# Distance Hereditary Graphs
# Characterizations and Recognition

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Part of the results are joint work with
**E. Gioan**

## Definition and examples

A graph $G = (V, E)$ is distance hereditary (DH) if in every connected induced subgraph $H = G[S]$ the distance between any two vertices $x$ and $y$ of $S$ is the same in $H$ than in $G$.

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- Every graph on four vertices is a DH graph
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Examples of non DH graphs

- The cycles of length at least 5 are not DH graphs
- The house, the gem and the domino are note DH graphs

# Some characterizations (1)

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Theorem: A graph is a DH graph iff it can be built up from a single vertex by a sequence of the following operations:

1. Add a pendant vertex $y$ to a vertex $x$ of graph
2. Add a false twin $y$ to a vertex $x$: i.e $N(x) = N(y)$
3. Add a true twin $y$ to a vertex $x$: i.e. $N(x) \cup \{x\} = N(y) \cup \{y\}$

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3. for every four vertices $u$, $v$, $w$, and $x$, at least two of the three sums of distances $d(u, v) + d(w, x)$, $d(u, w) + d(v, x)$, and $d(u, x) + d(v, w)$ are equal to each other

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4. $G$ has rankwidth 1
5. $G$ is totally decomposed by the split decomposition

# Some characterizations (3)

Theorem [Bandelt, Muller'86]

Let $G$ be a connected graph and $L_1, \ldots, L_k$ be the distance layout from an arbitrary vertex $v$ of $G$. Then $G$ is a DH graph iff the following conditions are verified for any $1 \leq i \leq k$:

1. If $x$ and $y$ belong to the same connected component of $G[L_i]$, then $N_{i-1}(x) = N_{i-1}(y)$

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3. If $x \in N_{i-1}(u)$ and $y \in N_{i-1}(u)$ are in different connected components $X$ and $Y$ of $G[L_{i-1}]$, then $X \cup Y \subseteq N(u)$ and $N_{i-2}(x) = N_{i-2}(y)$

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5. If $x \in N_{i-1}(u)$ and $y \in N_{i-1}(u)$ are in the same connected component $C$ of $G[L_{i-1}]$, then the vertices of $C$ non-adjacent to $u$ are either adjacent to both $x$ and $y$ or none of them

## The recognition problem (1)

Theorem [Hammer and Maffray'90, Damiand,Habib and P.'05]
The recognition problem of DH graph can be solved in linear time.

- compute a BFS of the graph
- examine the structure of the distance layers
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**Theorem [Gioan, P.'07]**
There exists a linear time fully dynamic recognition algorithm for the DH recognition problem

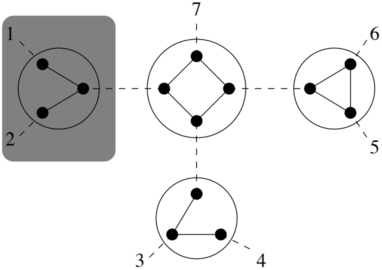
- the insertion of a new vertex $x$ costs $O(|N(x)|)$
- the deletion of a vertex $x$ costs $O(|N(x)|)$

$\longrightarrow$ based on the split decomposition characterization

## Split decomposition (1)

A graph-labelled tree (GLT) is a pair $(T, \mathcal{F})$ with $T$ a tree and $\mathcal{F}$ a set of graphs such that:

- each node $v$ of degree $k$ of $T$ is labelled by a graph $G_v \in \mathcal{F}$ on $k$ vertices



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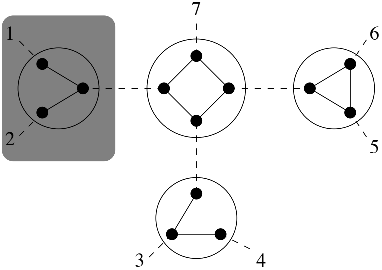
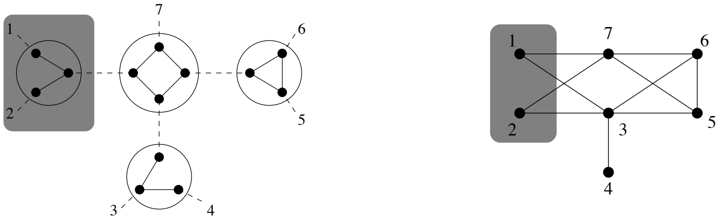
- each node $v$ of degree $k$ of $T$ is labelled by a graph $G_v \in \mathcal{F}$ on $k$ vertices
- there is a bijection $\rho_v$ from the tree-edges incident to $v$ to the vertices of $G_v$



## Split decomposition (2)

Given a GLT $(T, \mathcal{F})$, the accessibility graph $G_S(T, \mathcal{F})$ has

- its vertices in one-to-one correspondence with the leaves of $T$



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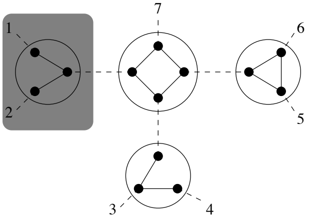
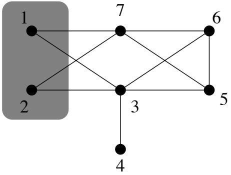
- its vertices in one-to-one correspondence with the leaves of $T$
- $xy \in E(G_S(T, \mathcal{F}))$ iff $\rho_v(uv)\rho_v(vw) \in E(G_v)$,
  $\forall$ tree-edges $uv$, $vw$ on the $x, y$-path in $T$



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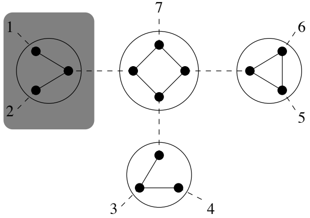


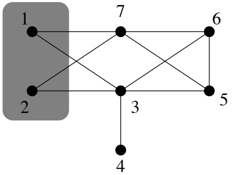


A bipartition $(A, B)$ of the vertices of a graph is a split iff

- $|A| \geqslant 2$, $|B| \geqslant 2$;
- for $x \in A$ and $y \in B$, $xy \in E$ iff $x \in N(B)$ and $y \in N(A)$.

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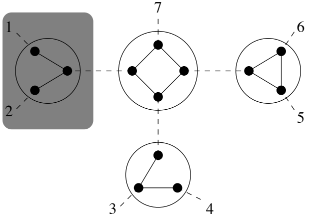


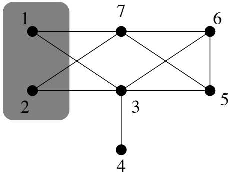


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Observation: every tree edge of a GLT defines a split of its accessiblity graph

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## Examples of splits

- any non-trivial bipartition of the clique
- any non-trivial bipartition of the $K_{1,n}$

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Theorem [Cunnigham'82 reformulated]
For any connected graph $G$, there exists a unique graph-labelled tree $(T, \mathcal{F})$ with a minimun number of nodes such that

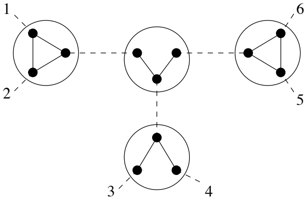
1. $G = G_S(T, \mathcal{F})$,
2. any graph of $\mathcal{F}$ is prime or degenerate (clique or star) for the split decomposition.

$$\rightarrow \text{We note } (T, \mathcal{F}) = ST(G)$$

# Split decomposition (5)

Definition A graph is totally decomposable (by the split decomposition) if every induced subgraph of size at least 4 has a split.

Theorem A graph is totally decomposable iff it is a DH graph.



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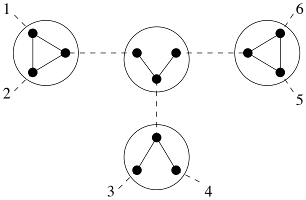
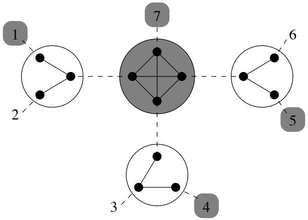



Theorem A graph is a DH graph iff its split tree only contains star and clique nodes.

# Incremental recognition (1)

Let $(T, \mathcal{F})$ be a graph-labelled tree, and $S$ be a subset of leaves of $T$. A node $u$ of $T(S)$ is:

- fully-marked by $S$ if any subtree of $T - u$ contains a leaf of $S$;



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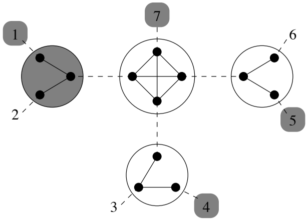
- singly-marked by $S$ if it is a star-node and exactly two subtrees of $T - u$ contain a leaf $l \in S$ among which the subtree containing the neighbor $v$ of $u$ such that $\rho_u(uv)$ is the centre of $G_u$;



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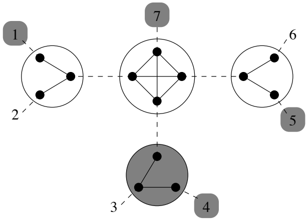
- partially-marked otherwise



# Incremental recognition (2)

Theorem (DH incremental characterization [Gioan, P.'07])
Let $G$ be a connected DH graph and $ST(G) = (T, \mathcal{F})$ be its split tree. Then $G + (x, S)$ is a DH graph iff:

1. At most one node of $T(S)$ is partially-marked.

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3. If there exists a partially-marked node $u$, then any star node $v \neq u$ of $T(S)$ is oriented towards $u$ if and only if it is fully-marked.

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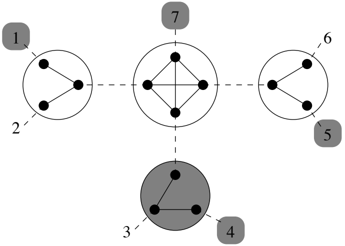
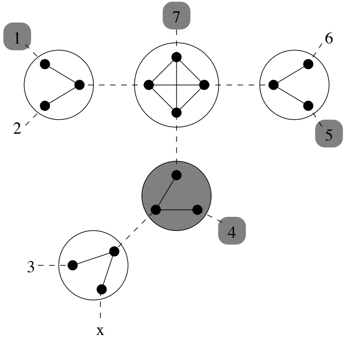
3. If there exists a partially-marked node $u$, then any star node $v \neq u$ of $T(S)$ is oriented towards $u$ if and only if it is fully-marked.
4. Otherwise, there exists a tree-edge $e$ of $T(S)$ towards which any star node of $T(S)$ is oriented if and only if it is fully-marked.

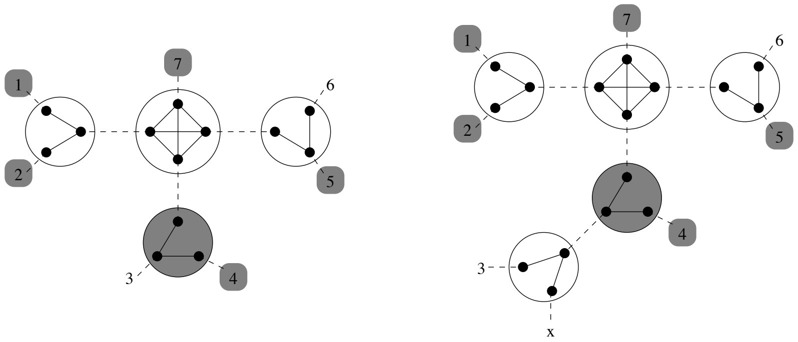
# Incremental recognition (3)




**The insertion fails:** the two singly-marked nodes are oriented towards the partially-marked node !

# Incremental recognition (4)




**The insertion succeeds:** in $G_S(T, \mathcal{F})$, we have $N(x) = S$

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Given $G$ and its split tree, the algorithm can be implemented in $O(|S|)$ time.

# Intersection model (1)

Definition Let $\Sigma = \{S_1, \ldots, S_n\}$ be a set family. The **intersection graph** of $\Sigma$ is the graph the vertices of which are in one-to-one correspondence with $\mathcal{F}$ and two vertices $x_i$, $x_y$ are adjacent if $S_i \cap S_j \neq \emptyset$.

Observation 1 [Marczewski] Every graph is an intersection graph.

Definition Let $\Sigma$ be a set family, then $\mathcal{I}(\Sigma)$ is the family of graphs containing the intersection graphs of subfamilies of $\Sigma$.

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Definition Let $\Sigma$ be a set family, then $\mathcal{I}(\Sigma)$ is the family of graphs containing the intersection graphs of subfamilies of $\Sigma$.

Observation 2 Every intersection graph family is hereditary

Observation 3 Every intersection graph family is closed under true twin addition

Observation 4 If $\mathcal{F}$ is an intersection graph family, then there exists an enumerable family $\Sigma$ such that $\mathcal{F} = \mathcal{I}(\Sigma)$.

# Intersection model (2)

Theorem [Scheinermann'86]

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2. $\mathcal{F}$ is closed under true twin addition
3. $\mathcal{F}$ has a series ordering

$$\exists\ G_1 \dots G_i, G_{i+1} \dots G_k \text{ such that}$$

- $\forall i,\ G_i \in \mathcal{F}$
- $\forall i,\ G_i$ is an induced subgraph of $G_{i+1}$
- $\forall H \in \mathcal{F}, \exists i\ H$ is a subgraph of $G_i$

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Lemma: the family of DH graphs has an intersection model

BUT find the model ...

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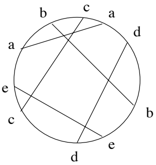


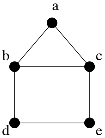


Observation Every DH graphs is the intersection graph of a set of chord in a circle

(that is DH graphs form a subfamily of the circle graphs)

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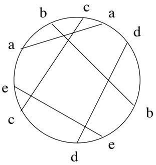


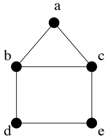


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BUT ... this is not a characterization

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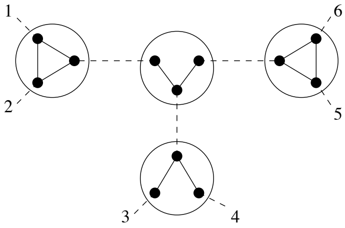
Idea: use the GLT representation



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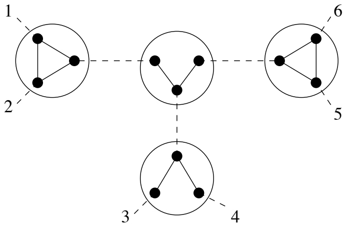
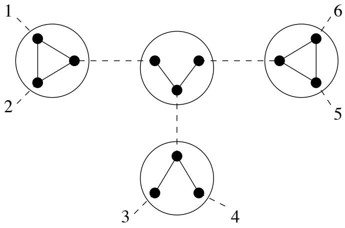



- each vertex $x$ receives a set $S(x)$ of path of $T$:
  $P_{x \to y} \in S$ if it is an alternating path from $x$ to leaf $y$.

# Intersection model (4)

Idea: use the GLT representation




- each vertex $x$ receives a set $S(x)$ of path of $T$:
  $P_{x \to y} \in S$ if it is an alternating path from $x$ to leaf $y$.
- vertices $x$ and $y$ are adjacent iff $S(x) \cap S(y) \neq \emptyset$

# Intersection model (5)

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e.g. compare with the intersection model of chordal graphs

**Theorem**

Let $G$ be a graph. The following condition are equivalent:

1. $G$ has no chordless cycle of length at least four ($G$ is chordal)
2. $G$ has a tree-decomposition, the bags of which are cliques
3. $G$ is the intersection graph of subtrees of a tree