Split Decomposition and Dynamic Distance Hereditary Graphs

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Joint work with E. Gioan (CNRS LIRMM)

Dynamic graph representation problem:

Given a representation $R(G)$ of a graph G and a edge or vertex modification of G (insertion or deletion) update the representation $R(G)$.

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When restricted to a certain graph family $\mathcal{F}$, the algorithm should:

1. check whether the modified graph still belongs to $\mathcal{F}$;
2. if so, udpate the representation;
3. otherwise output a certificate (e.g. a forbidden subgraph).

Some keys of the problem

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   - modular tree decomposition [Gallai'67]
   - split decomposition [Cunningham'82]
   - PQ-tree [Booth and Lueker'76]

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   - PQ-tree [Booth and Lueker'76]
3. Need of an incremental (dynamic) characterization.

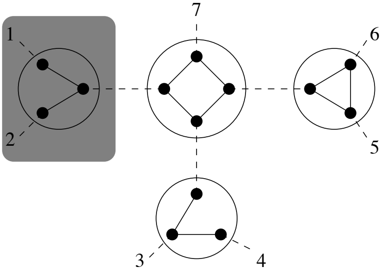
Some known results

	vertex modification	edge modification
proper intervals	$O(d + \log n)$ [HSS02]	$O(1)$ [HSS02]
cographs	$O(d)$ **[CPS85]**	$O(1)$ [SS04]
permutations	$O(n)$ [CP05]	$O(n)$ [CP05]
distance hereditary	$O(d)$ **[GP07]**	$O(1)$ [CT07]
intervals	$O(n)$ [C07]	$O(n)$ [C07]

Graph labeled tree

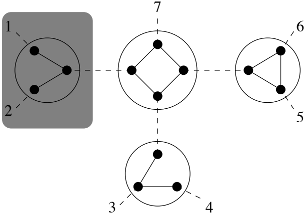
A *graph-labelled tree* is a pair $(T, \mathcal{F})$ with T a tree and $\mathcal{F}$ a set of graphs such that:

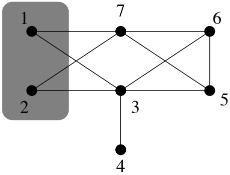
- each node v of degree k of T is labelled by a graph $G_v \in \mathcal{F}$ on k vertices
- there is a bijection ρ_v from the tree-edges incident to v to the vertices of G_v

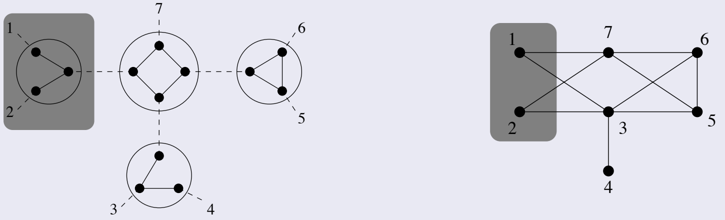


Given a graph labelled tree $(T, \mathcal{F})$, the graph $G_S(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$, $\forall$ tree-edges uv, vw on the x, y-path in T







Split

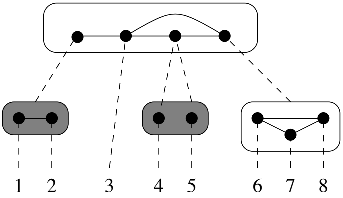
A bipartition (A, B) of the vertices of a graph $G = (V, E)$ is a **split** iff

- $|A| \geqslant 2$, $|B| \geqslant 2$;
- for $x \in A$ and $y \in B$, $xy \in E$ iff $x \in N(B)$ and $y \in N(A)$.

Rooted graph labeled tree

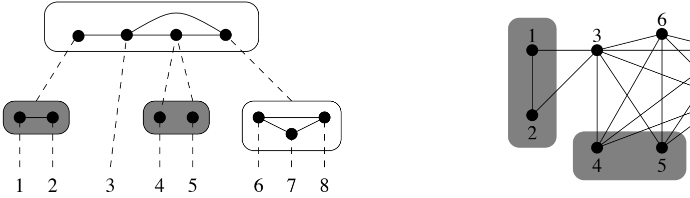
A *rooted graph-labelled tree* is a pair $(T, \mathcal{F})$ with T a rooted tree and $\mathcal{F}$ a set of graphs such that:

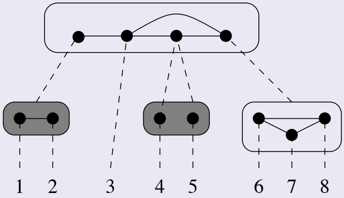
- each node v with k children of T is labelled by a graph $G_v \in \mathcal{F}$ on k vertices
- there is a bijection ρ_v from the tree-edges between v and its children to the vertices of G_v

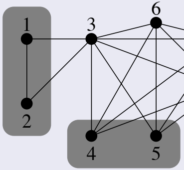


Given a rooted graph labelled tree $(T, \mathcal{F})$, the graph $G_M(T, \mathcal{F})$ has the leaves of T as vertices and

- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$, uv, vw tree-edges on the x, y-path in T and $v = lca_T(x, y)$.







Modules

A subset of vertices M of a graph $G = (V, E)$ is a **module** iff

$$\forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset$$

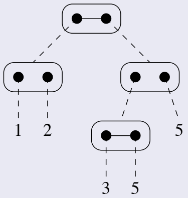
Modular decomposition

Examples of modules

- any subset of vertices of the clique
- any subset of vertices of the stable

Totally decomposable graphs

- Cographs



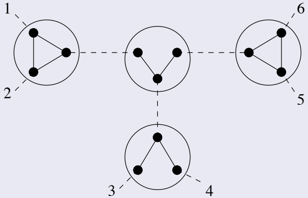
Split decomposition

Examples of splits

- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$

Totally decomposable graphs

- Distance hereditary grahs



Gallai'67 reformulated

For any graph G, there exists a unique rooted graph-labelled tree $(T, \mathcal{F})$ with a minimun number of nodes such that

1. $G = G_M(T, \mathcal{F})$ and
2. any graph of $\mathcal{F}$ is prime or degenerate for the modular decomposition.

$$\rightarrow \text{We note } (T, \mathcal{F}) = MD(G)$$

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For any connected graph G, there exists a unique graph-labelled tree $(T, \mathcal{F})$ with a minimun number of nodes such that

1. $G = G_S(T, \mathcal{F})$,
2. any graph of $\mathcal{F}$ is prime or degenerate for the split decomposition.

$$\rightarrow \text{We note } (T, \mathcal{F}) = ST(G)$$

1. Revisiting modular and split decomposition
2. Vertex modification of totally decomposable graphs
3. Conclusion and on-going work

Theorem (Tedder and Corneil '06, Gioan and Paul '07)

Let $G = (V, E)$ be a distance hereditary (DH) graph. It can be tested in

- $O(|S|)$ **whether** $G + (x, S)$**, with** $x \notin E$ **and** $N(x) = S$**, is a DH graph;**
- $O(|S|)$ *whether* $G - x$*, with* $S = N(x)$*, is a DH graph;*
- $O(1)$ *whether* $G + e$*, with* $e \notin E$*, is a DH graph;*
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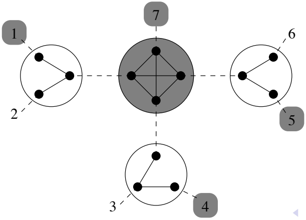
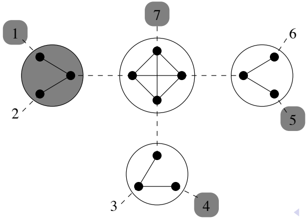
Let $(T, \mathcal{F})$ be a graph-labelled tree, and S be a subset of leaves of T. A node u of $T(S)$ is:

- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S;



Let $(T, \mathcal{F})$ be a graph-labelled tree, and S be a subset of leaves of T. A node u of $T(S)$ is:

- **fully-marked** by S if any subtree of $T - u$ contains a leaf of S;
- **singly-marked** by S if it is a star-node and exactly two subtrees of $T - u$ contain a leaf $l \in S$ among which the subtree containing the neighbor v of u such that $\rho_u(uv)$ is the centre of G_u;



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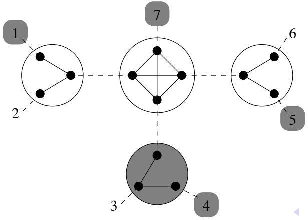
- **partially-marked** otherwise



Theorem (DH incremental characterization [Gioan, Paul '07])

Let G be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

1. *At most one node of $T(S)$ is partially-marked.*

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3. *If there exists a partially-marked node u, then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-marked.*

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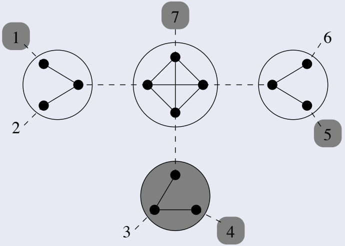
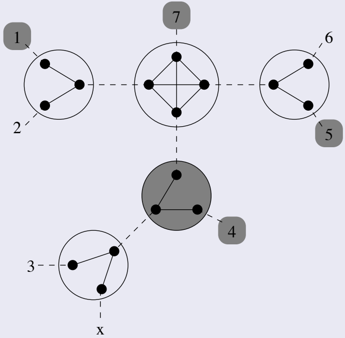
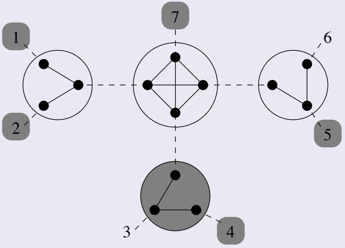
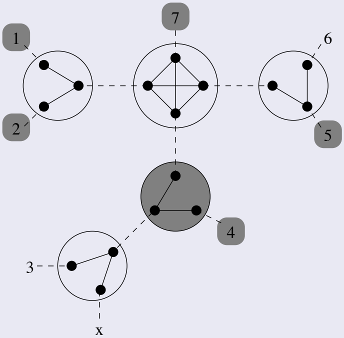
3. *If there exists a partially-marked node u, then any star node $v \neq u$ of $T(S)$ is oriented towards u if and only if it is fully-marked.*
4. *Otherwise, there exists a tree-edge e of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-marked.*





The insertion fails: the two singly-marked nodes are oriented towards the partially-marked node !





The insertion succeeds: in $G_S(T, \mathcal{F})$, we have $N(x) = S$

Insertion algorithm

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2. Check the mark-type of the nodes and look for an insertion node or edge;
3. Insert the node by either subdividing the insertion edge, or splitting the insertion node, or attaching x to the insertion node.

Cographs

Let $(T, \mathcal{F})$ be a rooted graph-labelled tree, and S be a subset of leaves of T. A node u of $T(S)$ is:

- **fully-marked** by S if for any children v of u, T_v contains some leaves of S;
- **singly-marked** by S if there is a unique child v such that T_v contains some leaves of S;
- **partially-marked** otherwise.

Theorem (Cograph incremental characterization [CPS'85])

Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

1. *At most one node of $T(S)$ is partially-marked.*
2. *Any series node of $T(S)$ is either fully or partially-marked.*

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Let G be a cograph and $MD(G) = (T, \mathcal{F})$ be its modular decomposition tree. Then $G + (x, S)$ is a cograph iff:

1. *At most one node of $T(S)$ is partially-marked.*
2. *Any series node of $T(S)$ is either fully or partially-marked.*
3. *If there exists a partially-accessible node u, then a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-marked.*
4. *Otherwise, there exists a tree-edge $e = uw$ of $T(S)$ such that a parallel node $v \neq u$ of $T(S)$ is a descendant of u if and only if it is fully-marked.*

Theorem (Corneil, Pearl and Stewart '85, Sharan and Shamir '04)

Let $G = (V, E)$ be a cograph. It can be tested in

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Concluding remarks

- There exists a characterization of cographs in terms of split tree.
- The graph-labelled tree representation of DH graphs yields an intersection model characterizing DH graphs.