

The complexity of partitioning into disjoint cliques and a triangle-free graph[☆]

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Abstract

Motivated by Chudnovsky's structure theorem of bull-free graphs, Abu-Khzam, Feghali, and Müller have recently proved that deciding if a graph has a vertex partition into disjoint cliques and a triangle-free graph is NP-complete for five graph classes. The problem is trivial for the intersection of these five classes. We prove that the problem is NP-complete for the intersection of two subsets of size four among the five classes. We also show NP-completeness for other small classes, such as graphs with maximum degree 4 and line graphs.

Keywords: Graph coloring, NP-completeness.

1. Introduction

In this paper we consider the problem of recognizing graphs having a vertex partition into disjoint cliques and a triangle-free graph. We say that a graph is *partitionable* if it has such a partition. The vertices in the P_3 -free part are colored blue and the vertices in the K_3 -free part are colored red. This problem is known to be NP-complete on general graphs [5]. The NP-completeness on bull-free graphs was motivated by an open question in [11] (after Thm 2.1) about the complexity of recognizing the class τ_1 introduced by Chudnovsky [3] in her characterization of bull-free graphs. Abu-Khzam, Feghali, and Müller [1] have then investigated the complexity of deciding whether a bull-free graph is partitionable. They have shown the following.

Theorem 1. [1] *Recognizing partitionable graphs is NP-complete even when restricted to the following classes:*

- (1) *planar graphs,*
- (2) *K_4 -free graphs,*
- (3) *bull-free graphs,*
- (4) *(C_5, \dots, C_t) -free graphs (for any fixed t),*
- (5) *perfect graphs.*

In Section 2, we prove Theorem 2 which improves Theorem 1. The classes h_1 and h_2 of Theorem 2 show that the problem remains NP-complete for the intersection of two subsets of size four among the five classes of Theorem 1 (graphs in the intersection of the five classes are partitionable). The class h_1 also answers the open question [1] of the complexity of recognizing partitionable Meyniel graphs, since every graph in h_1 is a parity graph and parity graphs correspond to gem-free Meyniel graphs. We also show NP-completeness for several other classes. The classes h_3 to h_9 are motivated by the introduction of other natural forbidden induced subgraphs (mainly C_4 , K_4^- , and $K_{1,3}$) and/or restriction on the maximum degree. The interesting feature of every class is briefly

[☆]To Nina (Pascal's daughter, born on Nov 1 2015) and Ilyas (Marin's son, born on Nov 6 2015)

discussed at the end of its dedicated subsection. We study the tightness of this result in Section 3 by considering all the intersections between every two graph classes of Theorem 2.

We use standard notations for graphs (see [12]), some of them are reminded in Figure 1. For interpretation of the references to color in the other figures, the reader is referred to the web version of this article.

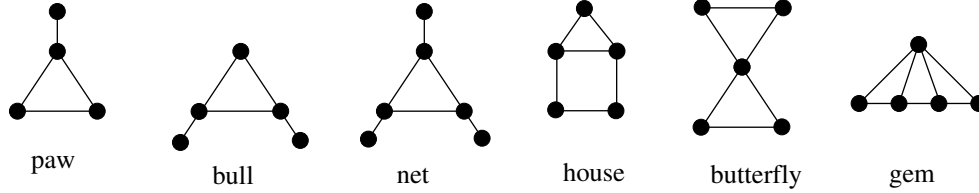


Figure 1: Some small graphs and their name.

A k -vertex is a vertex of degree k . Given a graph G , we denote its line graph by $L(G)$. Given a graph class \mathcal{C} , we denote by $L(\mathcal{C})$ the set of line graphs of graphs in \mathcal{C} .

2. Main result

In this section we prove the following result.

Theorem 2. *Recognizing partitionable graphs is NP-complete even when restricted to the following classes:*

- h_1 : planar $(C_4, \dots, C_t, \text{bull}, \text{gem}, \text{odd hole})$ -free graphs with maximum degree 8,
- h_2 : planar $(K_4, \text{bull}, \text{house}, C_5, \dots, C_t)$ -free graphs,
- h_3 : planar $(K_4, C_4, \text{gem}, C_7, \dots, C_t, \text{odd hole of length } \geq 7)$ -free graphs with maximum degree 7,
- h_4 : $(K_4, C_5, \dots, C_t, \text{net}, \text{odd hole})$ -free graphs with maximum degree 8,
- h_5 : $(K_4^-, \text{butterfly}, C_6, \dots, C_t)$ -free graphs with maximum degree 4,
- h_6 : $(K_4, K_4^-, C_4, \dots, C_t, \text{butterfly})$ -free graphs,
- h_7 : planar $(K_{1,3}, K_4^-, C_4, \dots, C_t, \text{odd hole})$ -free graphs with maximum degree 6,
- h_8 : planar $(K_{1,3}, K_4^-, C_9, \dots, C_t, \text{odd hole})$ -free graphs with maximum degree 5,
- h_9 : $(K_{1,3}, K_4^-, C_4, \dots, C_t, K_5, \text{odd hole})$ -free graphs with maximum degree 5,

Kratochvíl proved that PLANAR $(3, \leq 4)$ -SAT is NP-complete [6]. In this restricted version of SAT, the graph of variable-clause incidences of the input formula must be planar, every clause is a disjunction of exactly three literals, and every variable occurs in at most four clauses. For every class considered in Theorem 2, we provide a reduction from PLANAR $(3, \leq 4)$ -SAT. Given an instance formula I of PLANAR $(3, \leq 4)$ -SAT, we construct a graph G such that G is partitionable if and only if I is satisfiable.

For the classes h_1, h_2, h_3 , and h_4 , the boolean value true is associated to the color red, the boolean value false is associated to the color blue, and the clause gadget is P_3 . This way, an unsatisfied clause corresponds to a blue P_3 . For the classes h_5 and h_6 , the boolean value true is associated to the color blue, the boolean value false is associated to the color red, and the clause gadget is K_3 . This way, an unsatisfied clause corresponds to a red K_3 . For brevity, we say that a vertex with the color associated to the boolean value true (resp. false) is colored true (resp. false).

Given a variable x , a *variable gadget* is a graph G_x with two disjoint subsets of vertices S_x and $S_{\bar{x}}$ such that:

- There exists an involutive automorphism of G_x which swaps S_x and $S_{\bar{x}}$.
- There exists a partition of G_x such that every vertex in S_x is colored true and no blue vertex in $S_x \cup S_{\bar{x}}$ is adjacent to a blue vertex.
- No partition of G_x is such that both a vertex in S_x and a vertex in $S_{\bar{x}}$ are colored true.

The variable gadget depends on the considered graph class and is built on *forcers*. A forcer is a partitionable graph with a specified vertex q .

- A red forcer is such that q is red in every partition.
- A blue forcer is such that q is blue in every partition and there exists a partition such that every neighbor of q is red.

We construct G from I as follows. We take one copy of the variable gadget per variable. We take one copy of the clause gadget (either P_3 or K_3) per clause. Each of the 3 vertices of the clause gadget corresponds to a literal of the clause. The vertices in $S_x \cup S_{\bar{x}}$ are depicted in green in the representation of the variable gadgets (Fig. 4 and 8). A subset of these green vertices corresponds to the literals of the variable x . For every literal ℓ_x of I , one vertex corresponding to ℓ_x in G_x is identified with the vertex corresponding to ℓ_x in the clause gadget.

For the classes h_1 , h_2 , h_3 , and h_4 , we use a *parity labeling* to ensure that G has no induced odd hole. This labeling assigns a value in $\{1, 2\}$ to every vertex of a subgraph of G , such that the values alternate on every labeled induced path with at least 3 vertices. In every clause gadget P_3 of G , the two extremities of P_3 are labeled 1 and the middle vertex is labeled 2. Notice that in the variable gadget in Fig. 4, there exist green vertices labeled 1 and 2 both in S_x and $S_{\bar{x}}$. Thus, we can make sure that every vertex of a clause gadget is identified with a green vertex corresponding to the suitable literal having the same parity label. To check that G contains no odd hole, we consider the 2-connected components of G . They have a bounded number of vertices, except one large 2-connected component which is entirely labeled and contains every clause gadget.

Only a part of the variable gadget is depicted in the figures. The actual size of the variable gadget depends (linearly) on t . The variable gadget consists in sufficiently many copies of the depicted part that are arranged circularly, as suggested by the dashed edge. Thus, we can ensure that G is (C_i, \dots, C_t) -free by identifying every green vertex to at most one vertex of a clause gadget and by requiring that the distance in the variable gadget between two "used" green vertices is at least t . For subclasses of planar graphs, we also make sure to identify green vertices and vertices of the clause gadget so that G is planar.

Let us prove that this construction provides a reduction.

Lemma 3. *I is satisfiable if and only if G is partitionable.*

Proof. Suppose that I is satisfiable. For every variable x , if x is set to true (resp. false) in the satisfying assignment, then we partition G_x such that every vertex in S_x (resp. $S_{\bar{x}}$) is colored true and no blue vertex in $S_x \cup S_{\bar{x}}$ is adjacent to a blue vertex. This implies that every clause gadget of G contains a vertex colored true. Moreover, no vertex x nor \bar{x} can be the middle vertex of a blue P_3 with one extremity in the variable gadget. So G is partitionable. Conversely, suppose that G is partitionable and consider a partition of the vertices of G . Thus, every clause gadget of G contains a vertex colored true. By the property of the variable gadget, if a vertex corresponding to a literal is colored true, then every vertex corresponding to the opposite literal of the variable is colored false. Every variable x of I is set to true if and only if there exists a vertex colored true in S_x . By previous discussion, this gives a satisfying assignment of I . \square

We leave it to the reader to check that the variable gadgets satisfy the required properties and that G contains none of the induced subgraphs that the considered class forbids.

2.1. Class h_1

We use the blue forcer in Figure 2 and the variable gadget in Figure 4. The class h_1 is a subclass of parity graphs.

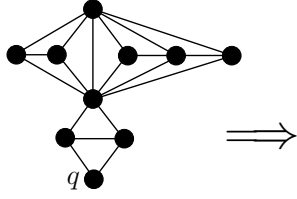


Figure 2: The blue forcer for h_1 .

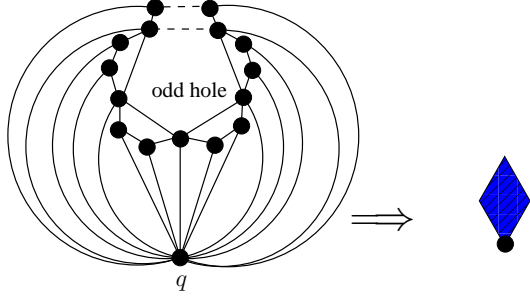


Figure 3: The blue forcer for h_2 .

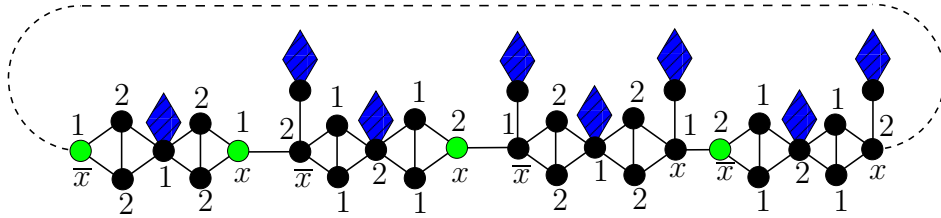


Figure 4: The variable gadget for h_1, h_2, h_3 , and h_4 . The corresponding clause gadget is P_3 .

2.2. Class h_2

We use the blue forcer in Figure 3 and the variable gadget in Figure 4. We make sure that the length of the odd-hole in the blue forcer is at least $t+1$, so that G is (C_5, \dots, C_t) -free. The maximum degree of G is a linear function of t . Notice that a graph in h_2 without large odd hole is perfect and K_4 -free, and thus partitionable.

2.3. Class h_3

The graph F on the left of Figure 5 admits no partition such that every neighbor of w is red. However, it has a partition such that w is blue and a partition such that w is red. We use five copies of F to obtain the red forcer, as depicted on the right of Figure 5. Suppose for contradiction that the red forcer admits a partition such that the specified vertex q is blue. By the property of F , at least one neighbor r of q is also blue. Again, by the property of F , at least one of neighbor s of r in the copy of F attached to r is also blue. Then qrs is a blue induced P_3 , which is a contradiction.

We obtain the blue forcer from the red forcer using Figure 6. We use the variable gadget in Figure 4. Notice that the red forcer contains C_5 . This is mandatory, since a C_5 -free graph in h_3 is perfect and K_4 -free, and thus partitionable.

2.4. Class h_4

We describe a blue forcer for h_4 . Consider the graph $\overline{C_7}$ with vertices v_0, \dots, v_6 such that v_i is adjacent to v_j unless $|i - j| \leq 1 \pmod{7}$. Every partition of $\overline{C_7}$ is such that three consecutive

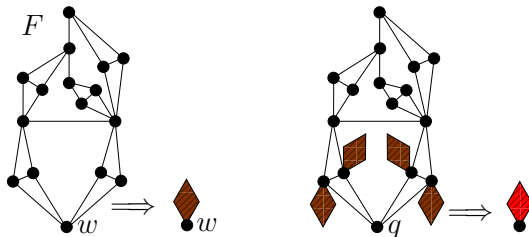


Figure 5: The red forcer for h_3 .

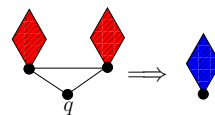


Figure 6: Obtaining a blue forcer from a red forcer for h_3, h_5 , and h_6 .

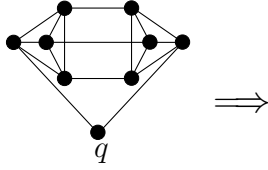


Figure 7: The red forcer for h_5 .

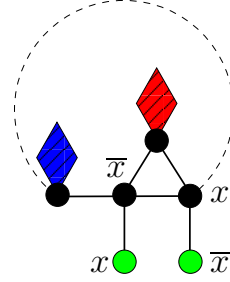


Figure 8: The variable gadget for h_5 and h_6 . The corresponding clause gadget is K_3 .

vertices are blue and the other vertices are red. So a partition is characterized by its monochromatic blue edge $v_t v_{t+2}$ (indices are taken modulo 7). Recall that two non-adjacent vertices are said to be *false twins* if they have the same set of neighbors. We add to \overline{C}_7 a false twin v'_i to v_i for $i \in \{3, 4, 6\}$. A vertex contained in a monochromatic blue edge cannot have a false twin, since it would create a blue P_3 . Thus, the monochromatic blue edge is either $v_5 v_0$ or $v_0 v_2$. The blue forcer is obtained by adding a copy of K_4^- and identifying a 2-vertex of K_4^- with v_0 . The specified vertex of the blue forcer is the 2-vertex (the "other" 2-vertex of K_4^-). Then we use the variable gadget in Figure 4. Notice that a \overline{C}_7 -free graph in h_4 is perfect and K_4 -free, and thus partitionable.

2.5. Class h_5

We use the red forcer in Figure 7. We obtain a blue forcer for h_5 from this red forcer using the construction of Figure 6. We use the variable gadget in Figure 8. Notice that degree 4 is best possible since graphs with maximum degree 3 are partitionable, as shown in the last section.

2.6. Class h_6

To obtain a red forcer for h_6 , we first prove that h_6 is not a subclass of partitionable graphs. By probabilistic arguments, there exists a graph J_1 among the random graphs $G_{n,p}$ with probability $p = n^{-1+1/2t}$ such that:

- J_1 contains at most $\frac{n}{100}$ cycles of length at most $2t$.
- J_1 contains a path of length at least $\frac{99}{100}n$ (Theorem 8.1 in [2]).
- J_1 contains no independent set of size $\frac{n}{7}$.

We call *bag* a subset of three vertices that induce either a P_3 or a K_3 . Let J_2 be the graph induced by the $\frac{99}{100}n$ vertices of the mentioned path in J_1 . Thus, J_2 can be split into $\frac{33}{100}n$ bags. Notice that we can destroy every cycle of length at most $2t$ in J_1 by removing at most $\frac{n}{100}$ vertices. For every such vertex v , we remove from J_2 the bag containing v . This way, we obtain a graph J_3 with girth at least $2t + 1$ and having at least $\frac{33}{100}n - \frac{n}{100} = \frac{8}{25}n$ bags. We obtain the graph J_4 by adding to J_3 the edge between the extremities of each of the $\frac{8}{25}n$ copies of P_3 contained in a bag. Every vertex in J_4 is contained in exactly one triangle. Also, every cycle of length at least $k \geq 4$ in J_4 has length at least $\frac{2}{3}k \geq \frac{2}{3}(2t + 1) \geq t + 1$. This implies that $J_4 \in h_6$. Notice that neither J_1 nor J_4 contains an independent set of size $\frac{n}{7}$. Suppose for contradiction that J_4 is partitionable. Without loss of generality, every triangle contains exactly one blue vertex. So, the graph induced by the blue vertices is (P_3, K_3) -free and is thus bipartite. Thus, there exists an induced blue independent set of size at least $\frac{1}{2} \times \frac{8}{25}n > \frac{n}{7}$. This contradiction shows that J_4 is a graph in h_6 that is not partitionable.

Consider a graph J_{min} in h_6 that is not partitionable and is minimal with respect to the number of edges. Since J_{min} is not partitionable, it contains a vertex u of degree at least 3. Moreover, since J_{min} is $(K_4, K_4^-, \text{butterfly})$ -free, the graph induced by the neighbors of u contains at most one edge. Thus, J_{min} contains an edge uv that is not contained in a triangle. The red forcer is obtained from J_{min} by subdividing once this edge uv . The specified vertex is the subdivision vertex. By

minimality, the graph $J_{min} \setminus \{uv\}$ is partitionable and both u and v are blue in every partition. Then the specified vertex must be red in order to avoid a blue P_3 .

We obtain a blue forcer for h_6 from this red forcer using the construction of Figure 6. We use the variable gadget in Figure 8.

The class h_6 is interesting because it avoids cycles of length 4 to t as (not necessarily induced) subgraphs. Moreover, every vertex is contained in at most one triangle since it is butterfly-free.

2.7. Classes h_7 , h_8 , and h_9

The classes h_7 , h_8 , and h_9 are subclasses of $(K_{1,3}, K_4^-, K_5, \text{odd hole})$ -free graphs, which correspond to the class of line graphs of bipartite graphs with maximum degree 4. For convenience, we thus consider the corresponding edge-partitioning problem described as follows. A triangle-free graph is *edge-partitionable* in red and blue if the blue edges induce a star forest and every vertex is incident to at most two red edges. We then have that a triangle-free graph G is edge-partitionable if and only if $L(G)$ is partitionable.

Lemma 4. *If a graph G is such that every edge is incident to a 4-vertex and a 2-vertex, then G is edge-partitionable and every edge-partition of G is such that every 4-vertex is incident to exactly two blue edges and every 2-vertex is incident to exactly one blue edge.*

Proof. In every edge-partition of G , every 4-vertex is incident to at least two blue edges. So, at least half of the edges of G are blue. Moreover, every 2-vertex is incident to at most one blue edge, since otherwise there would be a blue path on 4 edges or a blue C_4 . So, at most half of the edges of G are blue. Thus, there are equally many red and blue edges in every edge-partition of G . Moreover, every 4-vertex is incident to exactly two blue edges and every 2-vertex is incident to exactly one blue edge.

Now we show that G is edge-partitionable. We define the contraction of a 2-vertex v adjacent to u_1 and u_2 as the deletion of v and the addition of one (additional) edge u_1u_2 . Let G' be the multigraph obtained by contracting every 2-vertex of G . Since G' is 4-regular, we can orient the edges of G' such that the out-degree of every vertex is 2. We extend this orientation of G' to G such that the incidences of the arcs to the 4-vertices are unchanged. We assign red (resp. blue) to an edge of G if its tail (resp. head) is incident to a 4-vertex. This gives a valid edge-partition of G since every vertex is incident to at most two red edges and the graph induced by the blue edges is a star forest such that the 4-vertices are the centers of the stars. \square

Lemma 5. *Recognizing edge-partitionable bipartite graphs with maximum degree 4 is NP-complete even when restricted to:*

h'_7 : planar (C_3, \dots, C_t) -free graphs such that every 4-vertex is a cut vertex.

h'_8 : planar (C_9, \dots, C_t) -free graphs such that no edge is incident to two 4-vertices and such that every 4-vertex is a cut vertex.

h'_9 : (C_3, \dots, C_t) -free graphs such that no edge is incident to two 4-vertices.

Proof. For every class, we give a red edge forcer. The red edge forcers for h'_7 and h'_8 are depicted in Figure ???. Let us construct a red edge forcer for h'_9 . Recall that a (k, g) -cage is a k -regular graph with girth g and with the smallest number of vertices (see [4] for more information on cages). Consider a $(4, \lceil \frac{t+1}{2} \rceil)$ -cage J and subdivide every edge once to obtain the bipartite graph J' . We obtain a red edge forcer for h'_9 by adding a 1-vertex adjacent to one of the 2-vertices of J' . By Lemma 4, the added edge is necessarily red and J' is edge-partitionable.

For every class, we obtain a blue edge forcer from two copies of the red edge forcer of the class, as depicted in Figure 11a. This is the counterpart for edge-partition of the construction in Figure 6. These red and blue edge forcers are used in the construction of the variable gadget (Fig. 11b) and the clause gadget (Fig. 11c). To obtain the instance graph of the edge-partition problem, we identify every green edge corresponding to a literal in the variable gadget to the green edge corresponding to this literal in the clause gadget. The boolean value false is associated to the color red, so that the clause gadget is not edge-partitionable if the clause is not satisfied. An edge of the variable gadget

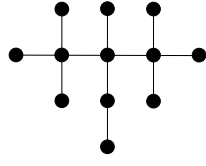


Figure 9: The red edge forcer for h'_7 .

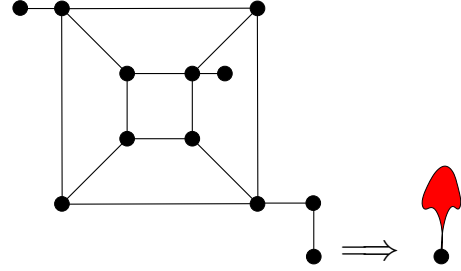


Figure 10: The red edge forcer for h'_8 .

that is adjacent to two red forciers and a green edge is said to be median. Notice that the color of median edges alternate, so that every median edge x gets the same color and every median edge \bar{x} gets the other color. The blue forcer in the clause gadget implies that if the edge corresponding to a literal is colored blue, then this edge is not incident to another blue edge in the variable gadget. Thus, if a green edge corresponds to a literal, then this green edge is part of a clause gadget and its color must be distinct from the color of its adjacent median edge. This means that every green edge x that corresponds to a literal gets the same color and every green edge \bar{x} that corresponds to a literal gets the other color.

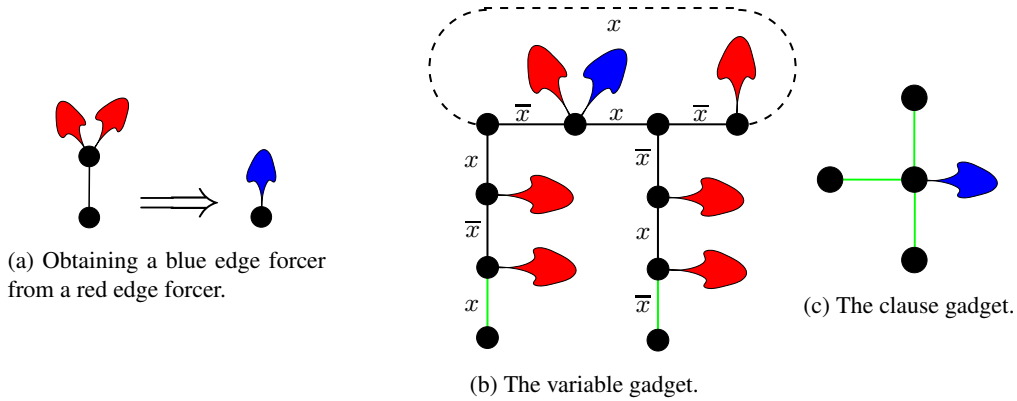


Figure 11: The gadgets for h'_7 , h'_8 , and h'_9 .

□

A line graph is planar if and only if it is the line graph of a planar graph with maximum degree 4 such that every 4-vertex is a cut vertex [10]. We thus have $L(h'_7) = h_7$, $L(h'_8) = h_8$, and $L(h'_9) = h_9$. So, deciding whether a graph in h_7 , h_8 , and h_9 is partitionable is NP-complete.

3. Polynomial classes

We gather here known graph classes for which deciding whether a member is partitionable is polynomial time solvable.

3.1. Chordal graphs

See [1].

3.2. P_4 -free graphs

Abu-Khzam, Feghali, and Müller [1] have shown that we can test in polynomial time whether a P_4 -free graph is partitionable.

3.3. Paw-free graphs

Olariu [9] has proved that every connected component of a paw-free graph is either triangle-free or a complete multipartite graph. Since a complete multipartite graph is P_4 -free, we can test in polynomial time whether a paw-free graph is partitionable.

3.4. $\overline{K_k}$ -free graphs

Recall that the Ramsey number $R(s, t)$ is the least r such that every graph on r vertices contains an independent set on s vertices or a clique on t vertices. In a $\overline{K_k}$ -free graph, every K_3 -free induced subgraph thus contains at most $R(k, 3) - 1$ vertices. Consider a $\overline{K_k}$ -free graph G with n vertices. For each of the $O(n^{R(k, 3) - 1})$ subsets S of at most $R(k, 3) - 1$ vertices of G , we can test in $O(n^3)$ time whether S induces a K_3 -free graph and $G \setminus S$ induces a P_3 -free graph. So, for any fixed k , we can test in polynomial time whether a $\overline{K_k}$ -free graph is partitionable.

4. Partitionable classes

We gather below known classes of partitionable graphs. The motivation is the conjecture that for every two of the classes considered in Theorem 2, their intersection is a subclass of partitionable graphs. For most pair of classes, we have identified a class of partitionable graphs containing the intersection, see Table 1. The remaining two open cases of the conjecture are denoted by a question mark.

	h_1	h_2	h_3	h_4	h_5	h_6	h_7	h_8
h_2	p_2							
h_3	p_2	p_2						
h_4	p_2	p_2	p_2					
h_5	p_3	p_3	?	p_2				
h_6	p_2	p_3	p_2	p_2	?			
h_7	p_3	p_2	p_2	p_2	p_4	p_2		
h_8	p_3	p_2	p_2	p_2	p_4	p_2	p_5	
h_9	p_3	p_2	p_2	p_2	p_4	p_2	p_5	p_5

Table 1: Intersections of the classes considered in Theorem 2.

A graph G is (d_1, \dots, d_l) -colorable if the vertex set of G can be partitioned into subsets V_1, \dots, V_l such that the graph induced by the vertices of V_i has maximum degree at most d_i for every $1 \leq i \leq l$. As it is well known, for every $a, b \geq 0$, every graph with maximum degree $a + b + 1$ is (a, b) -colorable [7].

4.1. p_1 : $(1, 0, 0)$ -colorable graphs

Every $(1, 0, 0)$ -colorable graph is partitionable since the color class of degree 1 induces a P_3 -free graph and the two color classes of degree 0 induce a bipartite graph, which is K_3 -free.

4.2. p_2 : $(K_4, \overline{C_7}, \text{odd hole})$ -free graphs

This class corresponds to perfect graphs with maximum clique size 3. By the strong perfect graph theorem, their chromatic number is at most 3. So, they are $(0, 0, 0)$ -colorable and thus $(1, 0, 0)$ -colorable. To quickly check that an entry in Table 1 is a subclass of p_2 , recall that K_4^- , C_4 , house, and gem are subgraphs of $\overline{C_7}$.

4.3. p_3 : $(K_4^-, \text{house}, \text{net})$ -free graphs

We define a *big clique* as a maximal clique with at least 3 vertices. Let G be a $(K_4^-, \text{house}, \text{net})$ -free graph and let B be a big clique in G . Consider a vertex $x \in G \setminus B$. Then x cannot be adjacent to every vertex in B since B is maximal. Also, x cannot be adjacent to at least two vertices in B since G is K_4^- -free. So every vertex in $G \setminus B$ is adjacent to at most one vertex in B . Since G is house-free, two vertices in $G \setminus B$ that are adjacent to distinct vertices in B must be non-adjacent. Since G is net-free, at most two vertices in B are adjacent to a vertex in $G \setminus B$. For every big clique in G , we color blue every vertex of the big clique that has no neighbor outside of the big clique. The remaining vertices of G are colored red. This gives a partition of G since the blue part is a disjoint union of cliques and the red part is triangle-free.

To quickly check that an entry in Table 1 is a subclass of p_3 , recall that C_4 is a subgraph of house and that bull is a subgraph of net.

4.4. p_4 : $(K_{1,3}, K_4^-, K_5, \text{butterfly})$ -free graphs

Let us call p'_4 the class of K_3 -free graphs with maximum degree 4 and such that no edge is incident to two vertices with degree at least 3. We show that every graph in p'_4 is edge-partitionable. Suppose that G is a counterexample to this statement that is minimal with respect to the number of edges. If G contains an edge e adjacent to at most 2 edges, then we can extend an edge-partition of $G \setminus e$ by assigning blue to e if and only if every edge adjacent to e is red. So, by minimality, every edge of G is adjacent to at least 3 edges. Since G is in p'_4 , every edge of G is incident to a vertex with degree 3 or 4 and a vertex with degree 1 or 2. Notice that G is a subgraph of some graph considered in Lemma 4.

So G is edge-partitionable, and thus every graph in p'_4 is edge-partitionable. Since $p_4 = L(p'_4)$, every graph in p_4 is partitionable.

To quickly check that an entry in Table 1 is a subclass of p_4 , recall that planar graphs are K_5 -free.

4.5. p_5 : planar $(K_{1,3}, K_4^-, C_4, \dots, C_{10})$ -free graphs with maximum degree 5

Let us call p'_5 the class of planar graphs with girth at least 11, maximum degree 4, and such that no edge is incident to two 4-vertices. We show that every graph in p'_5 is edge-partitionable. Suppose that G is a counterexample to this statement that is minimal with respect to the number of edges. So G is connected and contains at least 4 edges.

Firstly, we show that G does not contain a vertex v with degree at most 3 that is adjacent to a 1-vertex w . Otherwise, we can extend an edge-partition of $G \setminus \{w\}$ by assigning red to the edge vw , unless v is incident to two red edges in $G \setminus \{w\}$.

Secondly, we show that G does not contain a 4-vertex v adjacent to at least two 1-vertices w_1 and w_2 . Since G contains no edge incident to two 4-vertices, the other neighbors w_3 and w_4 of v have degree at most 3. If there exists a partition of $G \setminus \{w_1, w_2\}$ such that vw_3 and vw_4 have the same color, then we assign blue to vw_1 and vw_2 . Otherwise, we assume without loss of generality that vw_3 is red and vw_4 is forced to be blue. This means that the other two edges incident to w_4 are red. So we can assign blue to vw_1 and red to vw_2 .

By the previous properties, the graph G' obtained from G by removing every 1-vertex of G has minimum degree 2. It is well known that every planar graph with girth at least $5k + 1$ and minimum degree 2 contains a path with k 2-vertices (Lemma 5 in [8]). Consequently, G' contains two adjacent 2-vertices x and y . By the previous properties, x and y also have degree 2 in G . Then we can extend an edge-partition of $G \setminus xy$ by assigning red to xy . This contradiction shows that every graph in p'_5 is edge-partitionable.

Notice that $p_5 \subsetneq L(p'_5)$ since the line graph of a graph in p'_5 can be non-planar. So, every graph in p_5 is partitionable.

4.6. p_6 : graphs with maximum degree 3

Whereas the problem is NP-complete for graphs with maximum degree 4, graphs with maximum degree 3 are partitionable since they are $(1, 1)$ -colorable graphs and thus $(1, 0, 0)$ -colorable.

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