2-subcoloring is NP-complete for planar comparability graphs

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Abstract

A k-subcoloring of a graph is a partition of the vertex set into at most k cluster graphs, that is, graphs with no induced P_3. 2-subcoloring is known to be NP-complete for comparability graphs and three subclasses of planar graphs, namely triangle-free planar graphs with maximum degree 4, planar perfect graphs with maximum degree 4, and planar graphs with girth 5. We show that 2-subcoloring is also NP-complete for planar comparability graphs with maximum degree 4.

1. Introduction

A k-subcoloring of a graph is a partition of the vertex set into at most k cluster graphs, that is, graphs with no induced P_3. Unlike k-coloring, k-subcoloring is already NP-complete for $k = 2$:

Theorem 1. *2-subcoloring is NP-complete for the following classes:*

c_1: *(K_4, bull, house, butterfly, gem, odd-hole)-free graphs with maximum degree 5 [1],*

c_2: *triangle-free planar graphs with maximum degree 4 [2, 3],*

c_3: *($K_{1,3}, K_4, K_4^-, C_4$, odd-hole)-free planar graphs [4],*

c_4: *planar graphs with girth 5 [7].*

We refer to [5] for the description of the forbidden induced subgraphs. A graph G is $(d_1, \ldots, d_k)$-*colorable* if the vertex set of G can be partitioned into subsets $V_1, \ldots, V_k$ such that the graph induced by the vertices of V_i has maximum degree at most d_i for every $1 \leqslant i \leqslant k$. Notice that every $(1,1)$-colorable graph is 2-subcolorable. Moreover, on triangle-free graphs, $(1,1)$-colorable is equivalent to 2-subcolorable. As it is well known, for every $a, b \geqslant 0$, every graph with maximum degree $a + b + 1$ is (a, b)-colorable [6]. Thus, every graph with maximum degree 3 is 2-subcolorable, so that the degree bound of 4 in the classes c_2 and c_3 is best possible. Notice that the graphs in c_1 are comparability graphs since they are (bull, house, odd-hole)-free [5].

A natural question is whether 2-subcolorability is NP-complete for the intersection of two classes in Theorem 1. Except maybe for $c_2 \cap c_4$, that is, planar graphs with girth 5 and maximum degree 4, all other intersections contain only 2-subcolorable graphs:

- Graphs in c_1 and c_3 are odd-hole-free and graphs in c_2 and c_4 are triangle-free. So graphs in $c_1 \cap c_2$, $c_1 \cap c_4$, $c_2 \cap c_3$, and $c_3 \cap c_4$ are bipartite.
- A graph G in $c_1 \cap c_3$ is $(K_{1,3}, K_4, K_4^-$, butterfly)-free. So, the neighborhood of every vertex in G is $(3K_1, K_3, P_3, 2K_2)$-free. Thus, the maximum degree of G is at most 3.

Our result restricts the class c_1 to planar graphs and lowers the maximum degree from 5 to 4.

Theorem 2. *Let $\mathcal{G}$ denote the class of (K_4, bull, house, butterfly, gem, odd-hole)-free planar graphs with maximum degree 4. 2-subcoloring is NP-complete for $\mathcal{G}$.*

2. Main result

We reduce the problem of deciding whether a triangle-free planar graph with maximum degree 4 is $(1,1)$-colorable. As already mentioned, this is equivalent to decide whether such a graph is 2-subcolorable, which is NP-complete by the case of the class c_2 in Theorem 1 [2, 3]. From a graph G in c_2, we construct a graph G' in $\mathcal{G}$. Every vertex v of G is replaced by a copy H_v of the vertex gadget H depicted in Figure 1. The six vertices labeled $a_{i,j}$ in H are called *ports*. For every edge uv of G, we use two copies of the edge gadget E depicted in Figure 2 to connect H_u and H_v as follows:

- We identify the two vertices of degree 1 of one copy of E with the port $a_{p,0}$ of H_u and the port $a_{q,1}$ of H_v, with $0 \leqslant p \leqslant 3$ and $0 \leqslant q \leqslant 3$.
- We identify the two vertices of degree 1 of the other copy of E with the port $a_{p,1}$ of H_u and the port $a_{q,0}$ of H_v.

It is easy to check that G' can be made planar. Both E and H have maximum degree 4. The port corresponding to $a_{0,0}$ and $a_{0,1}$ has degree 2 in H and is connected to at most two edge gadgets, thus its degree in G' is at most 4. The port corresponding to $a_{1,0}$ has degree 3 in H and is connected to at most one edge gadget, thus its degree in G' is at most 4. This means that every port in G' has degree at most 4, so the maximum degree of G' is 4. Let S denote the set of vertices of G' whose neighborhood induces a P_3. Then $G' \setminus S$ is a bipartite graph such that all the ports of the vertex gadgets belong to the same part of the bipartition. By adding back S to $G' \setminus S$, we create triangles and induced C_4's, but every larger created induced cycle has the same length (and parity) as a cycle in $G' \setminus S$. Hence, G' is odd-hole free. Thus G' belongs to $\mathcal{G}$. Since G' is perfect and K_4-free, G' is expected to admit a proper 3-coloring: a 3-coloring is given by the partition into the bipartite subgraph $G' \setminus S$ and the independent set S.

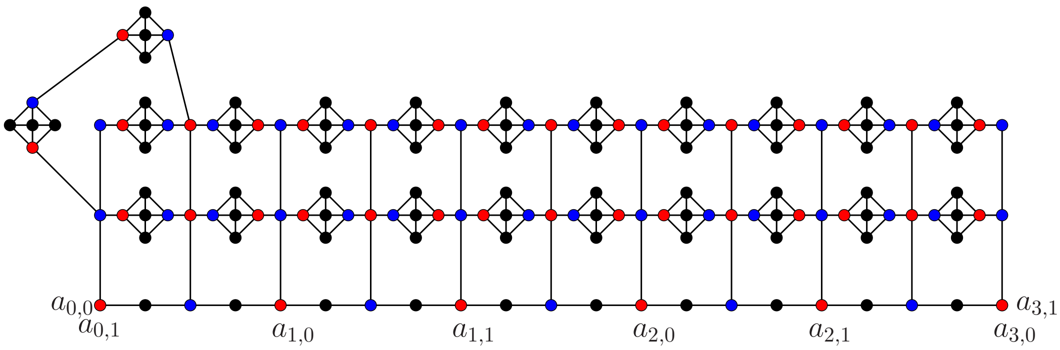

Figure 1: The vertex gadget H.

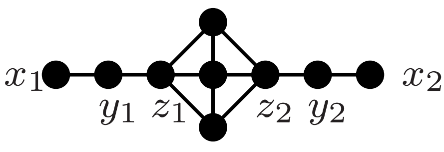

Figure 2: The edge gadget E.

Let us show that G' is 2-subcolorable if and only if G is $(1,1)$-colorable. Given a 2-subcoloring of a graph, we say that a vertex p is *saturated* if there exists a monochromatic edge pq and is unsaturated otherwise. We will need the following properties of E:

1. In every 2-subcoloring of $E \setminus \{x_1, x_2, y_1, y_2\}$, the vertices z_1 and z_2 get distinct colors and are saturated.

2. In every 2-subcoloring of $E \setminus \{x_1, x_2\}$, the vertices y_1 and y_2 get distinct colors and are unsaturated. This follows from Property (1).

3. There exists a 2-subcoloring of E such that the vertices x_1 and x_2 get distinct colors and are unsaturated. Just assign to x_i the color distinct from the color of y_i.

4. In every 2-subcoloring of E such that the vertices x_1 and x_2 get the same color, exactly one vertex in $\{x_1, x_2\}$ is saturated. This follows from Property (2).

The use of E and its properties were already one of the main ingredients in the reduction to the class c_1 in Theorem 1 [1].

We color blue the top right vertex in H. Then we greedily color the vertices whose color is forced by Properties (1), (2), and the absence of monochromatic P_3. This gives the partial 2-subcoloring of H depicted in colors red and blue in Figure 1. The top left part of H enforces that for every port, the two adjacent vertices above the port get the same color. Notice that all the ports in H get the same color. This common color is said to be the color of H. The color of H_v corresponds to the color of v in a $(1, 1)$-coloring of G. Suppose that one port of H is unsaturated. This forces the color of every black vertex on the bottom horizontal path in Figure 1. Then every other port is saturated. Thus, in every 2-subcoloring of H, at most one of the ports is unsaturated.

Suppose that uv is an edge in G. Consider the 2-subcolorings of the subgraph of G' induced by H_u, H_v, and the two edge gadgets for the edge uv. If distinct colors are given to H_u and H_v, then this 2-subcoloring can be extended to the edge gadgets using property (3). Since this extension does not saturate any of the considered ports of H_u and H_v, H_u can be connected to any number of vertex gadgets with the color distinct from the color of H_u. If the same color is given to H_u and H_v, then this 2-subcoloring can be extended using property (4). However, this coloring extension saturates the unique unsaturated port in both H_u and H_v. Thus, H_u can be connected to at most one vertex gadget with the same color as H_u.

Given a $(1, 1)$-coloring of G, we assign the color of every vertex u of G to the ports of H_u. If there exists a monochromatic edge uv in G, we extend the 2-subcoloring of H_u such that one port of H_u connecting H_u and H_v is unsaturated. Then we color the edge gadgets according to Property (4) in the case of a monochromatic edge and according to Property (3) otherwise.

Given a 2-subcoloring of G', we assign the color of H_u to the vertex u in G. Since H_u can be connected to at most one vertex gadget with the same color as H_u, the obtained coloring of G is a $(1, 1)$-coloring.

This shows that G' is 2-subcolorable if and only if G is $(1, 1)$-colorable.

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