A single exponential FPT algorithm for the K_4-minor cover problem

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Joint work with
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October 24, 2011

Parameterized K_4-Minor Cover

Given a graph $G = (V, E)$ and an integer k as parameter,

- is there a subset $S \subseteq V$ such that $G[V \setminus S]$ is K_4-minor free ?

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(a.k.a. Parameterized Treewidth-two Vertex Deletion)

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More generaly,

How fast can we solve Treewidth-t Vertex Deletion ?

Known results

1. Parameterized K_4-Minor Cover is FPT
   (by the Roberston and Seymour' graph minor theorem or by Courcelle's theorem)
2. Best algorithm runs is $2^{O(k \log k)}.n^{O(1)}$ [Fomin et al.'11]
3. No hope for a $2^{o(k)}.n^{O(1)}$ algorithm [???]

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Ingredients:

- iterative compression,
- protrusions, reduction and branching rules,
- Vertex Cover in circle graphs

Iterative compression for Feedback Vertex Set as a warm-up

Disjoint-K_4-Minor Cover

- Reduction Rules
- Protrusion Rule
- Branching Rules

Algorithm for the Disjoint-K_4-Minor Cover

Disjoint-Feedback Vertex Set (Disjoint-FVS)

- Given a graph $G = (V, E)$ a feedback vertex set S of size k
- Compute (if it exists) a feedback vertex set S' such that $S \cap S' = \emptyset$ and $|S'| < k$

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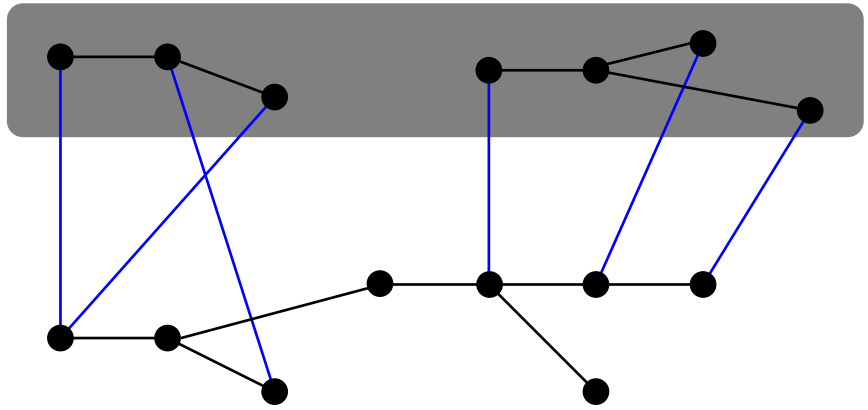
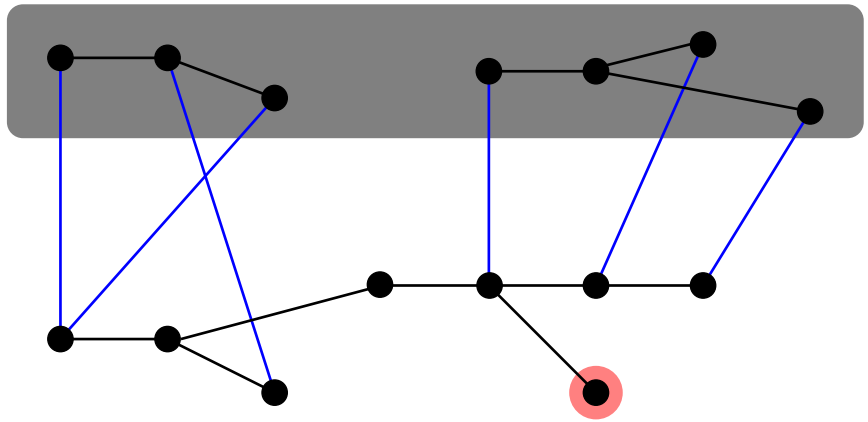
$$\boxed{\mu = k + \#cc(G[S])}$$

Skip example





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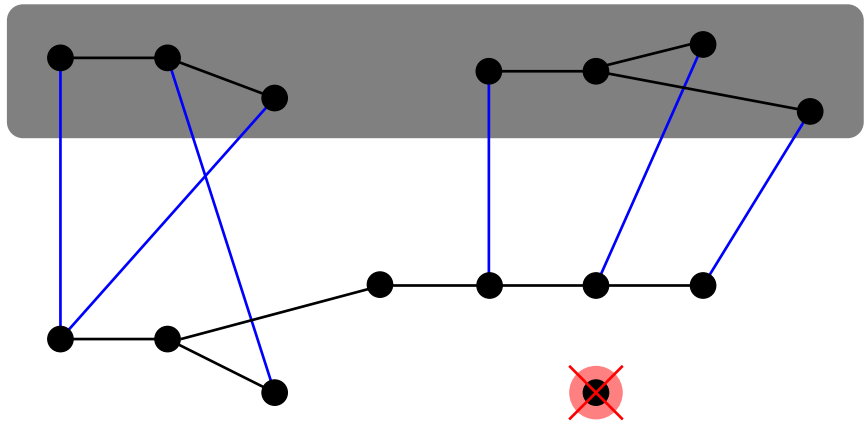

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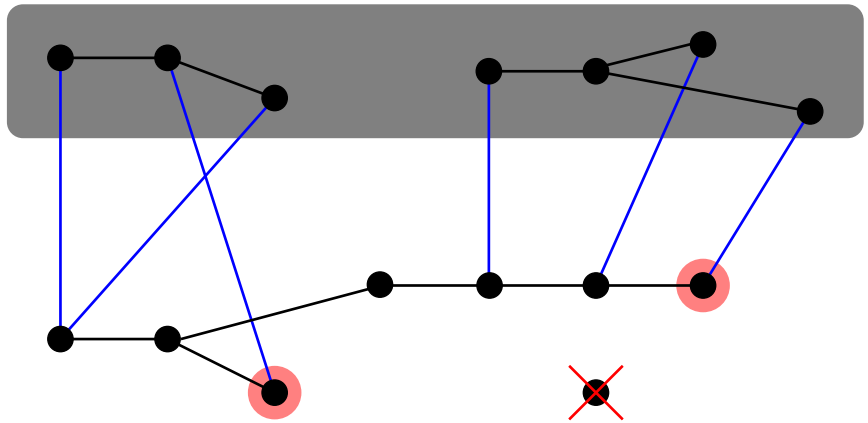

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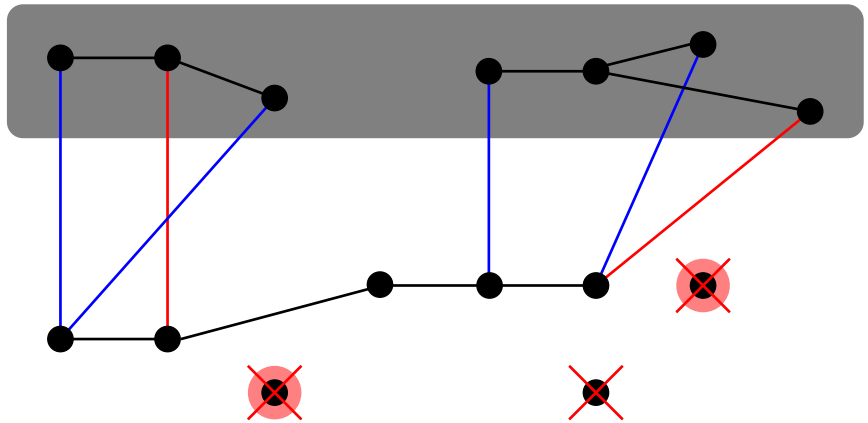

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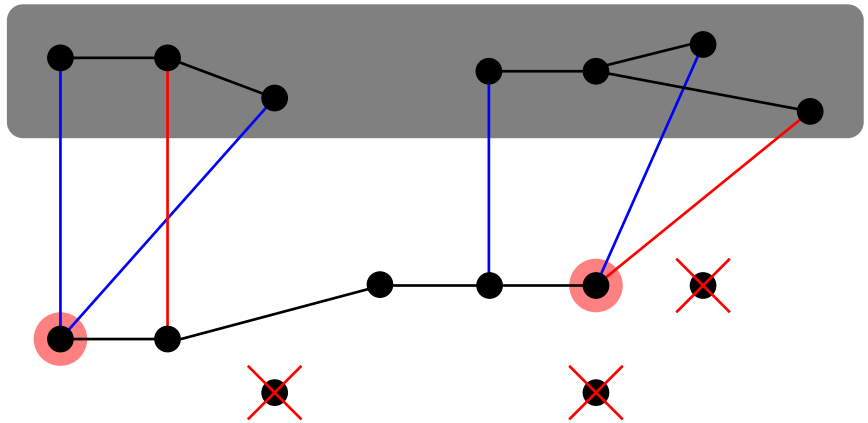

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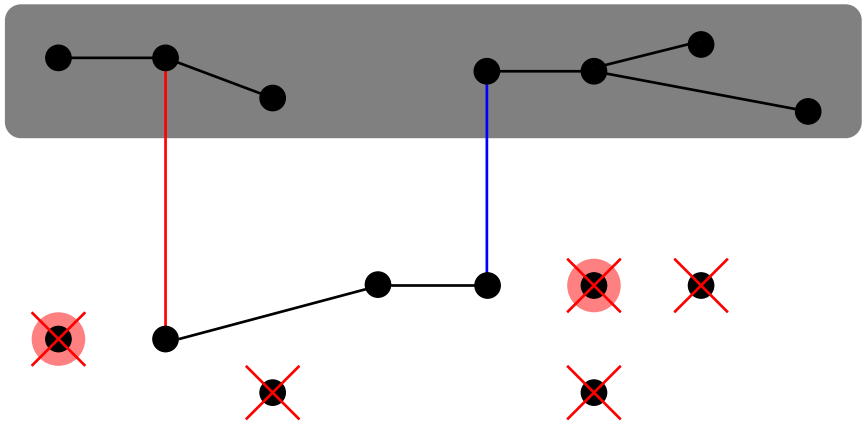

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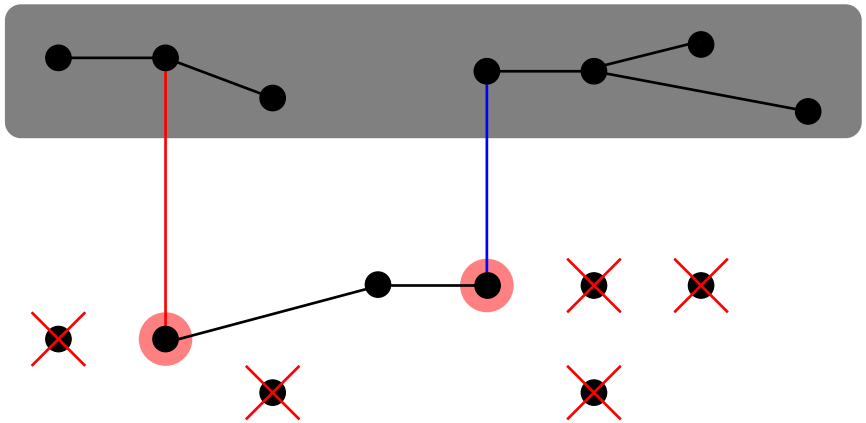

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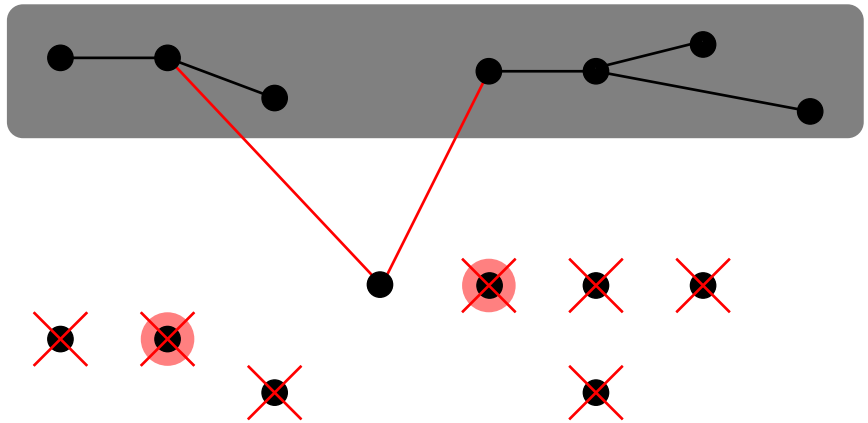

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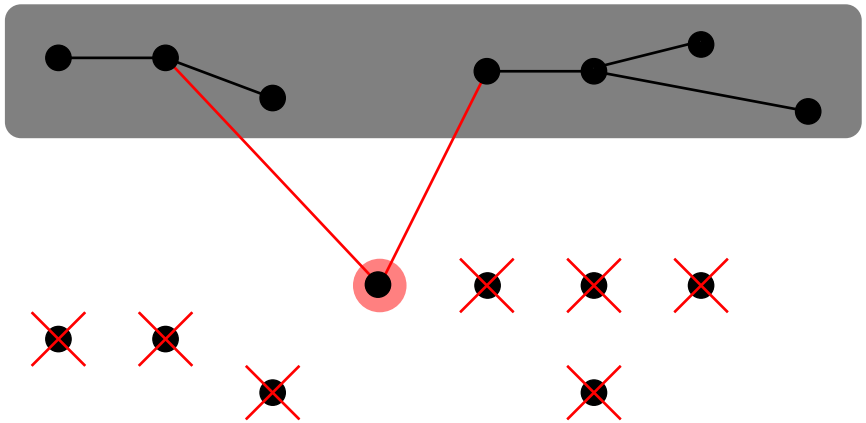

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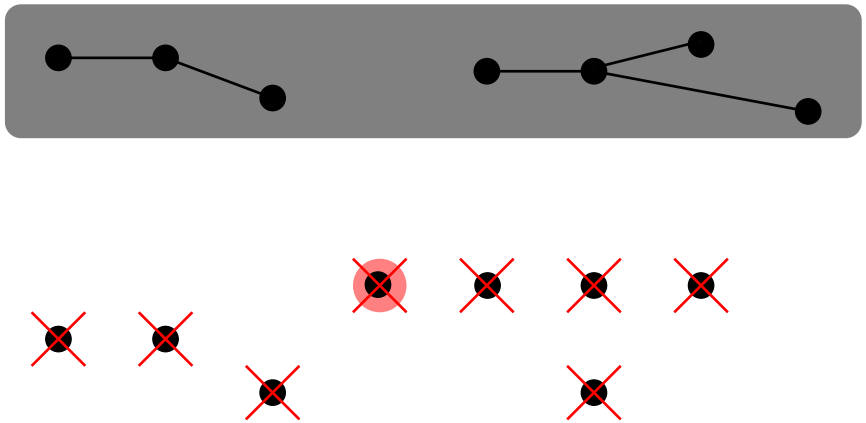
- $(G - \{x\}, S, k - 1)$ $\Rightarrow \mu$ decreases



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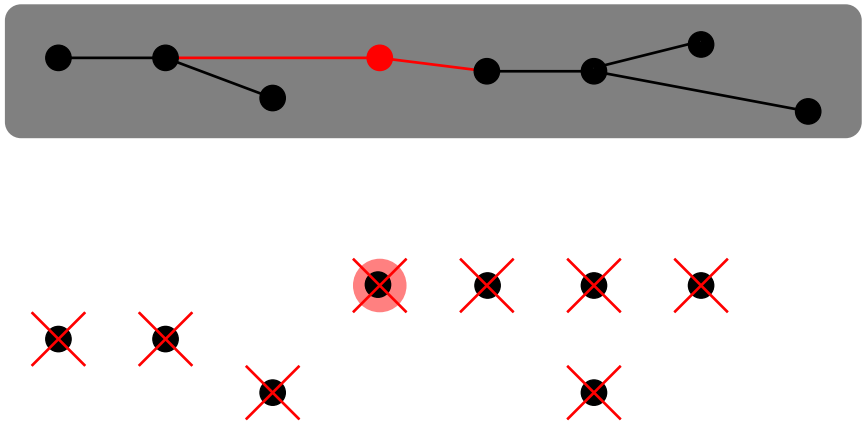

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$\Rightarrow$ Obs.: biconnected comp. of $G[S]$ and $G[V \setminus S]$ are SP-graphs

Iterative compression for Feedback Vertex Set as a warm-up

Disjoint-K_4-Minor Cover

- Reduction Rules
- Protrusion Rule
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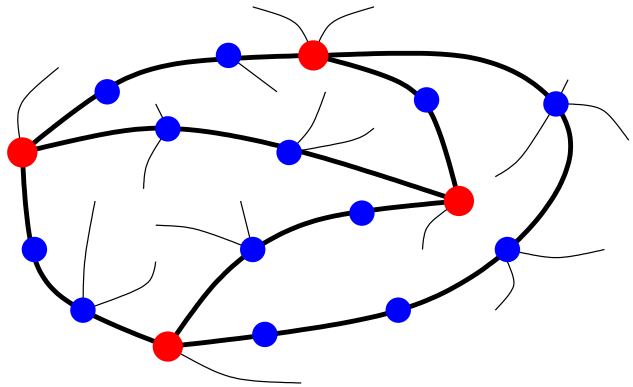
Algorithm for the Disjoint-K_4-Minor Cover

Some terminology

Observation: A graph contains a K_4-minor iff it contains a K_4-subdivision.



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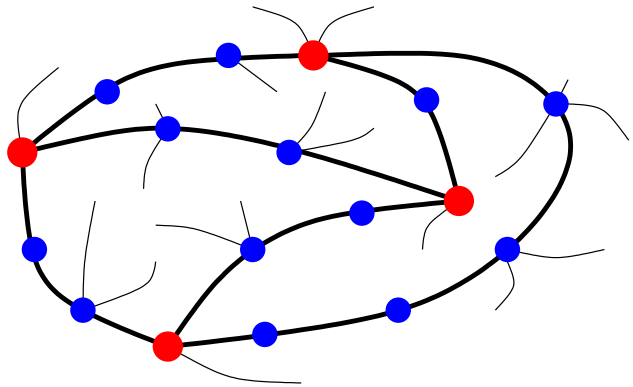

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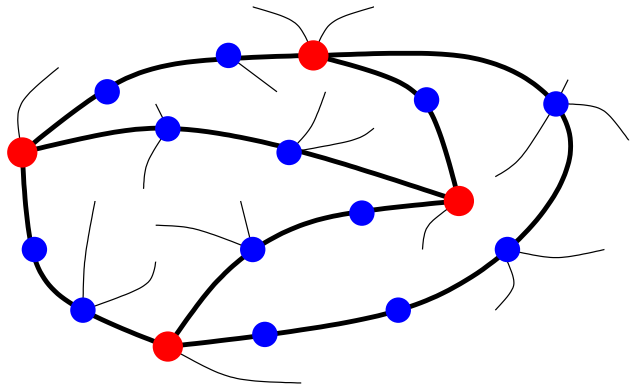

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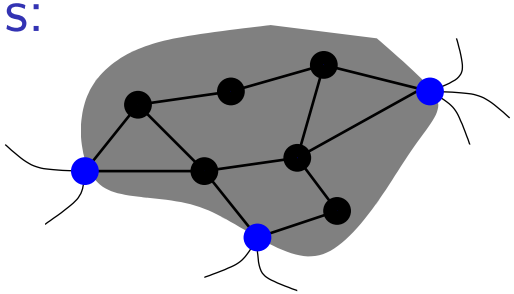
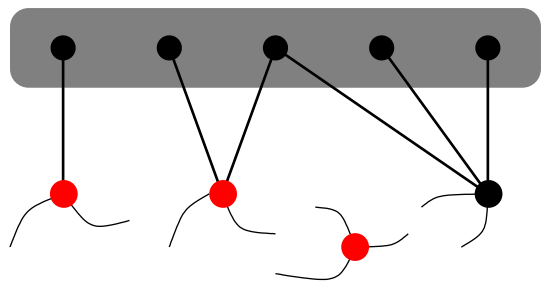
Notations:

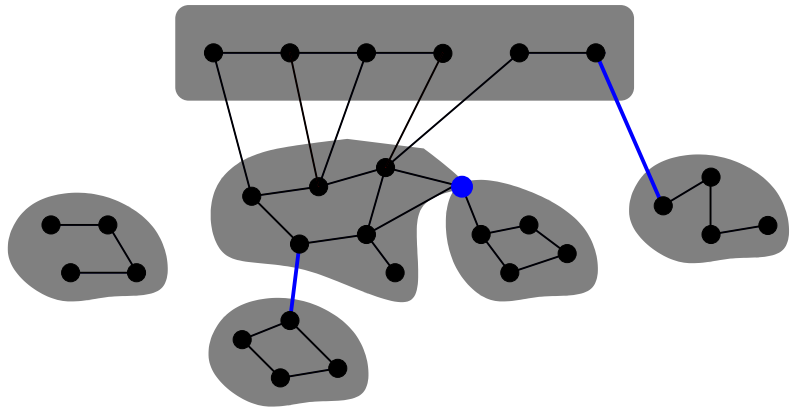


$\partial_G(X)$



$N_0,\ N_1,\ N_2$

Reduction Rules (1)



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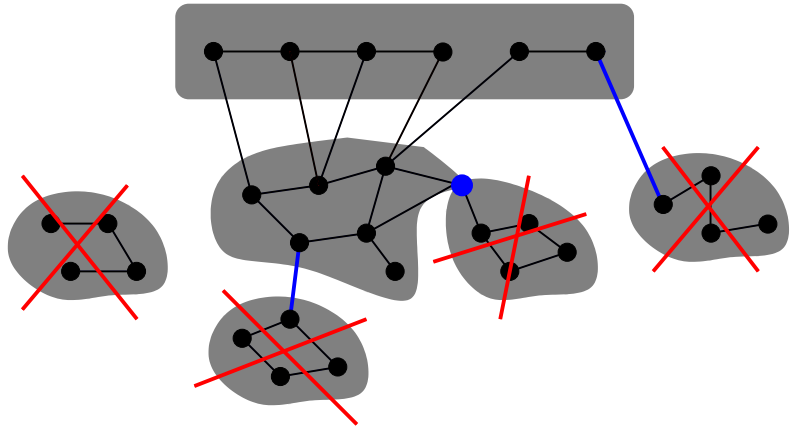

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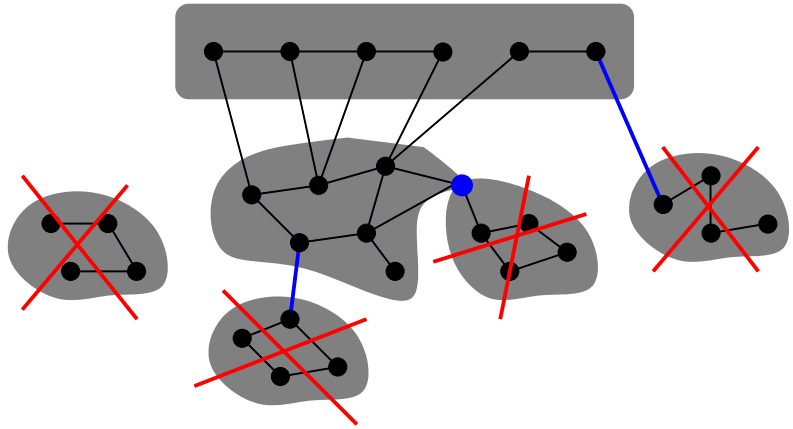

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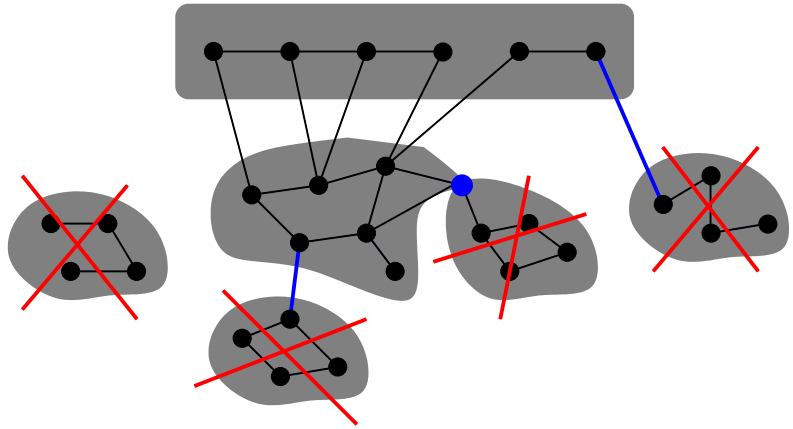

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Reduction Rule 2: Bypass every vertex of degree 2 in G with one neighbour in S and the other in $V \setminus S$.

Reduction Rule 3: Transform multiple edges into single edge.

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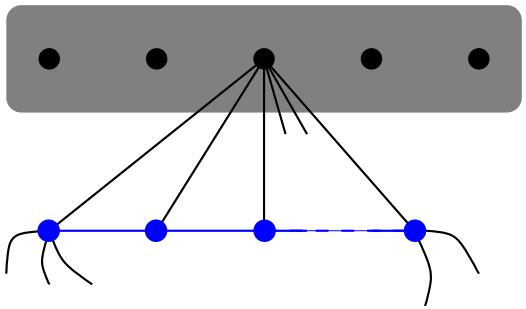

Reduction Rule 4: Let $X = \{x_1, x_2, x_3 \dots x_l\} \subseteq V \setminus S$ be a set of $l \geqslant 4$ vertices inducing a path in $G[V \setminus S]$ such that

- $\forall i \in [l]$, $N(x_i) \cap S = \{v\}$
- $\forall i \in [2, l-1]$, x_i has degree 3

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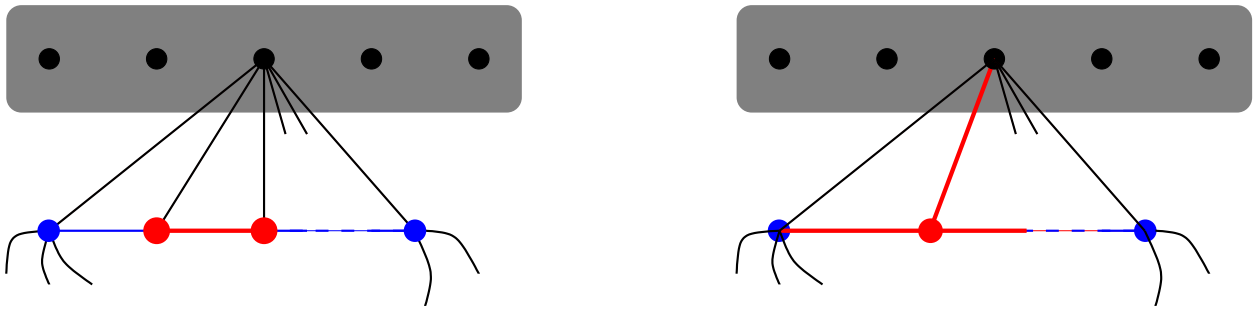

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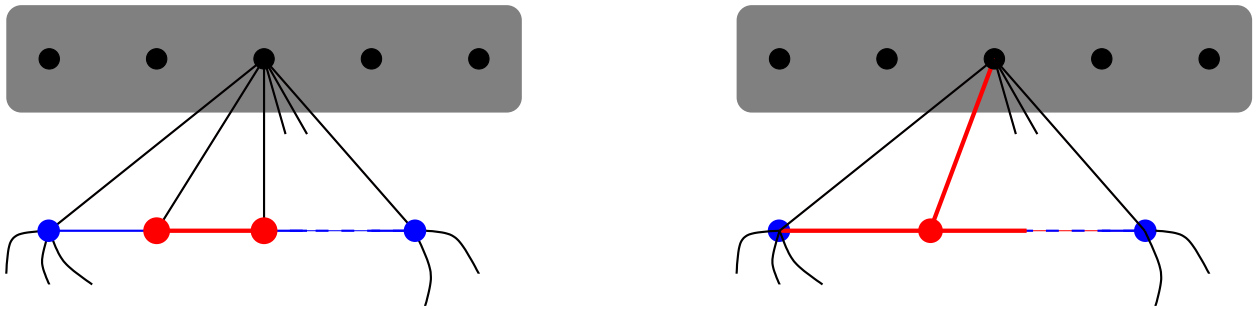

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- X contains at most 3 branching nodes of a K_4-subdivision.

  $\Rightarrow$ So one of its vertices is used as a subdivision node.

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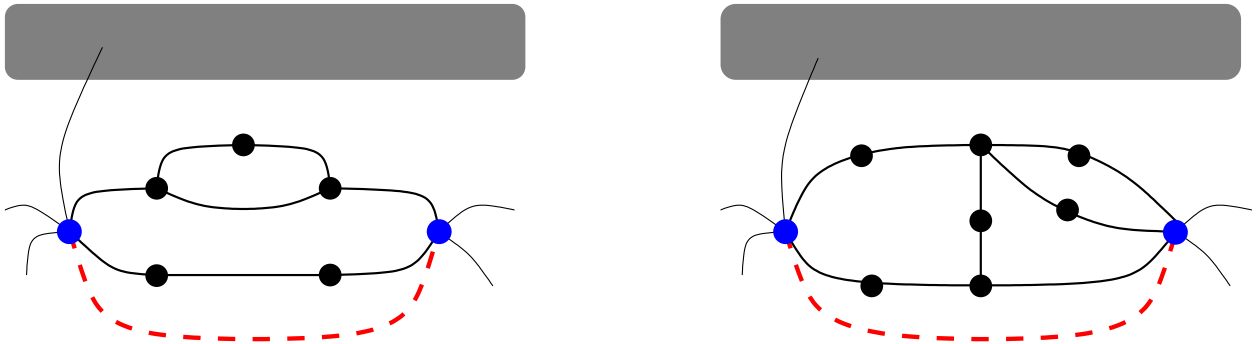

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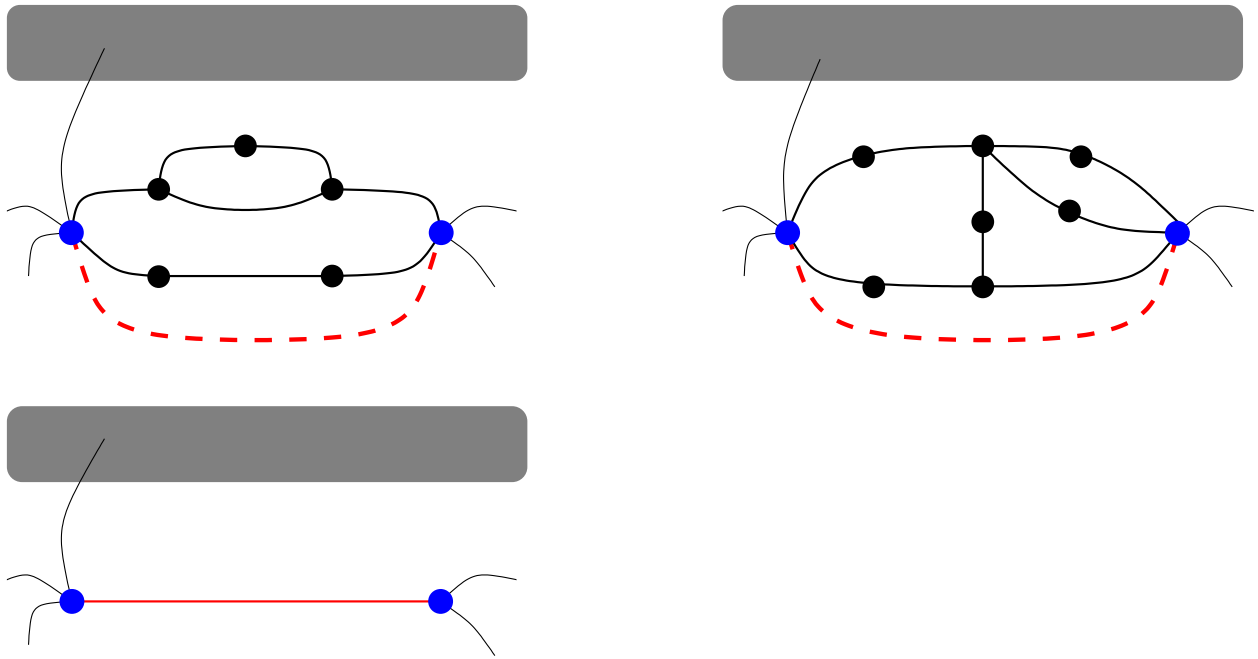

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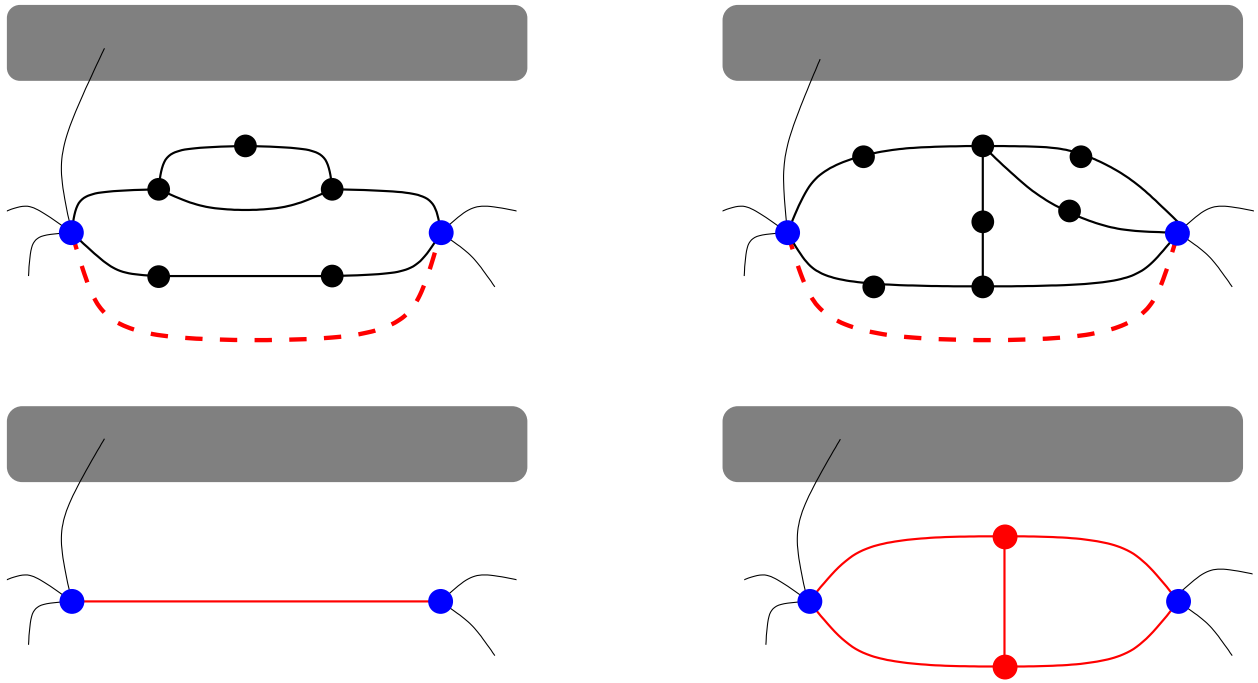

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- if $G[X] + (s, t)$ is K_4-minor free, replace $G[X]$ by the edge (s, t)
- otherwise, replace $G[X]$ by a 4-cycle containing s and t with a chord not incident to s nor t.

Iterative compression for Feedback Vertex Set as a warm-up

Disjoint-K_4-Minor Cover

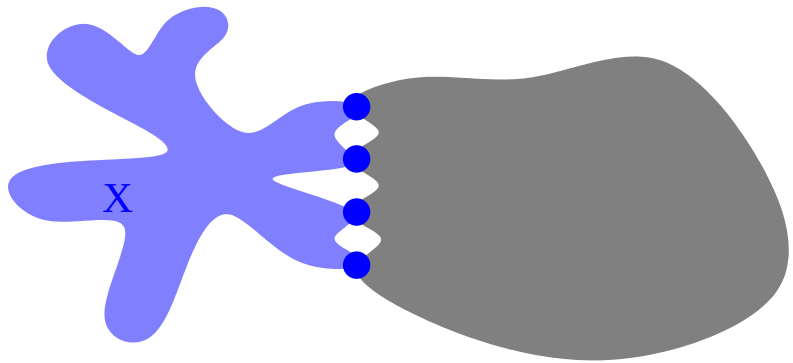
- Reduction Rules
- Protrusion Rule
- Branching Rules

Algorithm for the Disjoint-K_4-Minor Cover

Protrusion Rule (1)

A t-protrusion in a graph $G = (V, E)$ is a subset $X \subseteq V$ such that

$$tw(G[X]) \leqslant t \text{ and } |\partial_G(X)| \leqslant t$$



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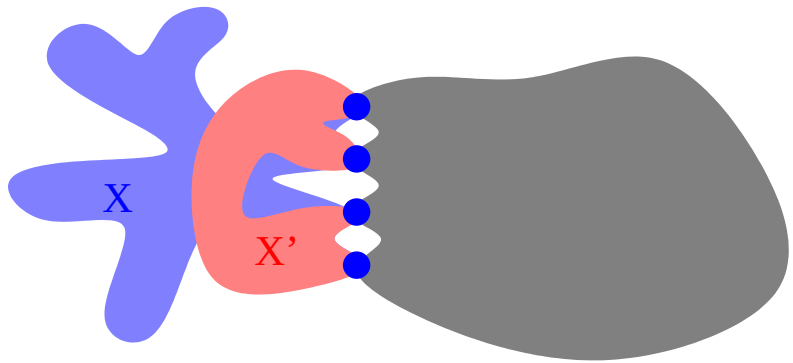

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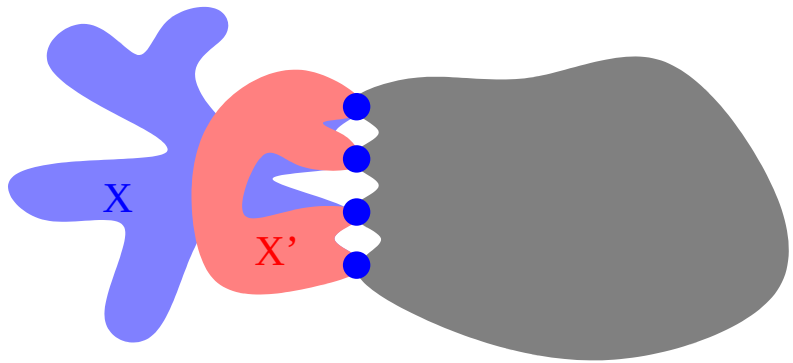

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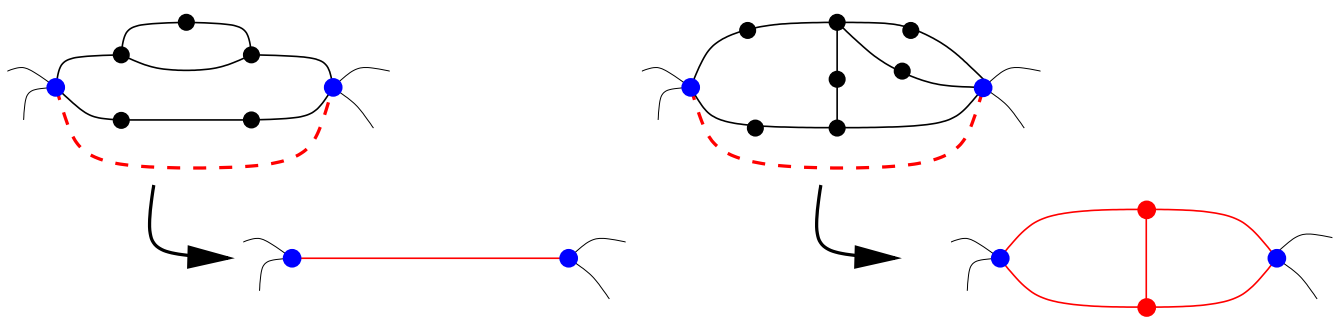
Observation: Reduction rule 5 is a example of protrusion rule



Protrusion Rule (2)

Let (G, S, k) be an instance of Disjoint-K_4-Minor Cover

Protrusion Rule: Let X be a t-protrusion of G such that

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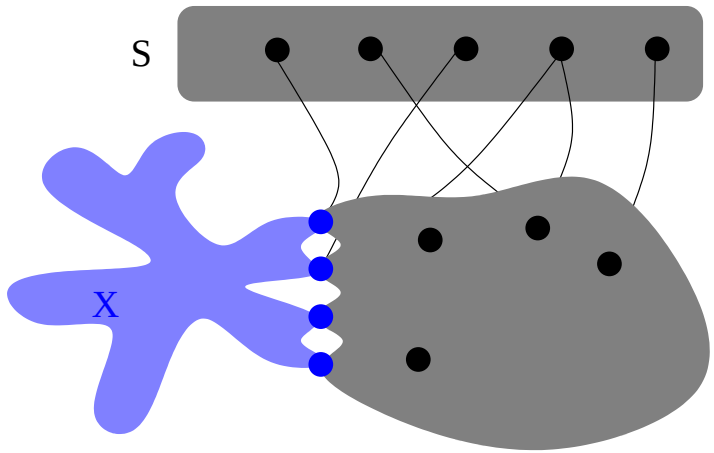

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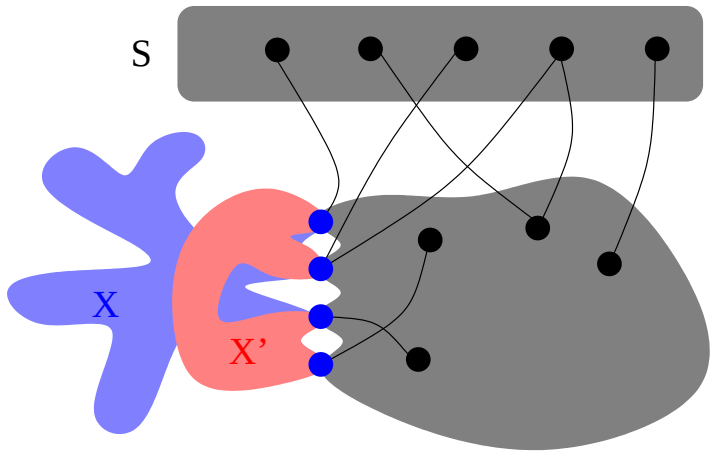

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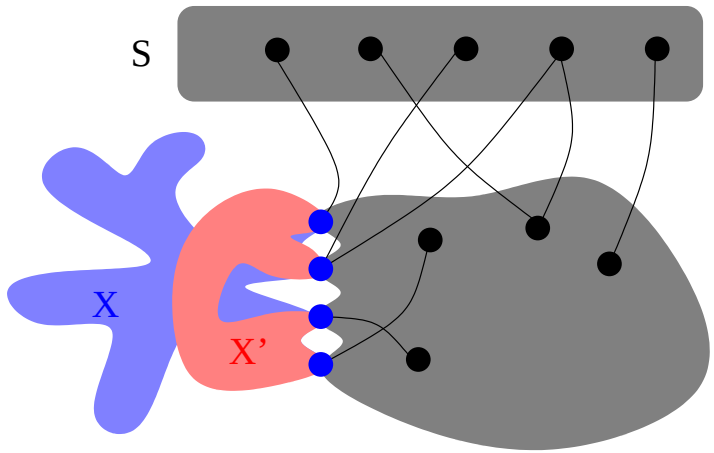

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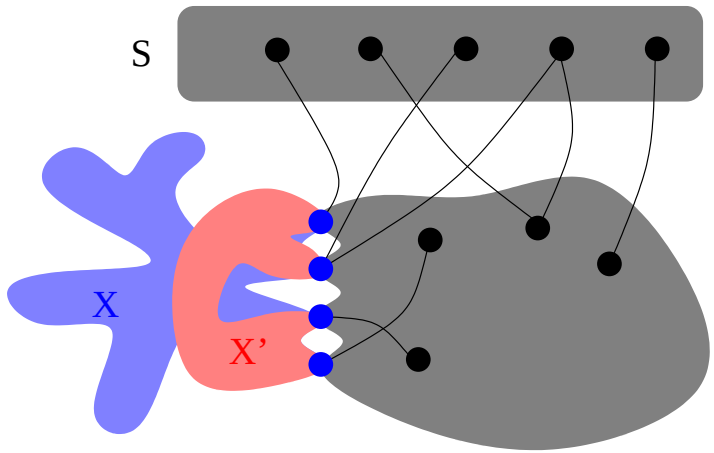

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Iterative compression for Feedback Vertex Set as a warm-up

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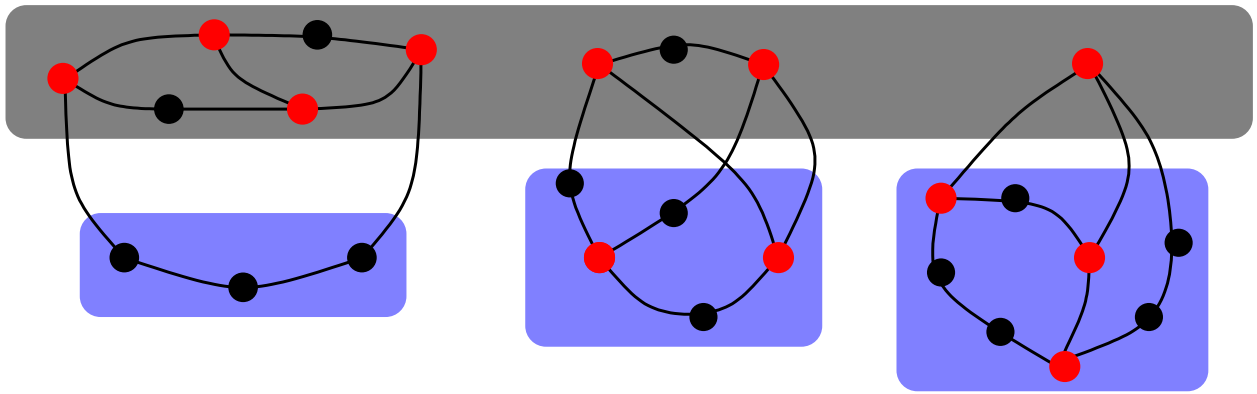

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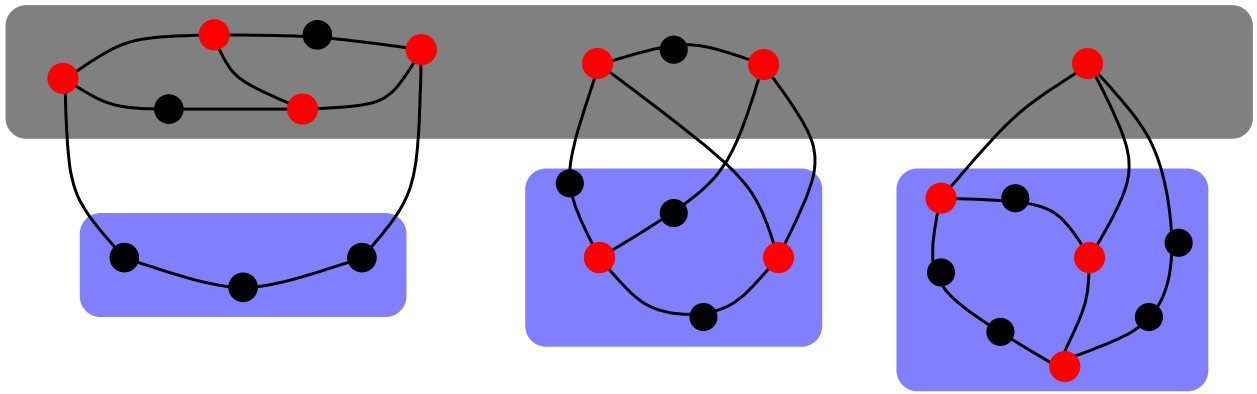

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Observation: To bound the branching degree, we have to guarantee that the above rule always applies on a set X such that $|X| \leqslant c$ for some constant c.

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We introduce a new measure function:

$$\mu = c_1 \times k + c_1 \times \#cc(G[S]) + \#bc(G[S])$$

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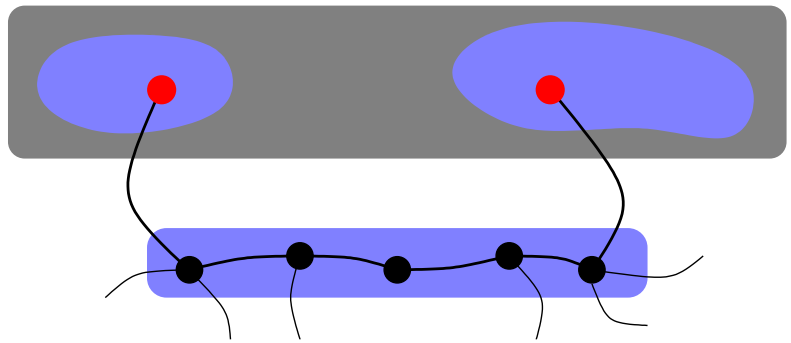

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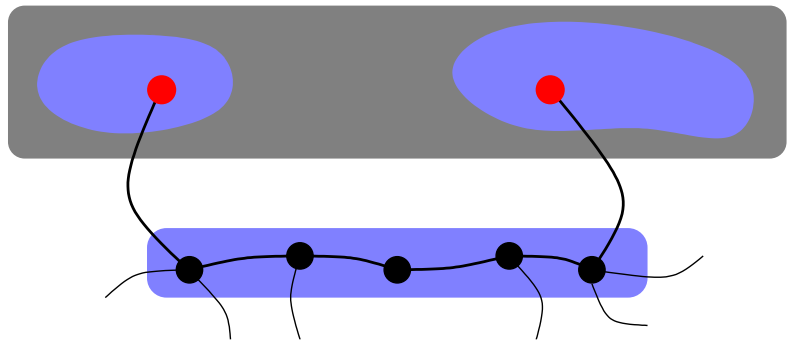

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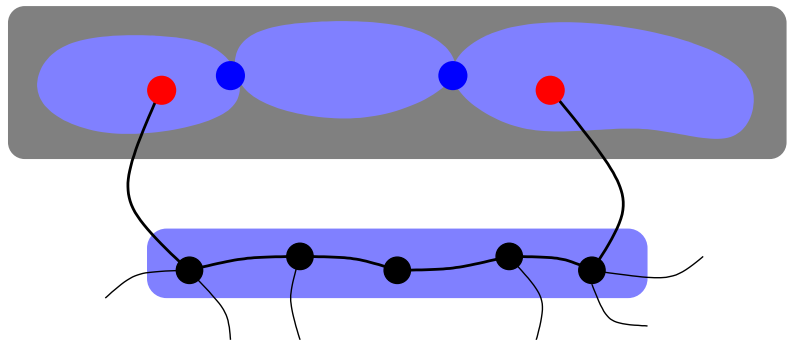

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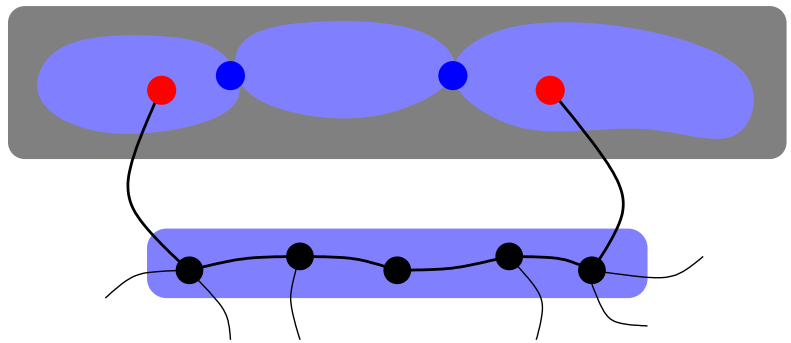

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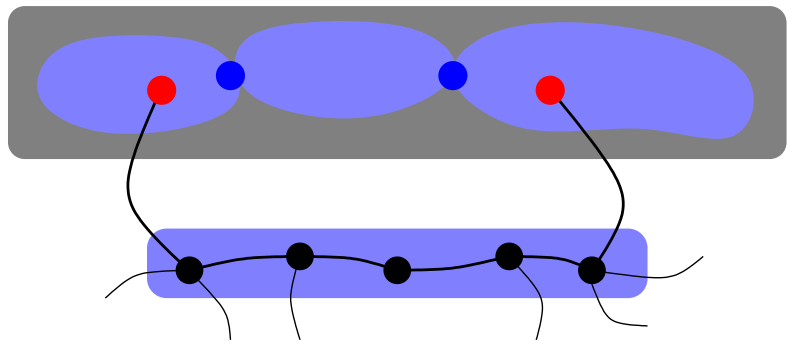

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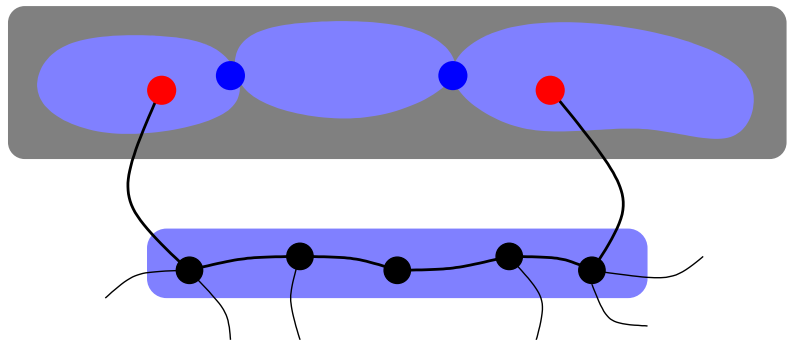

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- c_1 depends on the maximum size of the sets X on which the branching rules is applies

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Disjoint-K_4-Minor Cover

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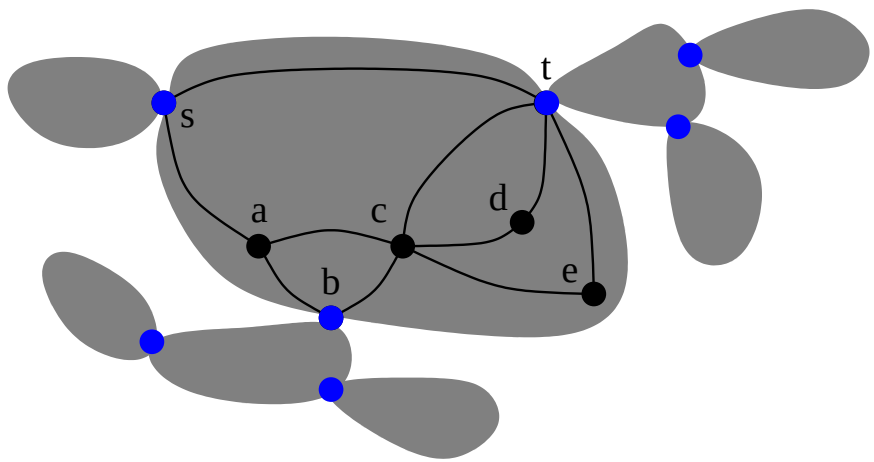

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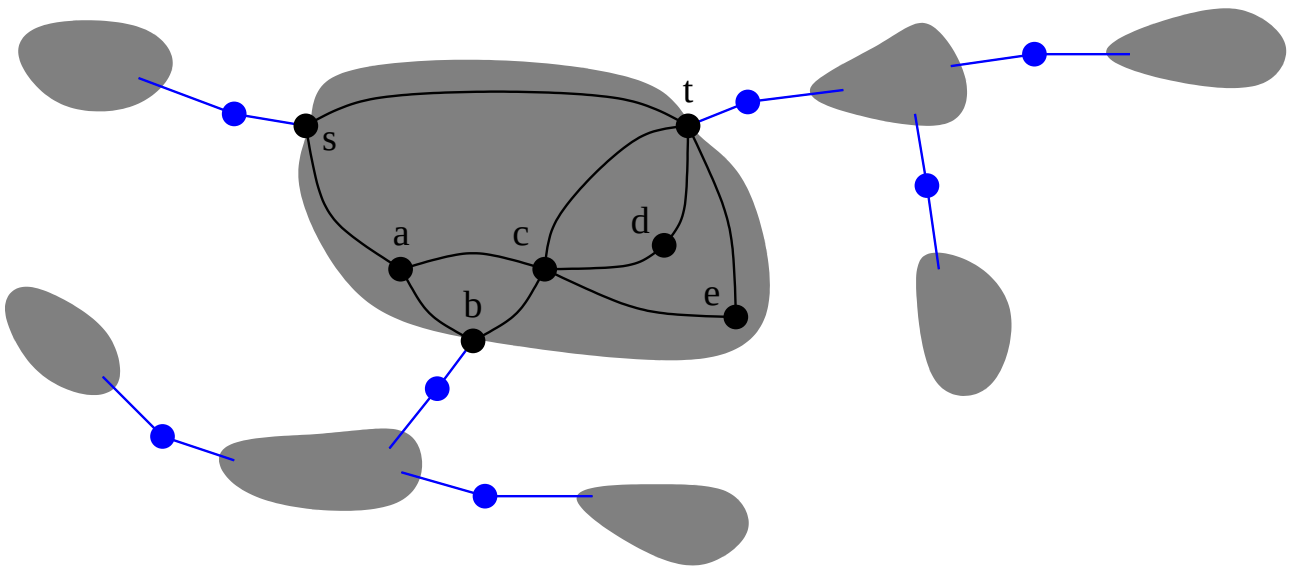

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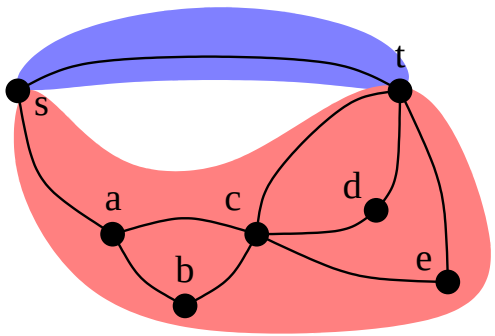

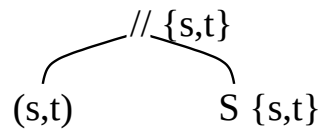

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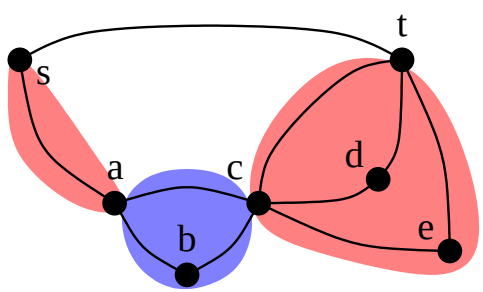

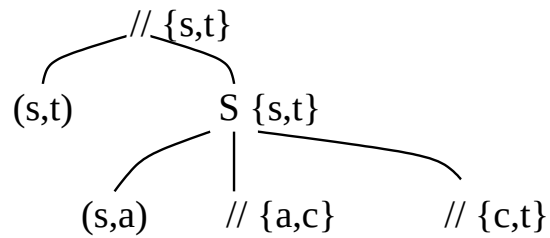

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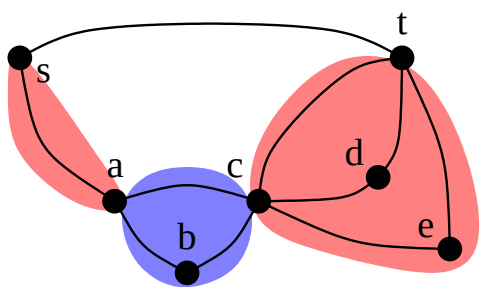

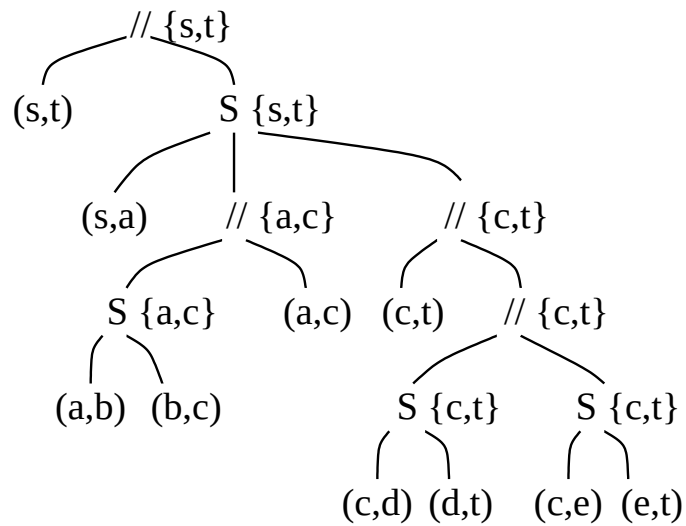

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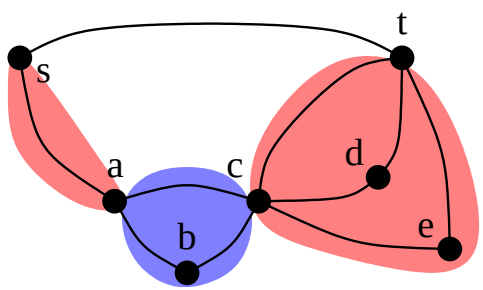

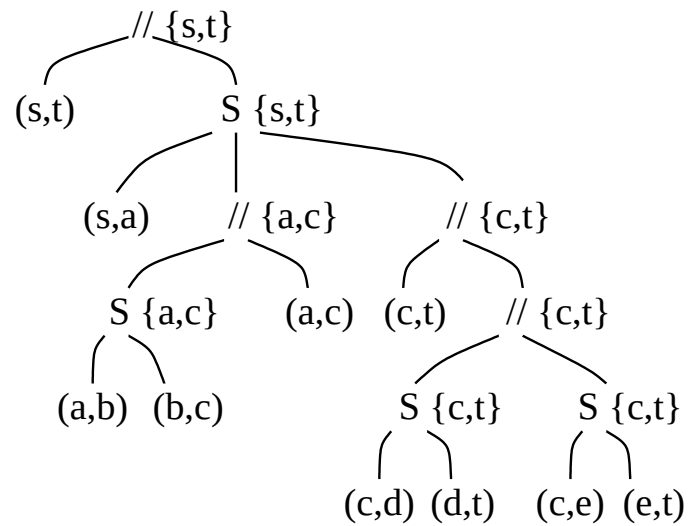

Obs.: $tw(G[V \setminus S] \leqslant 2$ and rules apply on t-protrusions with $t \leqslant 4$

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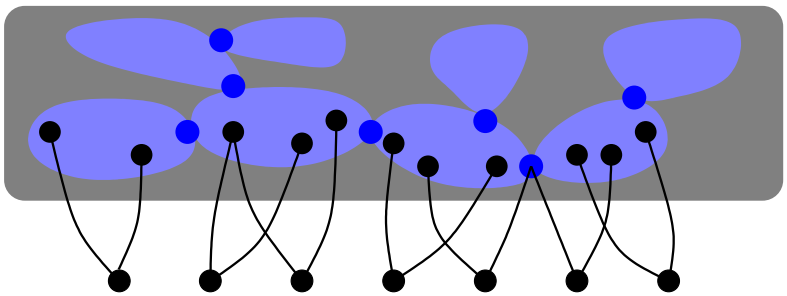

$\forall x \in V \setminus S$

- $d(x) = 2$
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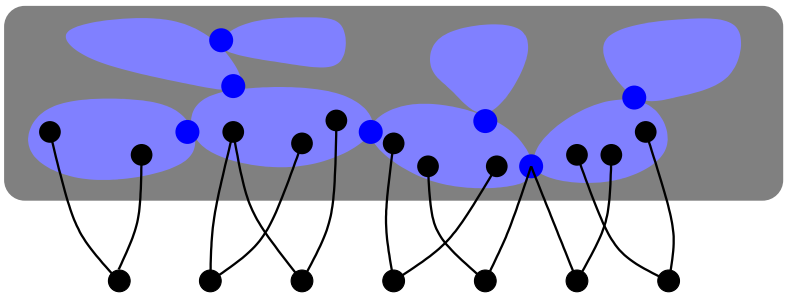

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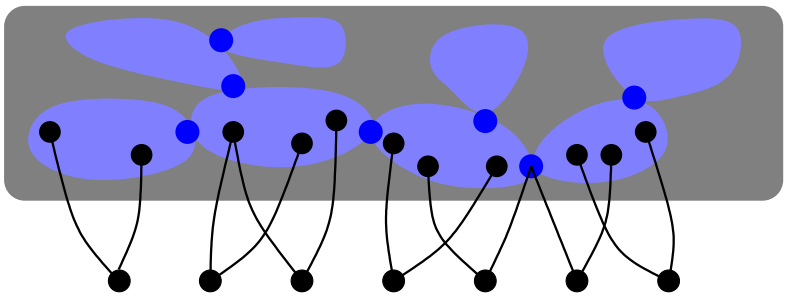

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Algorithm outline (3)

3. Solve each independent instance
   - Vertex Cover on the auxiliary graph $\tilde{G} = (V \setminus S, \tilde{E})$

$$\boxed{(x, y) \in \tilde{E} \text{ iff } G[G \cup \{x, y\}] \text{ contains a } K_4\text{-minor}}$$

Observation: this is compatible with single exponential time

Skip: $\tilde{G}$ is a circle graph

Algorithm outline (3')

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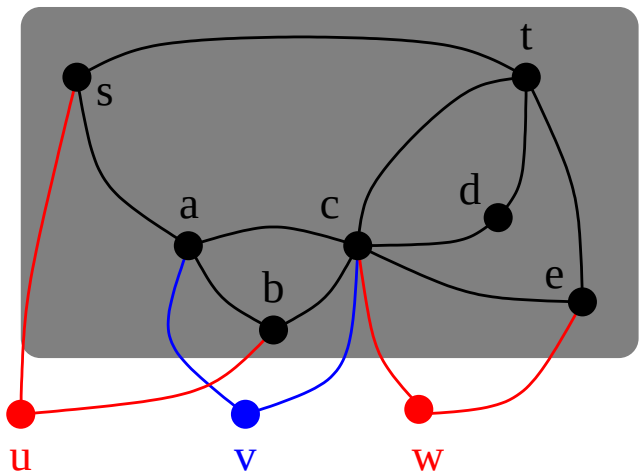

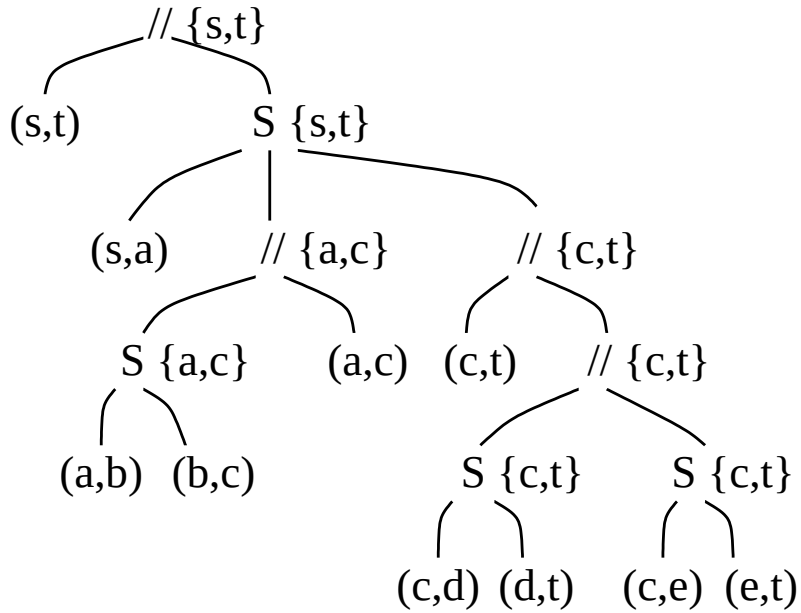

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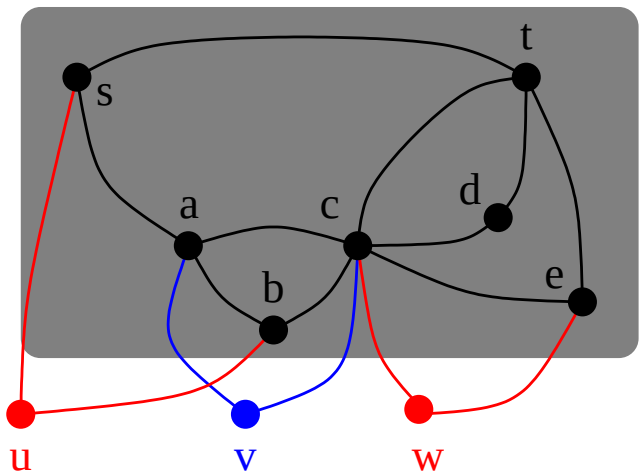

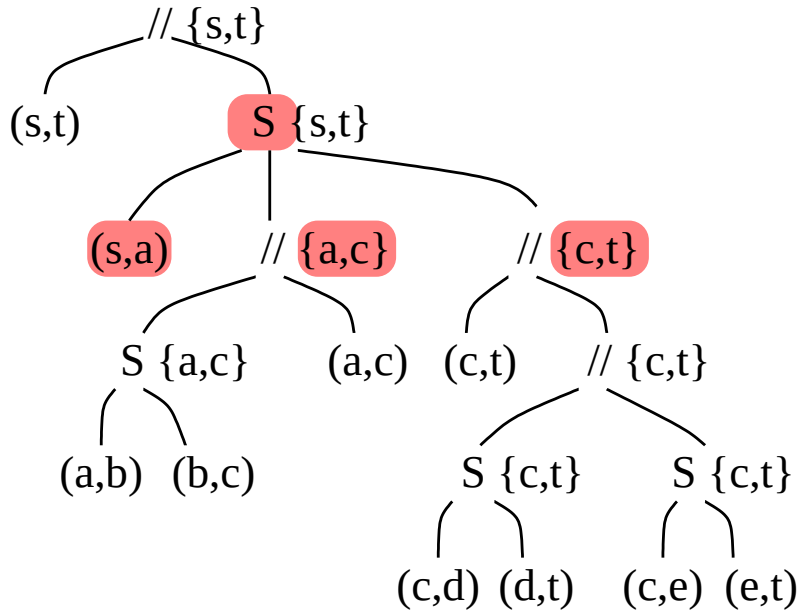

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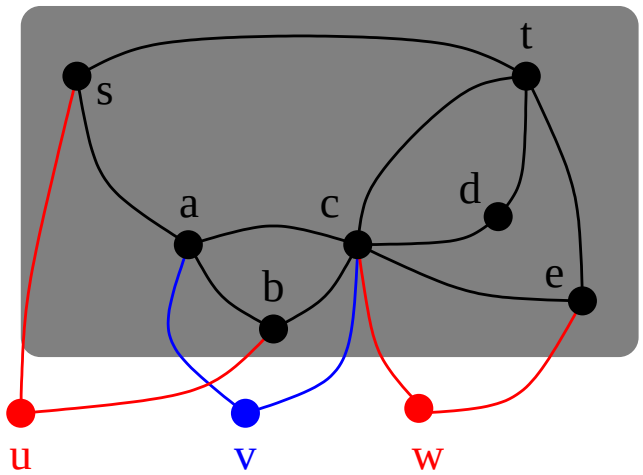

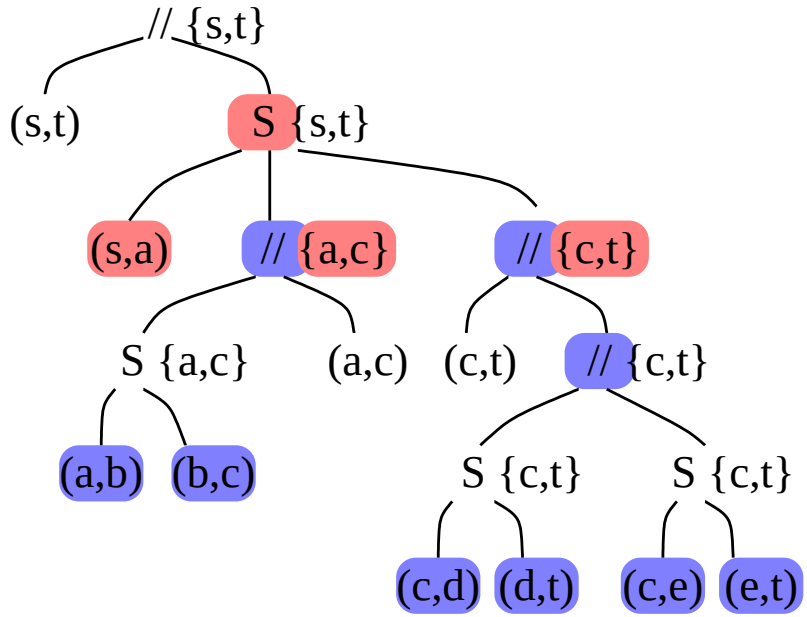

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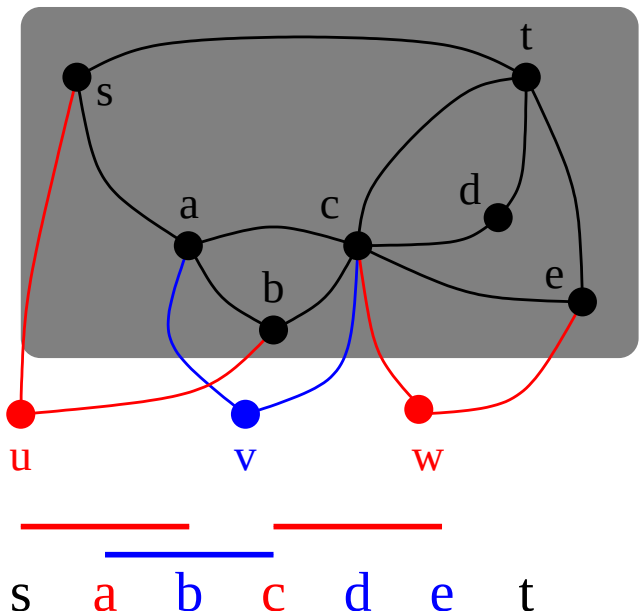
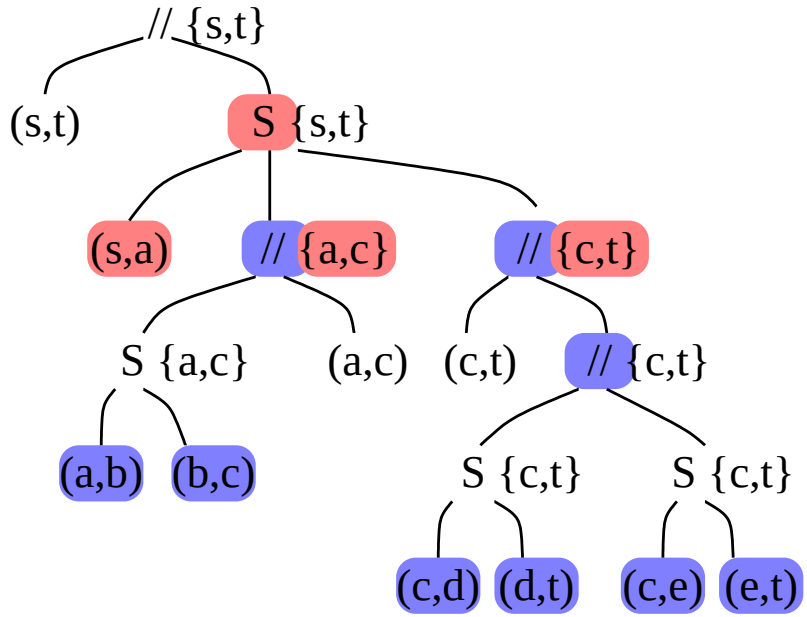
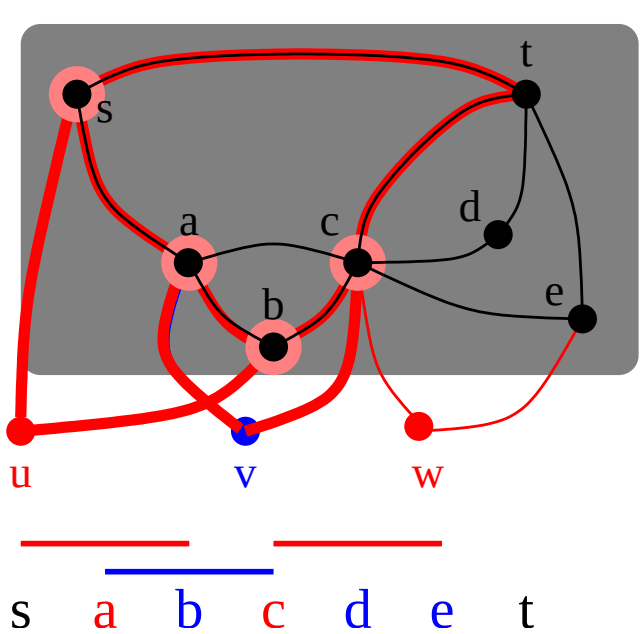


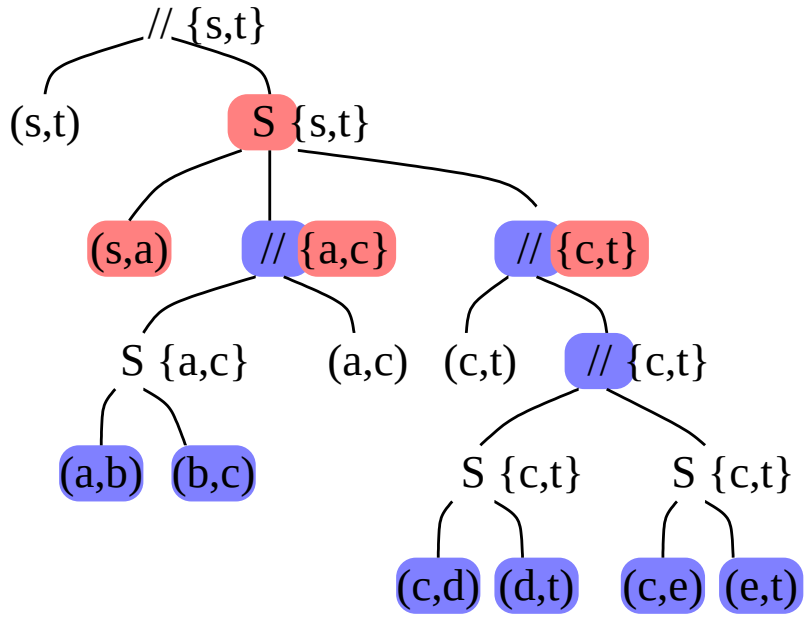


Algorithm outline (3')

3'. Solve each independent instance in polytime

Lemma: The auxiliary graph $\tilde{G}$ is a circle graph





Observation: $G[S \cup \{u, v\}]$ contains a K_4-minor
$\Leftrightarrow$ $I(u)$ and $I(v)$ overlap

Conclusion

Theorem: There exists a single exponential FPT time algorithm for the K_4-Minor Cover problem.

Further works:

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1. adpat the branch-or-process rule to address the Outerplanar Graph Vertex Deletion (or equiv. $\{K_{2,3}, K_4\}$-Minor Cover)

Conclusion

Theorem: There exists a single exponential FPT time algorithm for the K_4-Minor Cover problem.

Further works:

1. adpat the branch-or-process rule to address the Outerplanar Graph Vertex Deletion (or equiv. $\{K_{2,3}, K_4\}$-Minor Cover)

2. get rid of the protrusion rule to get an explicit algorithm

3. extend to higher treewidth or to K_t-minor for $t \geqslant 5$