Oriented coloring is NP-complete on a small class of graphs

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Abstract

A series of recent papers shows that it is NP-complete to decide whether an oriented graph admits a homomorphism to the tournament $T_4$ on 4 vertices containing a 4-circuit, each time on a smaller graph class. We improve these results by showing that homomorphism to $T_4$ is NP-complete for bipartite planar subcubic graphs of arbitrarily large fixed girth. We also show that push homomorphism is NP-complete for planar graphs with girth 9 and for bipartite planar graphs with girth 8.

1 Introduction

A series of recent papers \cite{6,1,3,2} considers the complexity of deciding homomorphism of an oriented graph to the tournament $T_4$ depicted in Figure 1. Homomorphism is decidable in polynomial time for every tournament with at most 4 vertices other than $T_4$. Each new paper shows that the problem is NP-complete on a smaller graph class. Theorem 1 improves again these results. Let $\mathcal{P}_g$ denote the class of planar graphs with girth least $g$. Therefore, $\mathcal{P}_{g+1}$ is proper subclass of $\mathcal{P}_g$.

**Theorem 1** For any fixed $g \geq 3$, deciding whether an oriented graph $G$ admits a homomorphism to $T_4$ is NP-complete, even if $G$ is restricted to be in $\mathcal{P}_g$, bipartite, subcubic, with DAG-depth 3, with maximum outdegree 2 and maximum indegree 2, and such that every 3-vertex is adjacent at most one 3-vertex.

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Klostermeyer and MacGillivray [6] consider a variation of oriented homomorphism. Given an oriented graph $G$ and a subset $X$ of vertices of $G$, the graph obtained from $G$ by reversing the direction of the arcs in the cut $(X, G \setminus X)$ is said to be push equivalent to $G$. The graph $G$ admits a push homomorphism to an oriented graph $T$ if there exists a graph $G'$ such that $G'$ is push equivalent to $G$ and $G'$ admits a homomorphism to $T$. The pushable chromatic number of $G$ is then defined as the minimum number of vertices of a graph $T$ such that $G$ has a push homomorphism to $T$.

Let $T_6$ be the oriented graph depicted in Figure 2 with vertex set $\{0, 1, \cdots, 5\}$ such that $ij$ is an arc if and only if $j = i + 1 \pmod 5$ or $j = i + 2 \pmod 5$. Guégan has shown [5] that an oriented graph has pushable chromatic number at most 3 if and only if it admits a homomorphism to $T_6$, and that every oriented graph in $P_{17}$ maps to $T_6$. In Section 3, we obtain complementary results and exhibit an oriented graph in $P_9$ and a bipartite oriented graph in $P_8$ that do not map to $T_6$.

Then we consider the complexity of deciding homomorphism to $T_6$, or equivalently, deciding whether the pushable chromatic number is at most 3. Notice that an oriented graph has pushable chromatic number at most 2 if and only if it admits a homomorphism to the 4-circuit, which is decidable in linear time. We prove the following result in Section 4.

**Theorem 2** Let $g$ be a fixed integer. Either every oriented graph in $P_g$ maps to $T_6$ or it is NP-complete to determine whether a graph in $P_g$ maps to $T_6$. Either every bipartite oriented graph in $P_g$ maps to $T_6$ or it is NP-complete to determine whether a bipartite graph in $P_g$ maps to $T_6$.

Finally, we obtain the following corollary of Theorem 2 using the graphs described in Section 3.
Corollary 3  Determining whether an oriented planar graph $G$ maps to $T_6$ is NP-complete, even if $G$ has girth 9, or if $G$ is bipartite with girth 8.

2  Proof of Theorem 1

Kratochvíl proved that PLANAR $(3, \leq 4)$-SAT is NP-complete [7]. In this restricted version of SAT, the graph of incidences variable-clause of the input formula is planar, every clause is a disjunction of exactly three literals, and every variable occurs in at most four clauses. We reduce PLANAR $(3, \leq 4)$-SAT to the problem of determining whether an oriented graph has a homomorphism to the tournament $T_4$.

Given an instance $I$ of PLANAR $(3, \leq 4)$-SAT, we construct a corresponding oriented graph $G$. We take one copy of the graph depicted in Figure 3 per variable of $I$ and one copy of the graph depicted on the left of Figure 4 per clause of $I$. Whenever a variable $v$ appears in a clause $c$ of $I$, we identify one vertex labelled $b$ or $\overline{b}$ (according to the literal of $v$ in $c$) of the vertex gadget of $v$ to a vertex $l_i$ of the clause gadget of $c$.

![Figure 3. Variable gadget.](image)

![Figure 4. Clause gadget.](image)

Every $T_4$-coloring of the vertex gadget is such that exactly one of following holds:

(1) Every vertex named $a$ is colored 1 and every vertex named $\overline{a}$ is colored 2 or 3.
(2) Every vertex named $\overline{a}$ is colored 1 and every vertex named $a$ is colored 2 or 3.
Moreover, if a vertex \( a \) or \( \bar{a} \) is colored 1, then the corresponding vertex \( b \) or \( \bar{b} \) is colored 1. If a vertex \( a \) or \( \bar{a} \) is colored 2 or 3, then the corresponding vertex \( b \) or \( \bar{b} \) is colored 2, 3, or 4.

The set of colors \( \{2, 3, 4\} \) is associated to the boolean value true and the set of colors \( \{1\} \) is associated to the boolean value false. Now let us assume that the vertices \( x \), \( y \), and \( z \) of the clause gadget are precolored according to their corresponding literal. On the right of Figure 4, we give the possible color extensions of the three paths of the clause gadget, both in the case of a true literal (above the path) and in the case of a false literal (below the path).

If a clause is satisfied, then at least one of its literal is true and the precoloring can be extended to a \( T_4 \)-coloring of the vertex gadget. Indeed, if the literal corresponding to \( x \) (resp. \( y \), \( z \)) is true, then the precoloring can be extended such that \( c(t) = 3 \) (resp. \( c(t) = 4 \), \( c(t) = 1 \)).

If a clause is not satisfied, then the precoloring cannot be extended to a \( T_4 \)-coloring of the vertex gadget. Indeed, we have \( c(x) = c(y) = c(z) = 1 \), thus \( c(t) \notin \{2, 3\} \) because \( c(x) = 1 \), \( c(t) \neq 4 \) because \( c(y) = 1 \), and \( c(t) \notin \{1, 2\} \) because \( c(z) = 1 \), so \( c(t) \notin \{1, 2, 3, 4\} \) and the clause gadget is not \( T_4 \)-colorable.

Now, we have to show that \( G \) satisfies the conditions of Theorem 1. It is easy to check that \( G \) is planar, bipartite, subcubic, with maximum outdegree 2 and maximum indegree 2, and that every 3-vertex is adjacent at most one 3-vertex. We can also assume that the girth is large, since we can increase both the length of the cycles in the gadgets forbidding color 4 and the distance between vertices corresponding to literals of a same variable (see the dotted arcs in Figure 3). We refer to [3] for the definition of DAG-depth. The vertex gadget contains the unique maximal reachable fragment of \( G \). It consists in a directed path \( p_1, \ldots, p_7 \) with an additional out-going arc at \( p_2 \) and \( p_4 \). This reachable fragment, and thus \( G \), has DAG-depth 3. Notice that the girth of \( G \) is large, whereas the length of directed paths is bounded since \( G \) has bounded DAG-depth. This implies that \( G \) has K-width 1, i.e., there exists at most one directed path between two vertices, and that \( G \) is acyclic, i.e., \( G \) has no circuit.

### 3 Planar graphs that do not map to \( T_6 \)

Figure 5 shows an oriented graph \( G_9 \) in \( P_9 \) that does not map to \( T_6 \) and Figure 6 shows a bipartite oriented graph \( G_8 \) in \( P_8 \) that does not map to \( T_6 \).

To see that \( G_9 \) does not map to \( T_6 \), consider first the subgraph depicted on the left of Figure 5. Suppose that this subgraph has a \( T_6 \)-coloring to such that \( t \) is colored 0. Since \( t \) and \( d \) have a common out-neighbor, \( d \) must be colored 5, 0, or 1. Suppose that \( d \) is colored 1. Then the directed paths starting from \( t \) (resp. \( v \)) forbid forbid that
Figure 5. The graph $G_9$ in $P_9$ that does not map to $T_6$.  

A vertex in the horizontal directed path is colored 3 (resp. 4). This is a contradiction since the directed path on 5 vertices does not map to $T_6 \setminus \{3, 4\}$. Hence, $v$ cannot be colored 1 and, by symmetry, $v$ cannot be colored 5. This implies that $t$ and $d$ get the same color in any $T_6$-coloring. Consider now the whole graph $G_9$ depicted on the left of Figure 5. It has no $T_6$-coloring since the represented arc would force its extremities to get distinct colors, whereas the four copies of the mentioned subgraph would force these extremities to get the same color.

Figure 6. The bipartite graph $G_8$ in $P_8$ that does not map to $T_6$.

To see that $G_8$ does not map to $T_6$, consider first the subgraph depicted on the left of Figure 6. If this subgraph has a $T_6$-coloring such that $t$ is colored 0, then $d$ is not colored 3 (resp. 0) because of the path on the left (resp. on the right). We associate to each vertex $i$ of $T_6$ its anti-twin $i + 3 \pmod{6}$. Consider now a $T_6$-coloring of the whole graph $G_8$ depicted on the left of Figure 5. For each represented vertex $v$, neither the color $c(v)$ of $v$ nor the anti-twin color $c(v) + 3 \pmod{6}$ can appear on another represented vertex. This is a contradiction since there are 4 represented vertices but only 3 pairs of anti-twins.

4 Proof of Theorem 2

We suppose that there exists a graph $H \in P_g$ that does not map to $T_6$ and is minimal for subgraph order. Notice that $T_6$ is circular, so $H$ must be 2-connected since otherwise we can obtain a $T_6$-coloring of $H$ from the $T_6$-colorings of the 2-connected components of $H$. Thanks to the graphs in Section 3 and the fact that graphs in $P_{17}$ map to $T_6$ [5], we can assume that $8 \leq g \leq 16$. It is well-known that graphs in $P_6$,
and therefore $H$, are 2-degenerate. So we have $\delta(H) = 2$. Let $v$ be a 2-vertex of $H$ and let $u_1$ and $u_2$ be the neighbors of $v$. A graph that is push equivalent to $H$ has a $T_6$-coloring if and only if $H$ has a $T_6$-coloring. So, by possibly replacing $H$ by a graph that is push equivalent to $H$, we can assume that $H$ contains the arcs $u_1v$ and $u_2v$.

The graph $H' = H \setminus v$ is a subgraph of $H$ and thus admits at least one $T_6$-coloring. Let $M$ be the set of $T_6$-colorings $m$ of $H'$ such that $m(v_1) = 0$. Let $S$ be the set $\{m(v_2) \mid m \in M\}$. Notice that we cannot have $m(v_1) = 0$ and $m(v_2) \in \{0, 1, 5\}$, since otherwise it would be possible to extend $m$ to $H$. So $S$ is non-empty subset of $\{2, 3, 4\}$.

Now we use $H'$ to construct a duplicator gadget $D$ with two specified vertices $z$ and $z'$ on its outerface such that $D$ maps to $T_6$ and every $T_6$-coloring of $D$ is such that $z$ and $z'$ have the same color. We consider two cases depending on $S$:

- If $S = \{3\}$, then the construction of $D$ is described in Figure 7, middle. If $z$ is colored 0, then $m$ is colored 3 and $z'$ is colored 0, so $z$ and $z'$ must have the same color.
- If $S \cap \{2, 4\} \neq \emptyset$, then the construction of $D$ is described in Figure 7, right. Suppose that $z$ is colored 0. Because of the copy of $H'$ between $z$ and $t$, $t$ cannot be colored 0, 1, or 5. Because of the directed 4-path between $t$ and $z$, $t$ cannot be colored 3. If $t$ is colored 2 then $S$ contains 2 and $d$ must be colored 4. The copy of $H'$ between $d$ and $z'$ forbids that $z'$ is colored 3, 4, or 5. The directed 2-path between $z'$ and $t$ forbids that $z'$ is colored 1, 2, or 3. The only remaining possibility is that $z'$ is colored 0, which is possible since $d$ is colored 4 and $S$ contains 2. A similar argument shows that if $t$ is colored 4, then we also obtain that $z$ and $z'$ must have the same color.

![Figure 7. Construction of D.](image)

The reduction is from PLANAR 3-COLORABILITY, which is known to be NP-complete [4] for planar graphs with maximum degree 4. Let $I$ be an instance of PLANAR 3-COLORABILITY. We construct an oriented graph $G$ based on $I$ as follows. The vertex gadget consists in a chain of 7 copies of $D$. It thus contains 8 particular vertices, namely the specified vertices of the copies of $D$, that must get the same color in any homomorphism to $T_6$. This common color is said to be the color of the vertex gadget. We replace every vertex of $I$ by a vertex gadget. For every edge $ab$ of $I$, we link a particular vertex of the gadget of $a$ to a particular vertex of the gadget of $b$ with a directed path of 2 arcs and we link another particular vertex of the gadget of $a$ to another particular vertex of the gadget of $b$ with a directed path of 4 arcs, as shown in Figure 8.
Suppose that $ab$ is an edge of $I$ and consider a $T_6$-coloring of the graph induced by the gadgets of $a$ and $b$ and the two paths between them. If the color of the gadget of $a$ is 0, then the color of the gadget of $b$ is then in $\{2, 4\}$, because the path with 2 arcs forces it to be in $\{2, 3, 4\}$ and the path with 4 arcs forces it to be distinct from 3. This shows that $I$ admits a 3-coloring using the colors $\{0, 1\}$ if and only if $G$ admits a $T_6$-coloring such that the gadget of a vertex colored $i$ is colored $2i$.

Now, we have to show that $G$ satisfies the conditions of Theorem 2. As already mentioned, the girth of $H$ satisfies $8 \leq g \leq 16$. The distance between $u_1$ and $u_2$ in $H'$ is at least $g - 2$. The distance between $z$ and $z'$ is $2(g - 2)$ in the case $S = \{3\}$ and is $\min (6, g)$ in the case $S \cap \{2, 4\} \neq \emptyset$, so this distance is at least 6. The shortest cycles of $G$ that are not contained in a copy of $D$ appear in Figure 8 and their length is at least $2 \times 6 + 2 + 4 = 18$. So $G$ contains no cycle of length strictly smaller than $g$. Moreover, if $H$ is bipartite, then $H'$ is bipartite and the distance between $u_1$ and $u_2$ in $H'$ is even. The paths joining particular vertices in distinct vertex gadgets also have even length, so that $G$ bipartite.

References


