On the number of prime factors of an odd perfect number

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Abstract

Let $\Omega(n)$ and $\omega(n)$ denote respectively the total number of prime factors and the number of distinct prime factors of the integer n. Euler proved that an odd perfect number N is of the form $N = p^e m^2$ where $p \equiv e \equiv 1 \pmod 4$, p is prime, and $p \nmid m$. This implies that $\Omega(N) \geq 2\omega(N) - 1$. We prove that $\Omega(N) \geq (18\omega(N) - 31)/7$ and $\Omega(N) \geq 2\omega(N) + 51$.

1 Introduction

A natural number N is said *perfect* if it is equal to the sum of its positive divisors (excluding N). It is well known that an even natural number N is perfect if and only if $N = 2^{k-1}(2^k - 1)$ for an integer k such that $2^k - 1$ is a Mersenne prime. On the other hand, it is a long-standing open question whether an odd perfect number exists.

In order to investigate this question, several authors gave necessary conditions for the existence of an odd perfect number N. Let $\Omega(n)$ and $\omega(n)$ denote respectively the total number of prime factors and the number of distinct prime factors of the integer n. Euler proved that $N = p^e m^2$ for a prime p, with $p \equiv e \equiv 1 \pmod 4$, p is prime, and $p \nmid m$. Moreover, recent results showed that $N > 10^{1500}$ [4], $\omega(N) \geq 9$ [3], and $\Omega(N) \geq 101$ [4].

In this paper, we study the relationship between $\Omega(N)$ and $\omega(N)$. By Euler's result, we have $\Omega(N) \geq 2\omega(N) - 1$. Steuerwald [6] proved that m is not square-free, that is, the exponents of the non-special primes cannot be all equal to 2. This implies that $\Omega(N) \geq 2\omega(N) + 1$. We improve this inequality in two ways:

Theorem 1. *If N is an odd perfect number, then $\Omega(N) \geq (18\omega(N) - 31)/7$.*

Theorem 2. *If N is an odd perfect number, then $\Omega(N) \geq 2\omega(N) + 51$.*

We prove Theorem 1 in Section 3 using standard arguments. We prove Theorem 2 in Section 4 via computations using the general method in [4].

To summarize the known results about $\Omega(N)$, we have

$$\Omega(N) \geq \max\{101, 2\omega(N) + 51, (18\omega(N) - 31)/7\}.$$

2 Preliminaries

Let n be a natural number. Let $\sigma(n)$ denote the sum of the positive divisors of n, and let $\sigma_{-1}(n) = \frac{\sigma(n)}{n}$ be the *abundancy* of n. Clearly, n is perfect if and only if $\sigma_{-1}(n) = 2$. We first recall some easy results on the functions σ and σ_{-1}. If p is prime, $\sigma(p^q) = \frac{p^{q+1}-1}{p-1}$, and $\sigma_{-1}(p^\infty) = \lim_{q\to+\infty} \sigma_{-1}(p^q) = \frac{p}{p-1}$. If $\gcd(a,b) = 1$, then $\sigma(ab) = \sigma(a)\sigma(b)$ and $\sigma_{-1}(ab) = \sigma_{-1}(a)\sigma_{-1}(b)$.

Euler proved that if an odd perfect number N exists, then it is of the form $N = p^e m^2$ where $p \equiv e \equiv 1 \pmod 4$, p is prime, and $p \nmid m$. The prime p is said to be the *special prime*.

3 Proof of $\Omega(N) \geq (18\omega(N) - 31)/7$

We want to obtain a result of the form $\Omega(N) \geq a\omega(N) - c$ for some $a > 2$ using the following idea. If a is close to 2, then N has a large amount of prime factors p such that both $p^2 \parallel N$ and $p \parallel \sigma(q^2)$ where $q^2 \parallel N$. It is well known (see [5]) that for primes t, r, and s such that $t \mid \sigma(r^{s-1})$, either $t = s$ or $t \equiv 1 \bmod s$. In particular, this gives $p \equiv 1 \bmod 3$ and thus $3 \mid \sigma(p^2)$. The exponent of the prime 3 is then large, so that $\Omega(N)$ is significantly greater than $2\omega(N)$.

Now we detail the number of certain types of factors of N and obtain the results by contradiction with the involved quantities.

- $p = \omega(N)$: number of distinct prime factors,
- $f = \Omega(N)$: total number of prime factors,
- p_2: number of distinct prime factors with exponent 2, distinct from 3,
- $p_{2,1}$: number of distinct prime factors with exponent 2 congruent to 1 mod 3,
- p_4: number of distinct prime factors with exponent at least 4, distinct from 3 and the special prime,
- f_4: total number of prime factors with exponent at least 4, distinct from 3 and the special prime,
- e: exponent of the special prime,
- f_3: exponent of the prime 3.

Now we obtain useful inequalities among these quantities. The special exponent is at least 1:

$$1 \leq e. \tag{1}$$

By detailing the total number of prime factors, we have

$$e + f_3 + 2p_2 + f_4 = f. \tag{2}$$

By considering the prime factors (distinct from 3 and the special prime) with exponent at least 4, we have

$$4p_4 \leq f_4. \tag{3}$$

As already mentioned, if $p \equiv 1 \bmod 3$ and $p^2 \parallel N$, then $3 \mid \sigma(p^2)$, so that

$$p_{2,1} \leq f_3. \tag{4}$$

Let us consider the number of distinct prime factors. We have the special prime, the primes from p_2 and p_4, and maybe the prime 3. So it is $1+p_2+p_4$ if $f_3 = 0$ and $2 + p_2 + p_4$ if $f_3 \geq 2$. We thus have

$$p \leq f_3/2 + 1 + p_2 + p_4 \tag{5}$$

and

$$p \leq 2 + p_2 + p_4. \tag{6}$$

For the sake of contradiction, we suppose that

$$7f \leq 18p - 32. \tag{7}$$

The following lemma is useful to obtain one last inequality:

Lemma 3. *Let p, q, and r be positive integers. If $p^2 + p + 1 = r$ and $q^2 + q + 1 = 3r$, then p is not an odd prime.*

Proof. Since $q^2 + q + 1 \equiv 0 \bmod 3$, then $q \equiv 1 \bmod 3$ and we set $q = 3s + 1$. The equality $q^2+q+1 = 3(p^2+p+1)$ reduces to $3s(s+1) = p(p+1)$. Notice that p divides $3s(s + 1)$, so that if p is an odd prime, then either $p \mid 3$, $p \mid s$, or $p \mid (s + 1)$. We have $p = 3$ in the first case, which gives no solution. We have $s \geq p-1$ in the other two cases, so that $p(p+1) = 3s(s+1) \geq 3(p-1)p$. This gives $p + 1 \geq 3(p - 1)$, so that $p \leq 2$, which is a contradiction. □

Let K be the multiset of all the primes distinct from 3 produced by all the components $\sigma(p^2)$ of N. The primes in K are 1 mod 3, so $|K| \leq e+2p_{2,1}+f_4$. For a prime $u > 3$, let $\alpha(u)$ be such that $\alpha(u) = \sigma(u^2)$ if $u \equiv 2 \bmod 3$ and $\alpha(u) = \sigma(u^2)/3$ if $u \equiv 1 \bmod 3$. By Lemma 3, $\alpha(u) = \alpha(v)$ implies $u = v$. So all primes from p_2 produce at least two prime factors, except for at most one per distinct prime from K. That is, $2p_2 - 1 - p_{2,1} - p_4 \leq |K|$. We thus have $2p_2 - 1 - p_{2,1} - p_4 \leq e + 2p_{2,1} + f_4$, which gives

$$2p_2 \leq 1 + e + 3p_{2,1} + p_4 + f_4. \tag{8}$$

The combination $5\times$**(1)**$+7\times$**(2)**$+5\times$**(3)**$+6\times$**(4)**$+2\times$**(5)**$+16\times$**(6)**$+$ **(7)** $+ 2 \times$ **(8)** gives $1 \leq 0$, a contradiction. This means that the assumption **(7)** that $7f \leq 18p - 32$ is false, and thus $\Omega(N) \geq (18\omega(N) - 31)/7$.

4 Proof of $\Omega(N) \geq 2\omega(N) + 51$

We use the general method and the computer program discussed in [4].

We use the following contradictions:

- The abundancy of the current number is strictly greater than 2.

- The current number n satisfies $\Omega(n) \geq 2\omega(n) + 51$.

We forbid the factors in $S = \{3, 5, 7, 11, 13, 17, 19\}$, in this order. We branch on the smallest available prime congruent to 1 mod 3. If there is no such prime, we branch on the smallest available prime congruent to 2 mod 3. We still use a combination of exact branchings and standard branchings, as in [4]. We use exact branchings only for the special components p^1 and for all the even powers 3^{2e} of 3.

By-passing roadblocks

A *roadblock* is a situation such that there is no contradiction and no possibility to branch on a prime. This happens when we have already made suppositions for the multiplicity of all the known primes and the other numbers are composites.

Given a roadblock M, we check that the composites involved are not divisible by an already considered prime, are not perfect powers, have no factor less than 10^{10}, and are pairwise coprime. Then we compute the following quantities:

- F: It is a lower bound on the number of distinct prime factors of M. We count the number of known prime factors of M plus two primes per composite number.

- A: It is an upper bound on the abundancy of M. For the abundancy of a component p^e, we use $\sigma_{-1}(p^e)$ for an exact branching and $\sigma_{-1}(p^\infty) = p/(p-1)$ for a standard branching.

  For a composite C, we know that C has at most $\left\lfloor \frac{\ln C}{10 \ln 10} \right\rfloor$ prime factors since C has no factor less than 10^{10}. So, the abundancy due to C is at most $(1 + 10^{-10})^{\left\lfloor \frac{\ln C}{10 \ln 10} \right\rfloor}$.

- T: It is the target lower bound on $\Omega(N) - 2\omega(N)$, thus an odd integer. We use $T = 51$ in the proof of Theorem 2.

For the sake of contradiction, we suppose that $\Omega(N) - 2\omega(N) \leq T - 2$. By Theorem 1, we have $\Omega(N) \geq (18\omega(N) - 31)/7$. So $(18\omega(N) - 31)/7 - 2\omega(N) \leq \Omega(N) - 2\omega(N) \leq T - 2$, which gives $\omega(N) \leq (7T + 17)/4$. Thus, N has at most $\omega(N) \leq (7T + 17)/4 - F$ prime factors that do not divide M. Let p be the smallest of these extra factors. We see that if

$$A(p/(p-1))^{(7T+17)/4-F} < 2 \tag{9}$$

then N cannot reach abundancy 2. This gives an upper bound on p. To get around the roadblock, we branch on every prime number p (except those that divide M or are already forbidden) in increasing order until **(9)** is satisfied.

Example:
$3^4 \Longrightarrow 11^2$
$11^{18} \Longrightarrow 6115909044841454629$
$6115909044841454629^{16} \Longrightarrow \sigma\left(6115909044841454629^{16}\right)$ Roadblock 1
$5^1 \Longrightarrow 2 \times 3$ Roadblock 2

We first branch on the components 3^4, 11^{18}, and $\sigma\left(11^{18}\right)^{16}$ and hit a first roadblock, as no factors of $C_1 = \sigma\left(\sigma\left(11^{18}\right)^{16}\right)$ are known. When trying to get around this roadblock, we first branch on 5^1 and hit a second roadblock. Consider this second roadblock:

- $F = 6$: We have the four primes 3, 5, 11, $\sigma\left(11^{18}\right)$, and at least two primes from C_1.
- $A = \sigma_{-1}\left(3^4 \times 5 \times 11^{\infty} \times \sigma\left(11^{18}\right)^{\infty}\right) \times (1 + 10^{-10})^{\left\lfloor \frac{\ln C_1}{10 \ln 10} \right\rfloor} = 1.9718518\cdots$
- $T = 51$.

Equation 9 is satisfied for $p \geq 6174$, so to circumvent M, we branch on every prime p between 7 and 6173, except 11.

When N has no factors in S.

If N has no factor in S, then it must have at least 115 distinct prime factors. We obtain this by considering the product $\Pi_{23 \leq p \leq 673} \frac{p}{p-1} = 1.99807632\ldots$ over the first 114 primes p greater than 19, which is an upper bound on the abundancy and is smaller than 2.

Using Theorem 1, we obtain

$$\begin{aligned}
\Omega(N) - 2\omega(N) &\geq (18\omega(N) - 31)/7 - 2\omega(N) \\
&= (4\omega(N) - 31)/7 \\
&\geq (4 \times 115 - 31)/7 \\
&= 61 + 2/7.
\end{aligned}$$

So, we have $\Omega(N) \geq 2\omega(N) + 62$, which concludes the proof of Theorem 2.

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References

[1] G.L. Cohen. On the largest component of an odd perfect number, *J. Austral. Math. Soc. Ser. A* **42** (1987), pp 280–286.

[2] T. Goto, Y. Ohno. Odd perfect numbers have a prime factor exceeding 10^8, *Math. Comp.* **77** (2008), no. 263, pp 1859–1868.

[3] P.P. Nielsen. Odd perfect numbers have at least nine different prime factors, *Math. Comp.* **76** (2007), no. 160, pp 2109–2126.

[4] P. Ochem, M. Rao. Odd perfect numbers are greater than 10^{1500}, *Math. Comp.* **81** (2012), pp 1869–1877.

[5] T. Nagell. Introduction to Number Theory, *John Wiley & Sons Inc.*, New York, 1951.

[6] R. Steuerwald. Verschärfung einer notwendigen Bedingung für die Existenz einen ungeraden vollkommenen Zahl, *S.-B. Bayer. Akad. Wiss.* (1937), pp 69–72.