j. This in turn means that there is a repetition in φ_λ^{t-1} as every λ-block represents one term in φ_λ^{t-1}, a contradiction to the minimality of t.

Next, suppose $j = 4i+2$. By (L_3), we have that $x_{j-1} = x_{j+r-1}$, and hence $\rho_1'\rho_2' = x_{j-1}x_j \dots x_{j+r-1}x_{j+r} \dots x_{j+2r-2}$ is also a repetition in φ_λ^t, where $j-1 = 4i+1$, and hence the reasoning in the above paragraph applies.

Suppose $j = 4i+4$. Then, analogous to the previous case, we infer $x_{j+r} = x_{j+2r}$, and hence $\rho_1'\rho_2' = x_{j+1}\dots x_{j+r}x_{j+r+1}\dots x_{j+2r}$ is also a repetition in φ_λ^t, where $j+1 = 4(i+1)+1$, so the reasoning for $j = 4i+1$ applies again.

Finally, consider the case with $j = 4i+3$. If $r = 8$, from $x_{j+2}x_{j+3}x_{j+4}x_{j+5} = x_{j+10}x_{j+11}x_{j+12}x_{j+13}$ it follows that the λ-block $x_{j+6}x_{j+7}x_{j+8}x_{j+9}$ is surrounded by the same λ-blocks, which contradicts (L_4). Hence, we may assume $r \geqslant 12$. Since the λ-blocks $x_{j+6}x_{j+7}x_{j+8}x_{j+9}$ and $x_{j+r+6}x_{j+r+7}x_{j+r+8}x_{j+r+9}$ are equal, and r is divisible by 4, it follows that also $x_{j-2}x_{j-1}x_jx_{j+1} = x_{j+r-2}x_{j+r-1}x_{j+r}x_{j+r+1}$ and we may apply the reasoning for the case with $j = 4i+1$ on the repetition $x_{j-2}\dots x_{j+2r-3}$. Hence, φ_λ^t is Thue. □

Now, take the circular sequence $\varphi_\zeta^{(4,4^{t-1})}$, with $\varphi = 5\ 6\ 7\ 8$, and use sequence wreathing on φ_λ^{t+1} and $\varphi_\zeta^{(4,4^{t-1})}$ to obtain the sequence

$$\varphi_6^t = \varphi_\lambda^{t+1} \wr_4 \varphi_\zeta^{(4,4^{t-1})}.$$

Similarly as above, we refer to the base-blocks of φ_6^t as λ-blocks, and to the wrap-blocks as ζ-blocks. The terms of λ-blocks (resp. ζ-blocks) are referred to as λ-terms (resp. ζ-terms). The sequence φ_6^1 is hence:

$$\underbrace{1\,2\,3\,4}_{\lambda(1)}\ \ 5\,6\,7\,8\ \ \underbrace{2\,1\,4\,3}_{\lambda(2)}\ \ 6\,7\,8\,5\ \ \underbrace{1\,2\,4\,3}_{\lambda(3)}\ \ 7\,8\,5\,6\ \ \underbrace{2\,1\,3\,4}_{\lambda(4)}\ \ 8\,5\,6\,7$$

It remains to prove that φ_6^t is also 6-Thue.

Lemma 9. *The sequence φ_6^t is 6-Thue for every positive integer t.*

Proof. By Lemmas 4 and 8, we have that φ_6^t is Thue. Thus, we only need to prove that every d-subsequence of φ_6^t is also Thue, for every $d \in \{2,3,4,5,6\}$. First, we list some general properties and then consider d-subsequences separately regarding the values of d.

By (L_3) and the definition of circular sequences, every seven consecutive terms of φ_6^t are distinct. Hence,

(R_1) there are no repetitions of length 2 or 4 in any d-subsequence φ_6^t.

Furthermore, since the length of any λ-block and ζ-block in φ_6^t is 4, one can deduce that:

(R_2) *in every d-subsequence of φ_6^t there are at most two consecutive λ-terms or ζ-terms;*
(R_3) *in every d-subsequence of φ_6^t any repetition contains λ-terms and ζ-terms;*

Given a d-subsequence $\sigma = z_1z_2\dots z_n$ of φ_6^t consisting of n elements, we define a mapping $\vartheta : \Sigma \to \{N, C\}^n$, where Σ represents the set of all d-subsequences of φ_6^t, mapping σ to an n-component vector, ith component being N if z_i belongs to a λ-block and C otherwise (N and C standing for a **n**on-cyclic and **c**yclic element, respectively). We call $\vartheta(\sigma)$ the *type vector* of σ.

(T_1) The type vector of any 2-subsequence contains $CCNN$ or $NNCC$ in the first five components (depending on the position of the first term in the sequence).
(T_2) The type vector of any 4-subsequence equals $NCNC$ or $CNCN$ in the first four components.

Now, suppose the contrary, and let $\rho = \rho_1\rho_2 = y_1\dots y_ry_{r+1}\dots y_{2r}$ be a repetition in some d-subsequence of φ_6^t. We start by analyzing possible values of r.

By (R_1), $r \geqslant 3$, so suppose first that $r = 3$. We will consider the cases regarding the type vectors of ρ. By (R_2) and (R_3), there are six possible type vectors for ρ, namely: $CCN\ CCN$, $CNC\ CNC$, $CNN\ CNN$, $NCC\ NCC$, $NCN\ NCN$, and $NNC\ NNC$. By (T_1) and (T_2), such a sequence does not appear in any ℓ-sequence for $\ell \in \{2,4\}$. Hence, it remains to consider $\ell \in \{3,5,6\}$. Let j, $j \in \{1,2,3,4\}$, be the index of y_1 in the λ- or ζ-block it belongs to.

Table 1
The type vectors of ρ regarding j's and ℓ's in the case $r = 3$, assuming the first term y_1 lies in a λ-block. In the symmetric case, when y_1 is in a ζ-block, the type vector values are simply interchanged.

j / ℓ	3	5	6
1	NNC NCC	NCN CCN	NCC NNC
2	NCC NCN	**NCN NCN**	NCC NNC
3	**NCN NCN**	NCC NCN	NNC CNN
4	NCN CCN	NNC NCC	NNC CNN

In Table 1, we present type vectors regarding j's and ℓ's. The only two type vectors matching the possibilities for the type vectors of ρ are in the cases $(j, \ell) \in \{(3, 3), (2, 5)\}$. In the case $(3, 3)$, the indices of $y_1, \ldots, y_6$ within their blocks are respectively 3, 2, 1, 4, 3, 2. When y_1 belongs to a λ-block, y_2 and y_5 must belong to consecutive ζ-blocks. But, then $y_2 \neq y_5$, due to the construction of circular sequences. On the other hand, if y_1 belongs to a ζ-block, then y_2 is at index 2 in a λ-block and y_5 is at index 3 in a λ-block, so again $y_2 \neq y_5$, due to (L_3).

In the case $(2, 5)$, the indices of $y_1, \ldots, y_6$ within their blocks are respectively 2, 3, 4, 1, 2, 3. Suppose first that y_1 belongs to a ζ-block. Then, y_2 is at index 3 in a λ-block and y_5 is at index 2 in a λ-block, and hence $y_2 \neq y_5$, due to (L_3). Finally, suppose y_1 belongs to a λ-block. Then, y_2 is at index 3 in a ζ-block and y_5 is at index 2 in a ζ-block, however, the two ζ-blocks are not consecutive, and hence $y_2 \neq y_5$. It follows that $r \geqslant 4$.

Using the construction properties of circular sequences, we can obtain additional properties of r regarding the structure of type vectors.

Claim 3. *If there are two consecutive ζ-terms in ρ_1, then 32 divides $d \cdot r$.*

Note that we do not require the two terms being in the same ζ-block.

Proof. We prove the claim by showing that having two consecutive ζ-terms, x_i and x_{i+d}, in ρ_1 imply that the corresponding two ζ-terms , x_j and x_{j+d}, in ρ_2 must appear at the same indices in their ζ-blocks. This fact further implies that the difference between i and j is $(8 \cdot 4)t$ (8 since each pair of λ- and ζ-blocks has 8 terms, and 4, since ζ-blocks have period 4 in $\varphi_{\zeta}^{(4,4^t)}$), for some positive integer t. On the other hand, there are $d \cdot r$ terms between x_i and x_j, and hence 32 divides $d \cdot r$.

We consider the cases regarding d. For $d = 1$, the claim is trivial. For $d = 2$, the pair of terms x_i and x_{i+2} can appear twice in four distinct ζ-blocks. However, since the parity of the indices i and j must be the same in this case, they must appear in the same ζ-block in ρ_2.

In the case $d = 3$ a pair of two symbols appear only once in four distinct ζ-blocks, hence there is nothing to prove. In the case $d = 4$, it is not possible to have two consecutive ζ-terms.

In the cases $d = 5$ and $d = 6$, the two terms belong to two consecutive ζ-blocks. In the former, there is again only one appearance of each pair per four blocks, so it remains to consider the case $d = 6$. There are two possible appearances of a pair, but since the indices must have the same parity, the pair must appear in the same two ζ-blocks. This completes the proof of the claim. ♦

We continue by considering the cases regarding d.

- $d = 2$.
  If there are no two consecutive terms of ρ_1 that belong to the same ζ-block, then $r = 4$ and y_1 is a part of a ζ-block. But in this case, there are two consecutive λ-blocks, uniquely determined by y_2, y_3 and y_6, y_7, which must be equal as $y_2 = y_6$ and $y_3 = y_7$, a contradiction to Lemma 8.
  So, there is at least one ζ-block which contains two terms of ρ_1. By Claim 3, r is divisible by $32/2 = 16$. Suppose y_1 is the first (resp. the second) term of some λ-block. Then, y_2, y_{r+1}, and y_{r+2} are also λ-terms, and hence every λ-block of ρ is uniquely determined. By Lemma 5, it follows there is a repetition in φ_{λ}^t, a contradiction to Lemma 8. Now, suppose y_1 is the third (resp. the fourth) term of some λ-block γ_1. Let γ_2 be the λ-block determined by y_r and $y_r + 1$. Clearly, $\gamma_1 \neq \gamma_2$, otherwise there is a repetition in φ_{λ}^t, by Lemma 5. However, since the third and fourth terms of γ_1 and γ_2 are equal, they are either mapped by λ from $\{1, 4\}$ or $\{2, 3\}$. As the λ-block in σ_1 following γ_1 is the same as the λ-block in σ_2 following γ_2, we obtain a contradiction due to (L_6).
  Finally, suppose y_1 is a part of a ζ-block. Since r is divisible by 16, the number of λ-blocks in each of ρ_1 and ρ_2 is divisible by 4, and since all of them are uniquely determined, we have a repetition in φ_{λ}^t (in fact already in φ_{λ}^{t-1}), a contradiction.
- $d = 3$.
  We first show that there are two consecutive ζ-terms in ρ_1. Suppose the contrary. Then, since $r > 3$ and the fact that the type vectors of ρ_1 and ρ_2 must match, there are two consecutive λ-terms in ρ_1. But, in the type vector, between two pairs of two consecutive λ-terms, for $d = 3$, there are two consecutive ζ-terms, a contradiction.

Hence, we may assume there are two consecutive ζ-terms in ρ_1 and by Claim 3, 32 divides $3r$. Observe also that for $d = 3$, there is at least one term from ρ in every λ-block of the covering sequence of ρ. Then, by (L_7), we infer that all λ-blocks in the covering sequences of ρ_1 are equal to the corresponding λ-blocks in the covering sequences of ρ_2, and hence there is a repetition in φ_λ^{t-1}, a contradiction.

- $d = 4$.
  In this case, all the terms of ρ are at the same indices in their λ- and ζ-blocks. As there is at least one ζ-term, by construction of circular sequences, we have that 32 divides $4r$, and hence 8 divides r. Thus, by (L_7), all λ-blocks in the covering sequences of ρ_1 are equal to the corresponding ones in the covering sequences of ρ_2, and hence there is a repetition in φ_λ^{t-1}, a contradiction.
- $d = 5$.
  In this case, λ- and ζ-blocks of the covering sequence of ρ_1 contain precisely one term from ρ_1 with an exception of every fifth block, which is being skipped. Hence, there are two consecutive λ- or ζ-terms in ρ_1 as soon as $r > 4$. As $r > 3$, the only possible ρ_1 with no consecutive terms of the same type has length 4. However, in such a case, the terms y_1 and y_{r+1} are not of the same type, so $r > 4$.
  Suppose first there are no consecutive ζ-terms in ρ_1. In that case, there are two consecutive λ-terms in ρ_1, and hence also in ρ_2. Moreover, since between every pair of consecutive λ-terms there are two consecutive ζ-terms, the only possible r for such ρ, satisfying also that the type vectors of ρ_1 and ρ_2 are the same, is 8. However, then y_1 and y_{r+1} are both ζ-terms but the difference between their indices in the covering sequence is $5 \cdot 8 = 40$, meaning that $y_1 \neq y_{r+1}$.
  So, we may assume there are two consecutive ζ-terms in ρ_1 and, by Claim 3, 32 divides $5r$ (hence 32 divides r also). By (L_7), all λ-blocks in the covering sequence of ρ_1 that contain one term from ρ_1 are equal to the corresponding λ-blocks in the covering sequence of ρ_2. Furthermore, since in the covering sequence of ρ three out of every four λ-blocks contain one term from ρ, also the λ-block γ_0 without a term is uniquely determined, unless it is the first λ-block of $\hat{\rho}_1$ or the last λ-block of $\hat{\rho}_2$. In the case when γ_0 is determined, the covering sequence of ρ_1 contains the same sequence of λ-blocks as the covering sequence of ρ_2, and so there is a repetition in φ_λ^{t-1}, a contradiction.
  Hence, we may assume γ_0 is not uniquely determined, and without loss of generality, suppose it is the first λ-block of $\hat{\rho}_1$. Since γ_0 is not uniquely determined, it is mapped from either the third or the fourth symbol of some λ-block ξ_0 of φ_λ^{t-1}. In the former case, ξ_0 is completely determined, since $r \geqslant 32$ and one can determine the λ-block following ξ_0 in φ_λ^{t-1}, and hence also γ_0 is completely determined. In the latter case, observe that, $y_2 \dots y_{r+1} y_{r+2} \dots y_{2r} x_{j+5}$ (with $y_{2r} = x_j$) is also a repetition, and considering it, we have all λ-blocks in $\hat{\rho}$ determined, a contradiction.
- $d = 6$.
  In this case, ρ_1 alternately contains two consecutive λ- and two consecutive ζ-terms, with a possible shift in the beginning depending on the index of first term in the covering sequence of ρ. Hence, as $r > 3$, there are always two consecutive ζ-terms in ρ_1 unless $r = 4$ and the type vector of ρ_1 is *CNNC*. However, in that case $y_1 \neq y_{r+1}$ by the construction of circular sequences.
  Thus, we may assume there are two consecutive ζ-terms in ρ_1 and, by Claim 3, 32 divides $6r$, hence 16 divides r. Let $r = 16t$; then the length of the covering sequence of ρ_1 is $6 \cdot 16t = 96t$ and therefore there are $12t$ λ-blocks, where every two out of three consecutive λ-blocks contain a term from ρ_1. By (L_7), all λ-blocks in the covering sequence of ρ_1 that contain one term from ρ_1 are equal to the corresponding λ-blocks in the covering sequence of ρ_2. Recall that a λ-block is not uniquely determined by one term; it can be one of two possible (see (L_8)).
  Let σ^t be the covering sequence of ρ with all ζ-blocks removed and let $\sigma^{t-1} = \lambda^{-1}(\sigma)$. Clearly, σ^{t-1} is a subsequence of φ_λ^{t-1}. Let σ_1^{t-1} and σ_2^{t-1} be the sequences defined analogously for ρ_1 and ρ_2, respectively. As we deduced above, $\sigma_1^{t-1} = z_1 z_2 \dots z_{12t}$ has $12t$ elements. We consider four subcases regarding the index of z_1 in φ_λ^{t-1}. Note that in each of the four cases, for every z_i that is a preimage of some λ-block with one term from ρ, we can uniquely determine which symbol z_i represents simply by (L_8) and the position of z_i in the λ-block of φ_λ^{t-1}. Consequently, every "complete" λ-block of σ_1^{t-1} is uniquely determined by (L_3), since we know at least two of its terms, and in the case, when two terms are known, they are at indices 2 and 3.
  Suppose first z_1 is at index $4i + 1$ in λ^{t-1} for some i. Then, there are $3t$ complete uniquely determined λ-blocks in σ_1^{t-1}, and hence by Lemma 5, there is a repetition also in φ_λ^{t-1}, a contradiction.
  Next, suppose z_1 is at index $4i + 4$ in λ^{t-1}. There are $3t - 1$ complete uniquely determined λ-blocks in σ_1^{t-1} and one λ-block, with 3 terms $z_{12t-2} z_{12t-1} z_{12t}$. However, as argued above, the latter is also uniquely determined, which means that $z_2 \dots z_{24t} w$, where w is the element at index $4i + 4 + 24t + 1$ in φ_λ^{t-1}, is also a repetition, and hence we may use the argumentation for z_1 being at index $4i + 1$.
  Suppose z_1 is at index $4i+3$ in λ^{t-1}. In this case, there are $3t-2$ complete uniquely determined λ-blocks in σ_1^{t-1}, and two λ-blocks having two terms in σ_1^{t-1}. The second one, $z_{12t-1} z_{12t} z_{12t+1} z_{12t+2}$ has the other two terms in σ_2^{t-1}. Now, if $3t$ is divisible by 4, then the λ-block $z_{-1} z_0 z_1 z_2$ equals $z_{12t-1} z_{12t} z_{12t+1} z_{12t+2}$ and we again can shift the sequence to the left as above, obtaining a repetition. Hence $3t$ is not divisible by 4. Consider the λ-blocks $\alpha_1 = z_3 z_4 z_5 z_6$ and $\alpha_1 = z_7 z_8 z_9 z_{10}$. They are uniquely determined and they must be equal to the λ-blocks $z_{3+12t} z_{4+12t} z_{5+12t} z_{6+12t}$ and $z_{7+12t} z_{8+12t} z_{9+12t} z_{10+12t}$, which is not possible due to the parity condition and (L_6).
  Finally, suppose z_1 is at index $4i + 2$ in λ^{t-1}. Again, if $3t$ is divisible by 4, then we shift the sequence by one to the right as the first (incomplete) λ-block in σ_1^{t-1} must match the first (incomplete) λ-block in σ_2^{t-1}, and we obtain

a repetition in φ_{λ}^{t-1}. Otherwise, $3t$ is not divisible by 4, and we obtain a contradiction on the equality of first two complete λ-blocks in σ_1^{t-1} and σ_2^{t-1}. $\square$

6. Discussion

In this paper, we improve the current state of Conjecture 1 by showing that it is true for every integer k between 1 and 8. In particular, we present two proving techniques, which are in their essence similar, but very much different in practice. Namely, the proving technique presented in Section 3 is (provided there are available computing resources) efficient for confirming Conjecture 1 for small k's, since one can employ computing resources to verify small instances, while the statement of the conjecture then holds almost trivially for larger instances. However, to be able to prove Conjecture 1 in general or at least for an infinite number of integers, it will fail.

On the other hand, the method described in Section 5 is more promising. We are using a special construction of Thue sequences with properties allowing to prove that they are also k-Thue. This technique needs more argumentation for proving that the generated sequences are indeed k-Thue, but allows for establishing properties for a bigger set of k's, possibly infinite.

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