Graph operations characterizing rank-width and balanced graph expressions

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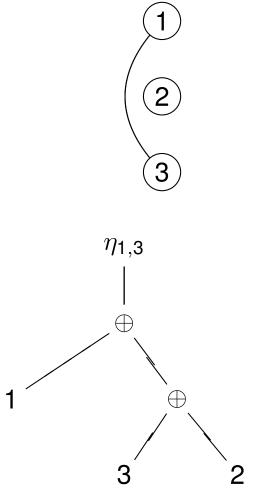
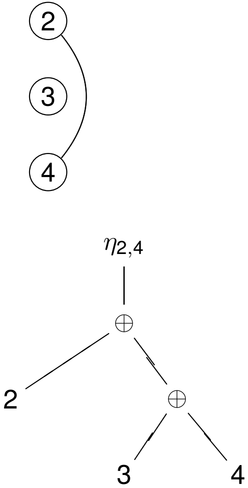
Introduction: Widths, decomposition and inductive computations

- In many cases G has width at most k means G is the value of a term $t \in T(F_k, C_k)$ where F_k is the related signature and C_k a set of constants.
- We have thus inductive computations based on the structure of the terms $\Longrightarrow$ linear time.
- For instance, when a term defining a cograph G ($cwd(G) \leq 2$) is given, we can compute its chromatic number $\chi(G)$ using the following rules:

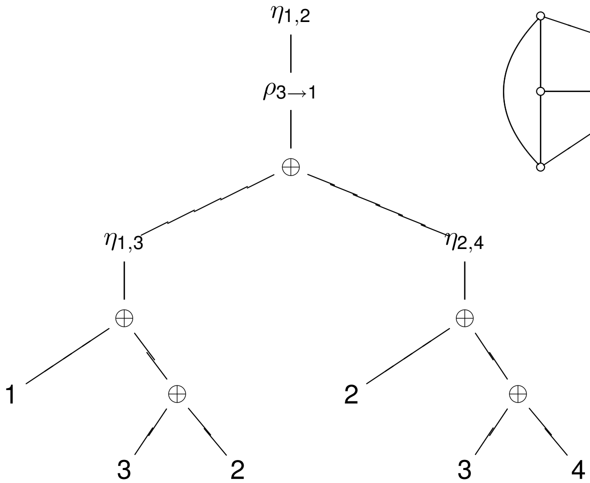
$$\begin{cases} \chi(G \oplus H) = \max\{\chi(G), \chi(H)\}, \\ \chi(G \otimes H) = \chi(G) + \chi(H), \\ \chi(\bullet) = 1 \end{cases}$$

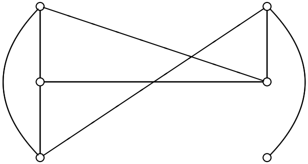
Clique-width(1)





Clique-width (2)





Overview(1)

- As tree-width, when k is part of the input, recognizing graphs of clique-width k is NP-complete (Fellows and al.)
- But contrary to tree-width, there is no polynomial algorithm that decides for fixed k, if a graph G has clique-width at most k.
- Oum and Seymour investigate the problem and define the notion of *rank-width* equivalent to clique-width.
- Rank-width is the parameter associated with a layout of a graph, which is defined as a *branch-decomposition* of a symmetric and submodular function.
- The disadvantage of rank-width is we need to transform the rank-decomposition into a clique-width expression to solve MS problems, which yields an exponential.
- We propose algebraic operations that characterize rank-width, operations based on linear transformations of the $GF(2)$-vector space.

Overview(2)

- The hidden constants yield by the general FPT algorithms based on MS logic are very huge (towers of exponents).
- It is sometimes better to use dynamic programming.
- We need sometimes balanced decompositions.
- It is the case for some applications (labeling schemes considered by Courcelle and al., Gavoille, ... and, NC-algorithms considered by Bodlaender for instance).
- Bodlaender proved that every tree-decomposition of width k can be transformed into a balanced tree-decomposition of width at most $3k+2$.
- We propose a general framework for obtaining balanced decompositions that handle many graph decompositions (rank-width, tree-width, clique-width, ...).

Plan

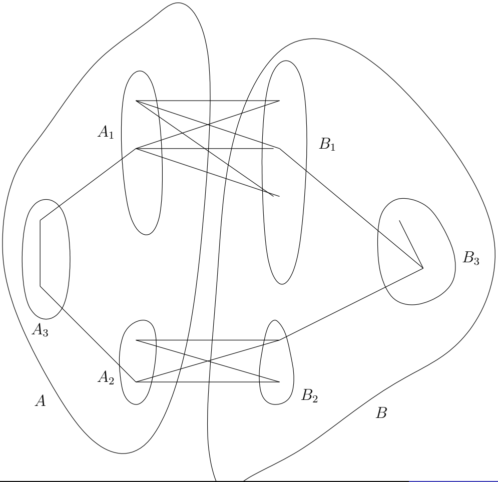
Notations

- Graphs are simple, loop-free and undirected unless specified. They are defined as $< V_G, A_G >$.
- We denote by A_G the adjacency-matrix of a graph G.
- A *sub-cubic tree* is a tree such that the degree of each node is at most 3.

Rank-width

- Let $X \subseteq V_G$. We let $\rho_G(X) = rk(A_G[X, V_G - X])$ where rk is the matrix rank function.
- A layout of a graph G is a pair (T, L) where T is a sub-cubic tree and $L : V_G \to N_T$ where N_T is the set of nodes of degree 1 in T.
- Any edge e of T induced a bipartition (X_e, Y_e) of the set N_T, thus a bipartition $(X, V - X)$ of V_G.
- the width of e is $\rho_G(X)$ and the width of the layout (T, L) is the maximum width over all edges e of T.
- The rank-width of G, denoted by $rwd(G)$, is the minimum width over all layouts of G

Example of layout of a graph(1)

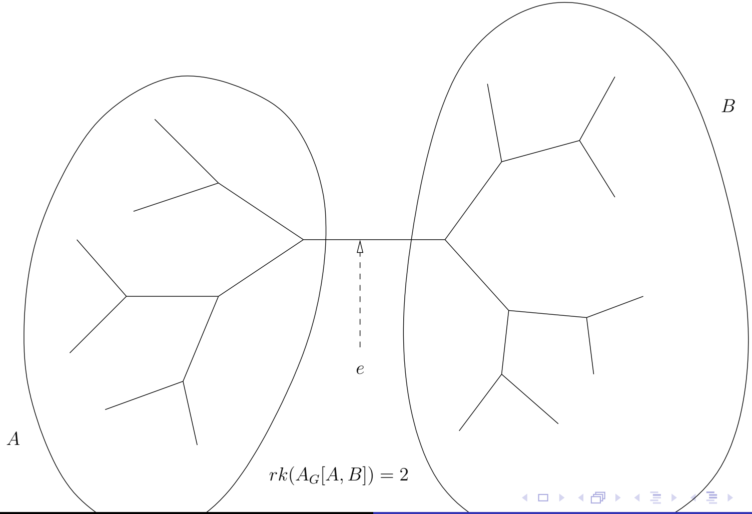


$rk(A_G[A, B]) = 2$

$A_G[A, B] =$

$A \backslash B$	B_1	B_2	B_3
A_1	1	0	0
A_2	0	1	0
A_3	0	0	0

Example of layout of a graph (2)



Properties of rank-width(1)

Clique-width

- $F_k = \{\oplus, \eta_{i,j}, \rho_{i \to j} \mid i, j \in [k], i \neq j\}$ and $C_k = \{\mathbf{i} \mid i \in [k]\}$
- The clique-width of a graph G is the minimum k such that G is the value (up to isomorphism) of a term t in $T(F_k, C_k)$.

Clique-width and Rank-width

$$rwd(G) \leq cwd(G) \leq 2^{rwd(G)+1} - 1.$$

Properties of rank-width(2)

- For fixed k, Oum proposed an $O(n^3)$ time “approximation” algorithm that for a graph G either outputs a layout of width at most $3k-1$ or outputs that G has rank-width at least $k+1$ (advantage over clique-width).
- But we need to transform the layout into a clique-width expression to solve FPT-problems (a disadvantage if it is important that the hidden constants stay small as possible).
- This is why characterization of rank-width by graph operations is interesting.

Plan

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Vectorial colorings

- Handling multiple colorings of vertices with k colors is clearly the same thing as handling colorings with colors in $\{0,1\}^k$ ($GF(2)$-vectors).
- Let $k \in \mathbb{N}$, $\mathbb{B} = \{0,1\}$. A $\mathbb{B}^k$-coloring of a graph G is a mapping $\gamma : V_G \rightarrow \mathbb{B}^k$.
- We consider that $x \in V_G$ has color i (among others) iff $\gamma(x)[i]$ (the i-th component of $\gamma(x)$) is 1.
- A $\mathbb{B}^k$-colored graph is a triple $G =< V_G, edg_G, \gamma_G >$ where γ_G is a $\mathbb{B}^k$-coloring of $< V_G, edg_G >$.
- Let us now define the operations ...

Linear recolorings

Definition

A recoloring $Recol_h$ is linear if $h : \mathbb{B}^k \to \mathbb{B}^\ell$ is linear, i.e., if for some $(k \times \ell)$-matrix N and all graphs G, we have, by letting $H = Recol_h(G)$:

$$\Gamma_H = \Gamma_G.N$$

i.e., $\gamma_H(x) = \gamma_G(x).N$ for each x in V_G.
If h and h' are linear recolorings, described respectively by N and N', then $h \circ h'$ is linear and is described by $N' \times N$.

We say that h is described by N and we note it $Recol_N$.

Remark

2^k different colors are handled efficiently via $(k \times \ell)$-matrices.

Edge creation from bilinear forms

For G, $\mathbb{B}^k$-colored and H, $\mathbb{B}^\ell$-colored, edges between G and H are created in 3 steps.

- The disjoint union of G and H.
- $x \in V_G$ colored by u and $y \in V_H$ colored by v are linked iff $u.M.v^T = 1$.
- Independent recolorings of G and H defined by N and P.

We denote this operations by $\otimes_{M,N,P}$.

Some remarks

Remark

It combines $\oplus$, η and ρ of clique-width and this is done in linear algebra.

Remark

As in the operations by Wanke, these operations add edges between two disjoint graphs, that are the two arguments. This is a difference with clique-width where a single binary operation is used, and $\eta_{i,j}$ applied to $G \oplus H$ may add edges to G and to H.

In terms of products of matrices...

$$A_K = \begin{pmatrix} A_G & \Gamma_G M \Gamma_H^T \\ \Gamma_H M^T \Gamma_G^T & A_H \end{pmatrix}$$

$$\Gamma_K = \begin{pmatrix} \Gamma_G . N \\ \Gamma_H . P \end{pmatrix}$$

where M^T denotes the transposition of the matrix M.

- We let R_n be the set of linear recolorings and bilinear products.
- We denote by $val(t)$ the graph defined by a term $t \in T(R_n, C_n)$. This graph is the value of the term in the corresponding algebra.

Plan

Main theorem

Theorem

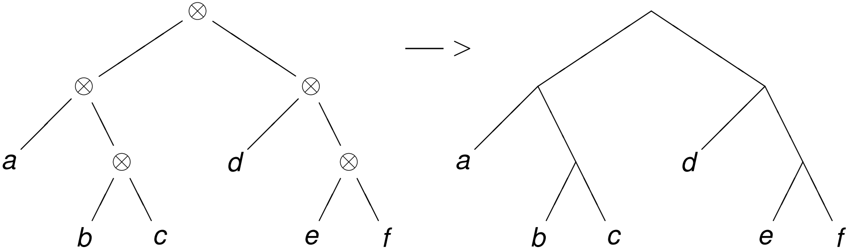
A graph G has rank-width at most n iff it is the value of a term in $T(R_n, C_n)$.

We can assume that a term in $T(R_n, C_n)$ is written with the binary operations $\otimes_{M,N,P}$ and the constants **u** where u is a row-vector in $\mathbb{B}^1 \cup \ldots \cup \mathbb{B}^n$ because it is clear that

$$Recol_Q(G \otimes_{M,N,P} H) = G \otimes_{M,N',P'} H$$

where $N' = N.Q$ and $P' = P.Q$.

Proof of “If direction”(1)



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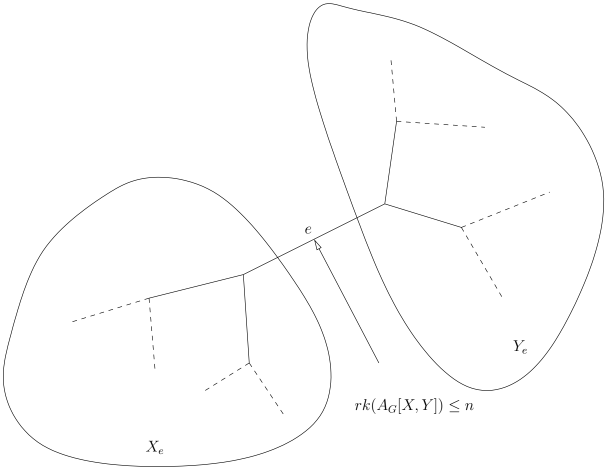
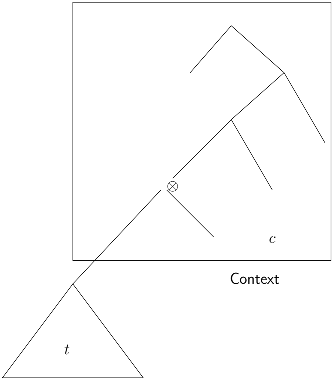
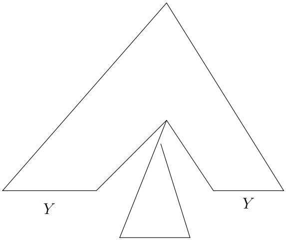
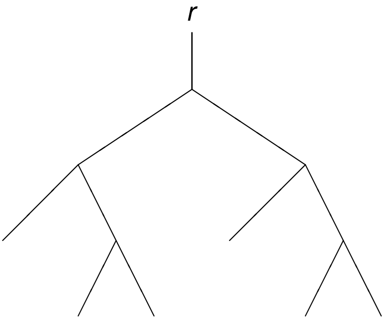


Proof of "If direction"(3)





Proof of converse direction(1)



Layout (T, L) of G

Layout (T, L) of G

- Choose a vertex r of degree 1 in T.
- Direct T with r as root.
- We claim that T directed, is the syntactic tree of a term t that defines G.
- We must show how to label each internal node of T by an operation $\otimes_{M,N,P}$ and how to color each vertex.

Proof of Converse direction (2)

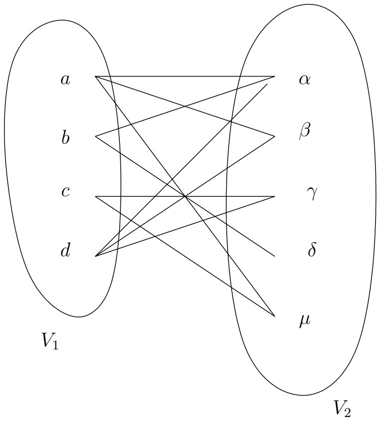
Lemma 1

Let G be a graph with a bipartition $V_G = V_1 \cup V_2$ of its vertices. Let $m = rk(A_G[V_1, V_2])$. Then $G = H \otimes_M K$ where M is a nonsingular $m \times m$ matrix, for some $\mathbb{B}^m$-colorings H and K of $G[V_1]$ and $G[V_2]$ respectively.

Idea of proof of Lemma 1 (1)

	α	β	γ	δ	μ
a	1	1	0	0	1
b	1	0	0	1	0
c	0	0	1	0	1
d	1	1	1	0	0

$M = A_G[\{a,b,c\},\{\alpha,\beta,\gamma\}]$
$A_G[d] = A_G[a] + A_G[c]$
$A_G^T[\delta] = A_G^T[\alpha] + A_G^T[\beta]$
$A_G^T[\mu] = A_G^T[\beta] + A_G^T[\gamma]$



Idea of Proof of Lemma 1(2)

- We call $\{a, b, c\}$ a vertex-basis of H with respect to K

- We call $\{\alpha, \beta, \gamma\}$ a vertex-basis of K with respect to H

$\mathbb{B}^3$-coloring of $G[V_1]$, i.e., H

a	1	0	0
b	0	1	0
c	0	0	1
d	1	0	1

$\mathbb{B}^3$-coloring of $G[V_2]$, i.e., K

α	1	0	0
β	0	1	0
γ	0	0	1
δ	1	1	0
μ	0	1	1

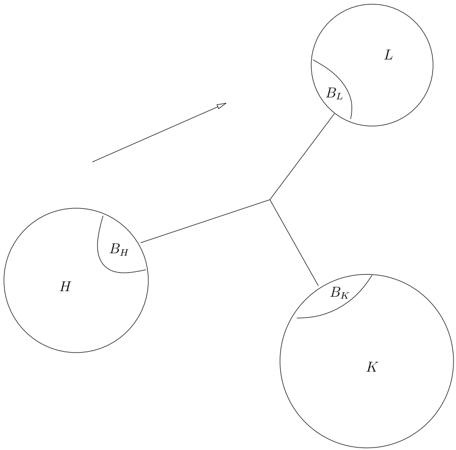
Proof of Converse direction (3)

Lemma 2

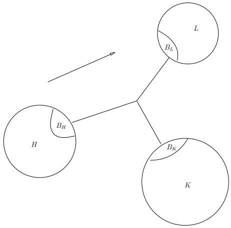
Let G be a graph, let H, K, L be induced subgraphs such that (V_H, V_K, V_L) is a 3-partition of V_G, with each component not empty. Let $h = \rho_G(V_H)$, $k = \rho_G(V_K)$, $\ell = \rho_G(V_L)$. There exist matrices of appropriate dimensions such that

$$G = (H \otimes_{M,N_1,N_2} K) \otimes_P L.$$

Idea of proof of Lemma 2 (1)



Idea of proof of Lemma 2(2)



Choice of vertex-basis

- Choose B_H a vertex-basis of H with respect to $K \cup L$
- Choose B_K a vertex-basis of K with respect to $H \cup L$
- Choose B_L a vertex-basis of L with respect to $H \cup K$
- Extract from $B_H \cup B_K$ a vertex-basis of $H \cup K$ with respect to L.

Idea of proof of Lemma 2 (3)

- By Lemma 1, there exist $\mathbb{B}^h$-coloring of H, $\mathbb{B}^k$-coloring of K and a $(h \times k)$-matrix M such that $G[H \cup K] = H \otimes_M K$.
- $B_H \cup B_K$ is included in a vertex-basis of $H \cup K$ with respect to G. By Lemma 1, there exists a matrix P such that $G = G' \otimes_P L$ where G' is a $\mathbb{B}^\ell$-coloring of $G[H \cup K]$.
- We can thus find a recoloring of H by N_1 and a recoloring of K by N_2 such that $G' = H \otimes_{M,N_1,N_2} K$.
- We have then $G = (H \otimes_{M,N_1,N_2} K) \otimes_P L$.

Plan

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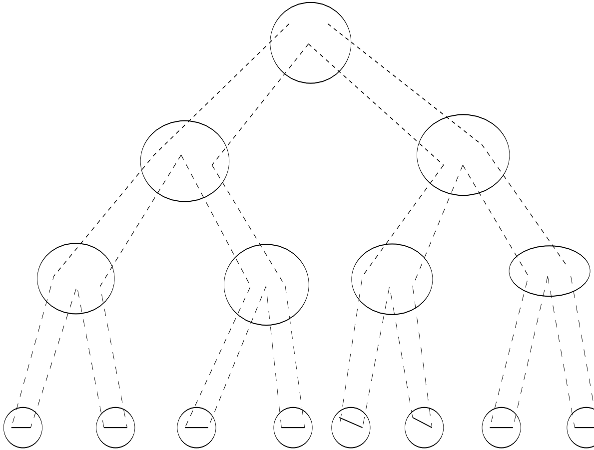
Balacing Theorems: General ideas



Generalisation

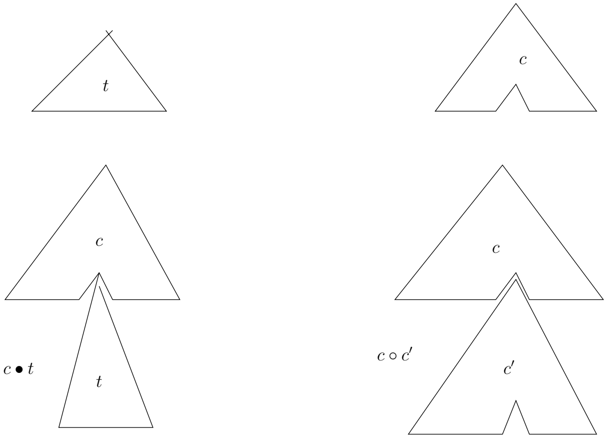
- Each tree has a balanced tree-decomposition of width 2 and of height at most $3\log(n)$.
- Bodlaender proves that every graph of tree-width k admits a tree-decomposition of width at most $3k+2$ and of height at most $\log(n)$).
- Courcelle and Vanicat proves that every graph of clique-width k admits a clique-width expression of height at most $3\log(n)$ and uses $k.2^k$ colors at most.
- Courcelle and Twigg proves that every graph of m-clique-width [variant of clique-width that uses several colors of each vertex] k has a m-clique-width expression of height at most $3\log(n)$ and uses $2k$ colors at most.
- Our result unifies all these results and as a corollary we get that every graph of rank-width k admits a layout of height at most $3\log(n)$ and of width at most $2k$.

Plan

Tools - Informal definition

- We let F_k (e.g. $\{\oplus, \eta_{i,j}, \rho_{i,j} \mid i,j \in [k], i \neq j\}$) be a set of operations and C_k (e.g. $\{\mathbf{i} \mid i \in [k]\}$) be a set of constants.
- A term is a well-formed expression in $T(F_k, C_k)$
- A context is a well-formed expression in $T(F_K, C_k \cup \{u\})$ with one and only one occurrence of u.
- $c \bullet t = c[t/u] \in T(F_k, C_k)$ for c a context and t a term
- $c \circ c' = c[c'/u] \in T(F_K, C_k \cup \{u\})$ for c, c' contexts.

Tools - Some visual intuitions



Intuition of the framework

Iterated splitting

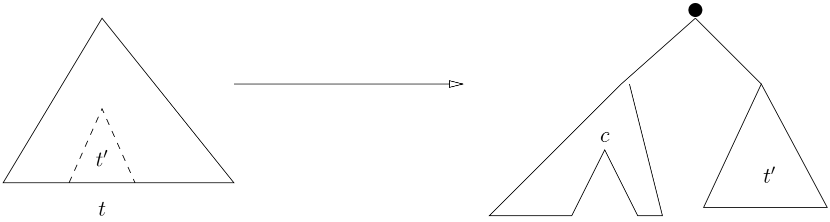
Transform the term t into a term t^b of height at most $3\log(n)$ written with symbols in $F_k \cup C_k \cup \{\bullet, \circ\}$.

Elimination of $\bullet$ and $\circ$ by operations

We replace the operations $\bullet$ and $\circ$ in t^b by operations that uses more colors, but not so much.

Iterated splitting(1)

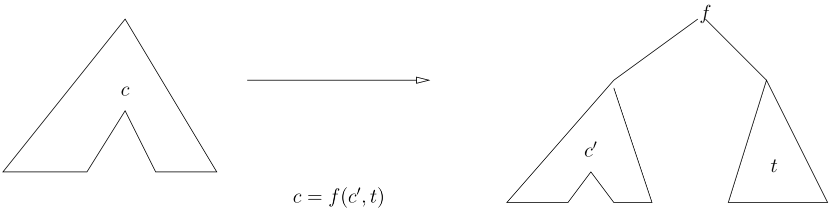
- We split a term t into a context c and a term t' such that c and t' are as equal as possible => $t = c \bullet t'$.



$$t = c \bullet t'$$

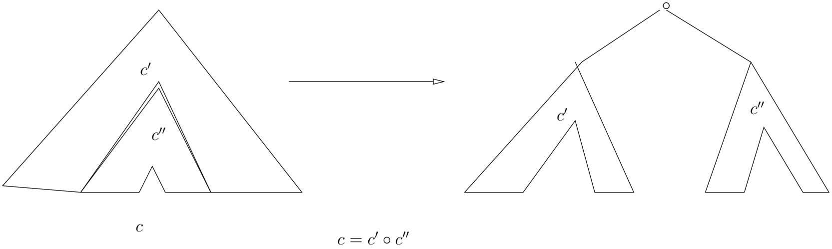
Iterated splitting(2)

- A context c is written $c = f(c', t)$ if $|t| \simeq |c'|$ or splitted (next slide).



Iterated splitting(3)

- A context c can be splitted into context c', c'' with c' and c'' as "equal" as possible and $c = c' \circ c''$.



Elimination of • and ∘

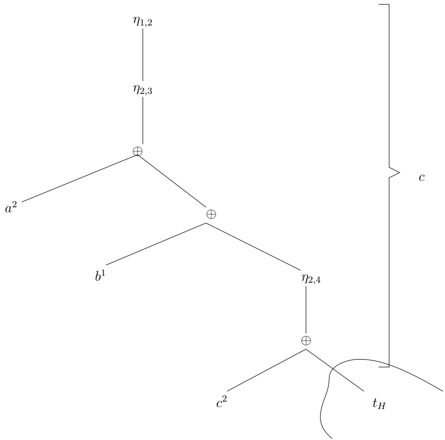
For terms and contexts over graph operations, we use the following rules to transform a term written with symbols in $F_k \cup C_k \cup \{\bullet, \circ\}$ into a term in $T(F_{k'}, C_{k'})$ with $k' \geq k$:

- A term t is replaced by $val(t)$.
- $c \bullet t_H$ is replaced by $f'(G_c, H)$ where $H = val(t_H)$ and G_c is a colored graph that represents the context c.
- $c \circ c'$ is replaced by $f''(G_c, G_{c'})$ (inductivity of the construction)

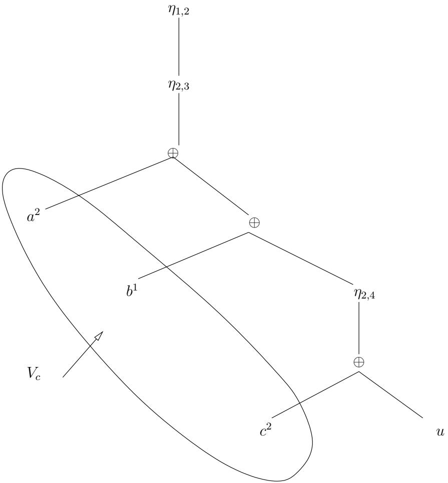
$\Longrightarrow$

With the rules above, one can eliminate • and ∘ and replace them by operations using more colors.

What is G_c: Example of clique-width over $[k]$? (1)



What is G_c: Example of clique-width over $[k]$? (2)



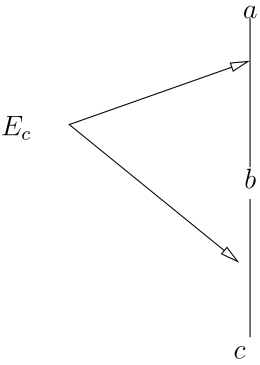
The set of vertices V_c

V_c is the set of leaves in c different from u.

What is G_c: Example of clique-width over $[k]$? (3)

The set of edges E_c

$$E_c = E_{G[V_c]}.$$

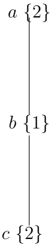


What is G_c: Example of clique-width over $[k]$? (4)

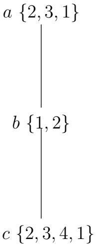
The color of a vertex

Each vertex v in V_c is colored by its color in $c[\emptyset/u]$ + a set J, which is the set of labels $j \in [k]$ such that v is linked to a vertex in H colored by j in H.

$val(c[\emptyset/u]) =$



$G_c =$



Conclusion

- This is why we have $k.2^k$ colors for clique-width.
- The same method works for m-clique-width and rank-width but we have $2k$ instead of the exponent.

Conclusion

- Rank-width is an interesting complexity measure because it is equivalent to clique-width, a powerful graph complexity measure, and also because it is increasing for the vertex-minor inclusion and has a polynomial approximation algorithm.
- We propose an algebraic characterization of the notion of rank-width by means of linear transformations of the $GF(2)$-vector space.
- Because in some applications, balanced decompositions are crucial, we propose here a general framework that handle many graph decompositions (tree-decomposition, clique-width, NLC-width, m-clique-width, Boolean-width and rank-width.

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Perspectives

- The tightness of the bound of the balanced terms.
- Generalize the general framework of balanced terms to arbitrary structures (not only binary ones).
- Can we do the balancing in linear time instead of $O(n\log(n))$?
- We propose a definition of rank-width for directed graphs, study it.

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Plan

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Proof (if direction)

- Let $t \in T(R_k, C_k)$ be such that $G = val(t)$.
- We let t have its root colored by a binary operation $\otimes_{M,N,P}$. We take the syntactic tree of t as a layout of G.

Claim

If $t = c \bullet t'$, $t' \in T(R_n, C_n)$, $c \in Cxt(R_n, C_n) - \{Id\}$, $G = val(t)$, $H = val(t')$ then we have:

1. $A_G[V_H, V_G - V_H] = \Gamma_H.B$ *and* $\Gamma_{G[V_H]} = \Gamma_H.C$ *for some matrices* B *and* C,
2. $rk(A_G[V_H, V_G - V_H]) \leq n$.

Proof of the claim

- *(2)* is an immediate consequence of (1).
- For *(1)*, we use an induction on the size of c. We have 2 cases:
  - $c = Id \otimes_{M,N,P} t''$ (similarly $c = t'' \otimes_{M,N,P} Id$), i.e., $G = K \otimes_{M,N,P} H$ where $K = val(t'')$.
  - $c = c' \otimes_{M,N,P} t''$ (similarly $c = t'' \otimes_{M,N,P} c'$) where $c' \neq Id$, i.e., $G = G' \otimes_{M,N,P} K$ where $G' = val(c' \bullet t')$ and $K = valt(t'')$.

The proof follows from the definition and inductive hypothesis.■

Proof Scketch of the lemma 1(1)

- We select m vertices $x_1, \ldots, x_m$ in V_H with associated row-vectors independent and m vertices $y_1, \ldots, y_m$ in V_K with associated column-vectors independent.
- We denote by $x_{m+1}, \ldots, x_p$ and $y_{m+1}, \ldots, y_q$ the other vertices of H and K respectively.
- We denote by $A_G[x_i]$ the row-vector associated with x_i. For each $j = 1, \ldots, p$ we have

$$A_G[x_j] = \sum_{1 \leq i \leq m} b_{ij} A_G[x_i]$$

- Similarly for $j = 1, \ldots, q$ we let

$$A_G^T[y_j] = \sum_{1 \leq i \leq m} c_{ij} A_G^T[y_i]$$

Proof Scketch of the lemma 1(2)

- We let $M = A_G[\{x_1, \ldots, x_m\}, \{y_1, \ldots, y_m\}]$ and $\gamma_H(x_j) = (b_{1j}, \ldots, b_{mj})$ and $\gamma_K(y_j) = (c_{1j}, \ldots, c_{mj})$.
- We prove that $G = H \otimes_M K$ using the definitions.
- By the choice of $x_1, \ldots, x_m$ and $y_1, \ldots, y_m$ the matrix M is non-singular.$\blacksquare$

Remark

- The construction above can be done if $\{A[x_i] \mid i = 1, \ldots, p\}$ where $p \geq rk(A)$ generates the row-vectors and $\{A^T[y_j] \mid j = 1, \ldots, q\},\ q \geq rk(A)$ generates the column-vectors. We call $\{x_1, \ldots, x_p\}$ a *vertex-generator* of H with respect to G.
- The linear combinations expressing $A[x_i]$ and $A^T[y_j]$ are not unique. Then we get $G = H \otimes_M K$ with M of dimension $p \times q$. We have the following from the proof of the previous lemma:

Proposition

Assume $G = H \otimes_A K$ with A of dimension $p \times q$ of rank k. Let M be a $k \times k$ sub-matrix of rank k of A. Then we have N of dimension $p \times k$, P of dimension $q \times k$ such that $A = N.M.P^T$ and $G = Recol_N(H) \otimes_M Recol_P(K)$.

Example 2

The same example can be used. We check $A = N.M.P^T$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \hline 1 & 0 & 1 \end{pmatrix} \times \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \times \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \times \left(\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{array}\right) = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}$$

Proof of Lemma 2(1)

We let $V_H = \{1, 2, \ldots, 5\}$, $V_K = \{a, b, c, d, e\}$, $V_L = \{\alpha, \beta, \gamma, \delta\}$. The portion of A_G relevant to the expression of G as $(H \otimes_{M,N_1,N_2} K) \otimes_P L$ is:

$A_G =$

	a	b	c	d	e	α	β	γ	δ
1	1	0	0	1	1	1	0	0	1
2	1	1	0	0	0	1	1	0	0
3	0	1	1	1	0	0	1	1	1
4	0	1	0	1	1	0	1	0	1
5	1	0	1	1	0	1	0	1	1
a						0	1	1	1
b						1	1	1	0
c						1	0	1	1
d						1	0	0	1
e						0	0	1	0

Proof of Lemma 2(2)

We have the following linear relation between rows and columns of $A_G[V_H, V_K \cup V_L]$, $A_G[V_K, V_H \cup V_L]$ and $A_G[V_L, V_H \cup V_K]$:

$$
\begin{aligned}
4 &= 1 + 2 \\
5 &= 2 + 3 \\
\delta &= \alpha + \beta \\
d &= a + b \\
e &= a + b + c
\end{aligned}
$$

Proof of Lemma 2(3)

Vertex bases are for $H : \{1, 2, 3\}$, for $K : \{a, b, c\}$, and for $L : \{\alpha, \beta, \gamma\}$. Among $\{1, 2, 3, a, b, c\}$, one can select $\{1, 2, a\}$.

$M =$

	a	b	c
1	1	0	0
2	1	1	0
3	0	1	1

$P =$

	α	β	γ
1	1	0	0
2	1	1	0
a	0	1	1

$N_1 =$

	1	2	a
1	1	0	0
2	0	1	0
3	0	0	1

$N_2 =$

	1	2	a
a	0	0	1
b	1	0	1
c	0	1	1

Proof of the “only if” direction(1)

- We let (t, L) be a layout of with n of G ($rwd(G) = n$).
- We root t with a leaf s as root.
- The weight of a node u of t is the number of the nodes of t/u, the subtree of the directed tree t rooted at u.
- For every edge of t of the form $\overrightarrow{vu}$, we let G_u be the induced subgraph of G, the vertices of which are the leaves of t/u, and G^v the induced subgraph of G, the vertices of which are the leaves not in t/u.

Claim

One can choose for each u different from the root of t, a vertex basis B_u of G_u relative to G; there corresponds to B_u a $\mathbb{B}^{r(u)}$-coloring of G_u where $r(u)$ is the width of $\overrightarrow{vu}$, and a term t_u in $T(R_n, C_n)$ defining G_u colored in this way.

Proof of the “only if” direction(2)

- The proof of the claim is by induction on the weight $w(u)$ of u, using the technical lemmas above.■
- For the case of u where $r \rightarrow u$, r is the root of t, we have G^r defined by $\mathbf{1}$ and G defined by $t_u \otimes_{P,O,O} \mathbf{1}$, where t_u is obtained by the claim above. This terminates the proof of the main theorem.■

Plan

Special Terms(1)

Definition

We let $S = T(F \cup \{\circ, \bullet\}, C \cup \{Id\})$. We let S_c and S_t be the least subsets of S such that:

$$S_t := S_c \bullet S_t \ \cup \ f(S_t, S_t) \ \cup \ b$$
$$S_c := S_c \circ S_c \ \cup \ f(S_t, S_c) \ \cup \ f(S_c, S_t) \ \cup \ f(S_t, Id) \ \cup \ f(Id, S_t)$$

with rules for each f in F, each b in C. We denote them by $SPE_t(F, C)$ and $SPE_c(F, C)$ if we need to specify F and C. Note that $Id \notin S_t \cup S_c$.

Every term t in $SPE_t(F, C)$ evaluates into a term $Eval(t)$ in $T(F, C)$ and every term c in $SPE_c(F, C)$ evaluates into a context $Eval(c)$ in $Cxt(F, C) - \{Id\}$.

Special terms(2)

Theorem

For every term t in $T(F, C) - C$ one can construct a term t^b in $SPE_t(F, C)$ such that $|t^b|_{FC} = |t|_{FC} = |t|$, $Eval(t^b) = t$ and $ht(t^b) \leq 3\log(|t| - 1)$. This term can be constructed in time $O(n\log(n))$ if $n = |t|$.

Flexibility

(F', C') is (F, C)-*flexible* if the following conditions hold:

1. F and F' are commutative.
2. There exist three mappings: $q \mapsto \hat{q}$, $q \mapsto f^q$ and $(q, q') \mapsto f^{q,q'}$ which satisfy the following properties:
   - (2.1) For every comb-term $q(x_1, \dots, x_n, u)$ over F with $n \geq 2$, $\hat{q}$ is a comb-term $\hat{q}(x_1, \dots, x_n)$ over F'.
   - (2.2) If $q(x_1, u)$ is the comb-term $g(x_1, u)$ then $\hat{q} = x_1$ and $f^q = g$.
   - (2.3) For every q as in (2.1), we have $f^q \in F'$ and $q \simeq f^q(\hat{q}, u)$.
   - (2.4) For every two comb-terms as in (2.1) or (2.2) $q(x_1, \dots, x_p, u)$ and $q'(x_1, \dots, x_n, u)$ we have $f^{q,q'} \in F'$ and

     $$\widehat{q''} \simeq f^{q,q'}(\hat{q}(x_1, \dots, x_p), \widehat{q'}(x_{p+1}, \dots, x_{p+n}))$$

     where $q'' = q(x_1, \dots, x_p, q'(x_{p+1}, \dots, x_{p+n}, u))$.

If q is a comb-term as in (2.2), Property (2.3) also holds from the definitions of $\hat{q}$ and f^q.

General theorem

Proposition

If (F', C') is (F, C)-flexible, then for every term t in $SPE_t(F, C)$ one can define a term $\tilde{t}$ in $T(F', C')$ that is equivalent to t and such that $|\tilde{t}|_{F'C'} = |t|_{FC}$ and $ht(\tilde{t}) \leq ht(t)$.

The main theorem of this section is:

Theorem

Let (F', C') be an (F, C)-flexible $\mathcal{S}$-signature. Every term t in $T(F, C)$ of size n is equivalent to a 3-balanced term t' in $T(F', C')$. This term can be constructed in time $O(n\log(n))$, if we assume that $\hat{q}$, f^q, $f^{q,q'}$ can be constructed in time $O(\max\{|q|, |q'|\})$.

Consequence of the above theoreom

Theorem

1. *Every graph of m-clique-width k is the value of a 3-balanced term of m-clique-width at most $2k$.*
2. *Every graph of rank-width k is the value of a 3-balanced term of rank-width at most $2k$.*
3. *Every graph of clique-width or NLC-width k is the value of a 3-balanced clique-width expression of clique-width or NLC-width at most $k \times 2^{k+1}$.*

Proof Sketch

- We show the method for rank-width.
- We let (R_k, C_k) denote the signature for graphs with vectors of colors in $\mathbb{B}^k$. We let (R'_{2k}, C_k) denote the signature for graphs with vectors of colors in $\mathbb{B}^k \cup \mathbb{B}^{2k}$.
- We must prove that the signature (R'_{2k}, C_k) is (R_k, C_k)-flexible.
- For that we show only how to construct f^q and $f^{q,q'}$ and the terms $\hat{q}$.
- Let $q = \otimes_1(x_1, \otimes_2(x_2, \ldots, \otimes_n x_n)) \ldots)$ where $\otimes_i = \otimes_{M_i,N_i,P_i}$.

Construction of $\hat{q}$

We let

$$\hat{q} = x_1 \otimes'_1 (x_2 \otimes'_2 (\dots (x_n \otimes'_n \emptyset_k)) \dots)$$

where $\otimes'_i = \otimes_{M'_i, N'_i, P'_i}$ with M'_i, N'_i are $(k \times 2k)$-matrices and P'_i are $(2k \times 2k)$-matrices and such that:

$$M'_i = \begin{pmatrix} M_i & 0_k \end{pmatrix} \quad N'_i = \begin{pmatrix} N_i & M_i.Q_i^T \end{pmatrix} \quad P'_i = \begin{pmatrix} P_i & 0_k \\ 0_k & I_k \end{pmatrix}$$

Construction of f^q

We let $\otimes^q = \otimes_{M,N,P}$ where:

$$M = \begin{pmatrix} O_k \\ I_k \end{pmatrix} \qquad N = \begin{pmatrix} I_k \\ 0_k \end{pmatrix} \qquad P = P_n.P_{n-1}.\dots.P_1$$

Construction of $f^{q,q'}$:general case

Let $q = x_1 \otimes_1 (x_2 \otimes_2 (\ldots(x_p \otimes_p u))\ldots)$ and $q' = x_{p+1} \otimes_{p+1} (x_{p+2} \otimes_{p+2} (\ldots(x_n \otimes_n u))\ldots)$ so that $q'' = x_1 \otimes_1 (x_2 \otimes_2 (\ldots(x_n \otimes_n u))\ldots)$. We let $f^{q,q'} = \otimes_{M,N,P}$ where:

$$M = \begin{pmatrix} O_k & 0_k \\ I_k & 0_k \end{pmatrix} \quad N = \begin{pmatrix} I_k & 0_k \\ 0_k & Q_{q'}^T \end{pmatrix} \quad P = \begin{pmatrix} Q_q & 0_k \\ 0_k & I_k \end{pmatrix}$$

where $Q_q = P_p.P_{p-1}.\ldots.P_1$ and $Q_{q'} = P_n.P_{n-1}.\ldots.P_{p+1}$.

The operations $\otimes^{q,q'}$ in particular cases

(1) $p = 1$, $n = 2$. Then $q = x_1 \otimes_1 u$, $q' = x_2 \otimes_2 u$, $\hat{q} = x_1$, $\widehat{q'} = x_2$. In this case $q'' = x_1 \otimes_1 (x_2 \otimes_2 u)$. We let $\widehat{q''} = x_1 \otimes^{q,q'} x_2$ with $\otimes^{q,q'} = \otimes_{M,N,P}$ where :

$$M = M_1.N_2^T \quad N = \begin{pmatrix} N_1 & M_1.P_2^T \end{pmatrix} \quad P = \begin{pmatrix} P_2.P_1 & M_2 \end{pmatrix}$$

It is clear that $\widehat{q''} \simeq \hat{q} \otimes^{q,q'} \widehat{q'}$.

The operations $\otimes^{q,q'}$ in particular cases

(2) $p = 1$, $n > 2$. Then $q = x_1 \otimes_1 u$,
$q' = x_2 \otimes_2 (x_3 \otimes_3 \ldots (x_n \otimes_n u)) \ldots)$, $\hat{q} = x_1 \otimes'_1 \emptyset_k$. In this case we can apply the construction of the general case which gives $\otimes^{q,q'}$.
In particular we have $Q_q = P$.
We have thus

$$\widehat{q''} = (G_1 \otimes'_1 \emptyset_k) \otimes^{q,q'} \widehat{q}'(G_2, \ldots, G_n).$$

But we can merge $G_1 \otimes'_1 \emptyset_k$ and $\otimes_{q,q'}$ into a single operation $\bar{\otimes}^{q,q'} = \otimes_{M,N,P}$ where:

$$M = \begin{pmatrix} M_1 & 0_k \end{pmatrix} \quad N = \begin{pmatrix} N_1 & M_1.Q_{q'}^T \end{pmatrix} \quad P = \begin{pmatrix} P_1 & 0_k \\ 0_k & I_k \end{pmatrix}$$

The operations $\otimes^{q,q'}$ in particular cases

(3) $p > 1$, $n = p + 1$. This case is similar. We have $\widehat{q'} = x_2 \otimes'_2 \emptyset_k$ and we get

$$\widehat{q''} = \hat{q} \otimes^{q,q'} (x_2 \otimes'_2 \emptyset_k)$$

and we can derive an operation $f^{q,q'}$ as wanted.

(4) By using the same technique as above, we can eliminate the constant $\emptyset_k$ from every $\hat{q}$. We have thus proved that (R'_{2k}, C_k) is (R_k, C_k)-flexible. ■

Plan

Contexts

Definition

A context is a term in $T(F, C \cup \{u\})$ having a single occurrence of the variable u (a nullary symbol). We denote by $Cxt(F, C)$ the set of contexts. We denote by Id the particular context u.

- Let s be a context and t be a term or a context, we denote by $s[t/u]$ the result of the substitution of t for u in s.
- $s \circ s' = s[s'/u]$, belonging to $Cxt(F, C)$ for s, s' in $Cxt(F, C)$.
- $s \bullet t = s[t/u]$, belonging to $T(F, C)$ for s in $Cxt(F, C)$ and t in $T(F, C)$.

M-clique-width

- The notion of m-clique-width is introduced by Courcelle and Twigg (STACS'07) for constraint labeling schemes.
- Let L be a finite set of colors. A multi-labeled graph is defined as $< V_G, edg_G, \delta_G >$ where $\delta_G(x) \subseteq L$ (a vertex may have zero, one or several colors).
- For $R \subseteq L \times L$, $g, h : L \to \mathscr{P}(L)$ and for G, H disjoint multi-labeled graphs, we let $K = G \otimes_{R,g,h} H$ where $V_K = V_G \cup V_H$ and :

$$\begin{aligned}
edg_K &= edg_G \cup edg_H \\
&\quad \cup \{xy \mid x \in V_G, y \in V_H, R \cap (\delta_G(x) \times \delta_H(y)) \neq \emptyset\}, \\
\delta_K(x) &= (g \circ \delta_G)(x) = \{a \mid a \in g(b), b \in \delta_G(x)\} \text{ if } x \in V_G, \\
\delta_K(x) &= (h \circ \delta_H)(x) \text{ if } x \in V_H.
\end{aligned}$$

Recolorings

Definition

For $h : \mathbb{B}^k \rightarrow \mathbb{B}^\ell$ and G a $\mathbb{B}^k$-colored graph, we let $Recol_h(G) = G' =< V_G, edg_G, \gamma_{G'} >$ where $\gamma_{G'} = h \circ \gamma_G$. Hence G' is a $\mathbb{B}^\ell$-colored graph.

Graph products

Definition

Let $f : \mathbb{B}^k \times \mathbb{B}^\ell \to \{0, 1\}$, let $g : \mathbb{B}^k \to \mathbb{B}^m$ and $h : \mathbb{B}^\ell \to \mathbb{B}^m$ be arbitrary mappings. For G, $\mathbb{B}^k$-colored and H, $\mathbb{B}^\ell$-colored, we let $K = G \otimes_{f,g,h} H$ be defined as follows, where, as usual, we assume $V_G \cap V_H = \emptyset$:

$$
\begin{aligned}
&V_K = V_G \cup V_H,\\
&edg_K = edg_G \cup edg_H \cup \{xy \mid x \in V_G, y \in V_H, f(\gamma_G(x), \gamma_H(y)) = 1\},\\
&\gamma_K(x) = (g \circ \gamma_G)(x) \text{ if } x \in V_G,\\
&\gamma_K(x) = (h \circ \gamma_H)(x) \text{ if } x \in V_H.
\end{aligned}
$$

Hence K is a $\mathbb{B}^m$-colored graph.

Multiple colorings (suite)

- For each $u \in \mathbb{B}^n$ we let **u** be a constant denoting a graph with one vertex colored by u and no edge.
- We denote by C_n the set of this constants. We use the constant $\emptyset_k$ to denote the empty $\mathbb{B}^k$-colored graph.
- We let B_n be the finite set of operations $Recol_h$, $\otimes_{f,g,h}$ where $k, \ell, m \in [n]$. It is not a loss of generality to assume $k, l, m \neq 0$.
- Every graph G is assigned a *default* $\mathbb{B}^1$*-coloring* with $\gamma_G(x) = (1)$ for every vertex x.
- Every term $t \in T(B_k, C_k)$ denotes a $\mathbb{B}^k$-colored graph for some $k \geq 1$, or actually the family of all graphs isomorphic to such a graph.
- The *Boolean-width* of a graph G, denoted by $Bwd(G)$ is the minimum n such that G is defined, up to isomorphism, by a term in $T(B_n, C_n)$.

Multiple colorings(suite)

The operations of B_n represent the "best we can do with n vertex colors" if we only allow:

- manipulations of vertex colors by quantifier-free operations, and done only in terms of existing vertex colors,
- creations of edges between two disjoint graphs based on vertex colors, independently on both sides.

What we do not allow:

- vertex color manipulations that depend on existence of loops incident with a vertex (actually we do not consider loops at all)
- edge deletions (and creations) of edges (in place of non-existing ones), like in the edge-complement operation.

Some comparisons

Proposition

For every uncolored undirected graph G,

1. $rwd(G) \leq cwd(G) \leq 2^{rwd(G)+1} - 1$.
2. $Bwd(G) \leq mcwd(G) \leq cwd(G) \leq 2^{Bwd(G)+1} \leq 2^{mcwd(G)+1}$.
3. $mcwd(G) \leq twd(G) + 3$.
4. $rwd(G) \leq 4 \times twd(G) + 2$.

$\mathbb{B}^k$-colored graphs as relational structures

- The meaning of $c_i(x) = true$ will be "x has color i".
- A $\mathbb{B}^n$-colored graph $G =< V_G, edg_G, \gamma_G >$ for $n \leq k$ is faithfully described by the relational structure with domain V_G that we will also denote by G:

$$< V_G, edg_G, c_{1G}, \ldots, c_{kG} > .$$

- Every relational structure of this form, and such that edg_G is symmetric and anti-reflexive ($edg_G(x, x)$ never holds) represents a $\mathbb{B}^k$-colored graph G.
- The notions of quantifier-free operations are well-known from the results of Courcelle and al.

FPT-algorithms

Proposition

1. *The operations $Recol_h$ are quantifier-free operations.*
2. *The operations $\otimes_{f,g,h}$ are expressible in terms of $\oplus$ and quantifier-free operations.*

Corollary

For each n, every MS graph property of a graph G can be decided in time $O(|t|)$, if G is the value of a given term $t \in T(B_n, C_n)$.