Algorithmics of Modular Decomposition

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Joint work with: A. Bergeron, S. Bérard, S. Bessy, B.M. Bui Xuan, C. Chauve, D. Corneil, F. Fomin, E. Gioan, M. Habib, A. Perez, S. Saurabh, S. Thomassé, M. Tedder, L. Viennot...
Modular decomposition of undirected graphs

Ehrenfeucht et al’s modular decomposition algorithm

Common Intervals of permutations

Modular decomposition of tournaments

Kernelization algorithm for FAST
Does a permutation graph have a unique representation?
Permutations and permutation graphs

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A subset of vertices $M$ of a graph $G = (V, E)$ is a module iff 
$\forall x \in V \setminus M$, either $M \subseteq N(x)$ or $M \cap N(x) = \emptyset$
Modules

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\[ \forall x \in V \setminus M, \text{ either } M \subseteq N(x) \text{ or } M \cap N(x) = \emptyset \]

Examples of modules:
- connected components
- connected components of $\overline{G}$
A subset of vertices $M$ of a graph $G = (V, E)$ is a module iff
$\forall x \in V \setminus M$, either $M \subseteq N(x)$ or $M \cap N(x) = \emptyset$

▶ A graph (a module) is prime if all its modules are trivial: e.g. the $P_4$. 

▶ A graph (a module) is degenerate if every subset of vertices is a module: cliques and stables.
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Permutation graph recognition

Theorem [Gallai’67]
A permutation graph has a unique representation (up to reversal) iff it is prime.
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Recognition algorithm

- Recursively solve the problem on modules
- Solve the prime case (with linear time transitive orientation algorithm)
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Recognition algorithm

- Recursively solve the problem on modules
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Theorem [McConnell and Spinrad’99]
The permutation graph recognition problem can be solved in $O(n + m)$ time

- we need a linear time modular decomposition algorithm
Partitive families

If $M$ and $M'$ are two overlapping modules then

- $M \setminus M'$ is a module
- $M \cap M'$ is a module
- $M \Delta M'$ is a module
Partitive families

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The set of modules of a graph forms a **partitive family**

A module is **strong** if it does not overlap any other module
Partitive families

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The set of modules of a graph forms a partitive family.

A module is strong if it does not overlap any other module.

Strong modules are nested into an inclusion tree: the modular decomposition tree $MD(G)$.
A partition $\mathcal{P}$ of the vertex set of a graph $G$ is a modular partition if every part is a module of $G$.
Modular partition and quotient graph

A partition $\mathcal{P}$ of the vertex set of a graph $G$ is a modular partition if every part is a module of $G$.

If $\mathcal{P}$ is a modular partition of $G$, the quotient graph $G/\mathcal{P}$ is the induced subgraph obtained by choosing one vertex per part of $\mathcal{P}$. 
Theorem [Gal’67,CHM81]

Let $G = (V, E)$ be a graph. Then either

1. (parallel) $G$ is not connected, or
2. (series) $\overline{G}$ is not connected, or
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Let $G = (V, E)$ be a graph. Then either

1. (parallel) $G$ is not connected, or
2. (series) $\overline{G}$ is not connected, or
3. (prime) $G/\mathcal{M}(G)$ is a prime graph, with $\mathcal{M}(G)$ the modular partition containing the maximal strong modules of $G$. 

![Diagram of a graph and its modular partition]

- Parallel-partition
- Series-graph
- Prime-partition

![Graph layout with nodes and edges]

- Nodes labeled from 1 to 11
- Edges connecting different nodes
Theorem [Gal’67,CHM81]

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3. (prime) $G/\mathcal{M}(G)$ is a prime graph, with $\mathcal{M}(G)$ the modular partition containing the maximal strong modules of $G$.

Observation: If a $P_4$ on $\{a, b, c, d\}$ overlap a module $M$, then

$$|M \cap \{a, b, c, d\}| = 1$$
Modular decomposition algorithms

- $O(n^4)$ [Cowan, James, Stanton’72]
- $O(n^3)$ [Blass, 1978], [Habib, Maurer’79]
- $O(n^2)$ [McConnell, Spinrad’89]
Modular decomposition algorithms

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▶ \(O(n^2)\) [McConnell, Spinrad’89]
▶ \(O(n + m\alpha(m, n))\) [Spinrad’92], [Cournier, Habib’93]
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- $O(n + m)$ [McConnell, Spinrad’94], [Cournier, Habib’94]
- $O(n + m\log n)$ [Habib, Paul, Viennot’99] (factoring permutation), [McConnell, Spinrad’00]
Modular decomposition algorithms

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- $O(n + m \log n)$ [Habib, Paul, Viennot’99] (factoring permutation), [McConnell, Spinrad’00]
- $O(n + m)$ [Capelle, Habib’97] (factoring permutation) [Dahlhaus, Gustedt, McConnell’97], [Tedder, Corneil, Habib, Paul’08]

- other many others for variants of modular decomposition
Cographs - Totally decomposable graphs

Theorem: A graph is a cograph (a $P_4$-free graph $\bullet - \bullet - \bullet - \bullet$) iff its modular decomposition tree does not contain any prime node.
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Cographs can be built from the single vertex with the disjoint union and series composition.

Exercice: prove that cographs are permutation graphs.
**Cographs - Totally decomposable graphs**

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**Linear time recognition algorithms**
- incremental [Corneil, Pearl, Stewart’85]
- partition refinement [Habib, P.’05]
- LexBFS [Bretscher, Corneil, Habib, P.’08]
Modular decomposition of undirected graphs

Ehrenfeucht et al’s modular decomposition algorithm

Common Intervals of permutations

Modular decomposition of tournaments

Kernelization algorithm for FAST
Ehrenfeucht et al’s modular decomposition algorithm

\( \mathcal{M}(G, v) \) is the modular partition composed by

- \( \{v\} \) and the maximal modules of \( G \) not containing \( v \).

1. Compute \( \mathcal{M}(G, v) \)
2. Compute \( MD(G/\mathcal{M}(G,v)) \)
3. For each \( \mathcal{X} \in \mathcal{M}(G, v) \) compute \( MD(G[\mathcal{X}]) \)
Computation of $\mathcal{M}(G, v)$ (1)

Lemma [MR84] Let $\mathcal{P}$ be a modular partition of $G = (V, E)$. $\mathcal{X} \subseteq \mathcal{P}$ is a module of $G/\mathcal{P}$ iff $\cup_{M \in \mathcal{X}} M$ is a module of $G$. 

\begin{figure}
\centering
\includegraphics[width=\textwidth]{diagram.png}
\end{figure}
Computation of $\mathcal{M}(G, v)$ (1)

Lemma [MR84] Let $\mathcal{P}$ be a modular partition of $G = (V, E)$.
$X \subseteq \mathcal{P}$ is a module of $G/\mathcal{P}$ iff $\bigcup_{M \in X} M$ is a module of $G$.

A vertex $x$ is a splitter for a set $S$ of vertices if
$$\exists y, z \in S \text{ with } xy \in S \text{ and } xz \notin E$$
We say that $x$ separate $y$ and $z$. 

![Diagram](https://via.placeholder.com/150)
Computation of $\mathcal{M}(G, \nu)$ (1)

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\[ \exists y, z \in S \text{ with } xy \in S \text{ and } xz \notin E \]
We say that $x$ separate $y$ and $z$.

Lemma If $x$ is a splitter for the set $S$, then any module $M$ containing $S$ must also contain $x$. 
Computation of $\mathcal{M}(G, \nu)$ (2)

Lemma If $\nu$ is a splitter of a set $S$, then for any module $M \subseteq S$ either $M \subseteq S \cap N(\nu)$ or $M \subseteq M \cap \overline{N}(\nu)$.
Computation of $M(G, v)$ (2)

Lemma If $v$ is a splitter of a set $S$, then for any module $M \subseteq S$ either $M \subseteq S \cap N(v)$ or $M \subseteq M \cap \overline{N}(v)$

$O(n + m \log n)$ time using vertex partitioning algorithm
Computation of $\mathcal{M}(G, v)$ (2)

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Computation of $MD(G/\mathcal{M}(G,v))$ (3)

- The modules of $G/\mathcal{M}(G,v)$ are linearly nested:
  any non-trivial module contains $v$
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- The forcing graph $\mathcal{F}(G,v)$ has edge $\overrightarrow{xy}$ iff $y$ separates $x$ and $v$
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- The modules of $G/\mathcal{M}(G,v)$ are linearly nested: any non-trivial module contains $v$.
- The forcing graph $\mathcal{F}(G, v)$ has edge $\overrightarrow{xy}$ iff $y$ separates $x$ and $v$.
Computation of $MD(G_{/M(G,v)})$ (4)

Complexity

- [Ehrenfeucht et al.’94] gives a $O(n^2)$ complexity.
- [MS00]: simple $O(n + m \log n)$ vertex partitioning algorithm
- [DGM’01]: $O(n + m.\alpha(n, m))$ and a more complicated $O(n + m)$ implementation.

Other algorithms

- [CH94] and [MS94]: the first linear algorithms.
- [MS99]: $O(n + m)$ algorithm which extends to transitive orientation.
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[Spinrad’03] The new [linear time] algorithm [MS99] is currently too complex to describe easily [...] I hope and believe that in a number of years the linear algorithm can be simplified as well.

- [Tedder, Corneil, Habib, P.’08] simple linear time algorithm.
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Kernelization algorithm for FAST
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Obs.: common intervals of $\sigma_1, \sigma_2$ are not modules of $G(\sigma_1, \sigma_2)$
Back to permutations: common intervals

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- Computation of all common intervals in linear time - $O(n^2)$ - [Uno, Yagura’00]
Back to permutations: common intervals

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Strong (common) interval doesn’t overlap other common intervals

Lemma [de Montgolfier] A set $S$ is a strong interval of $\sigma_1$ and $\sigma_2$ iff it is a strong module of the permutation graph $G(\sigma_1, \sigma_2)$
The family of common intervals is weakly partitive:

If $I_1$ and $I_2$ are two common intervals then

- $I_1 \cup I_2$ is a common interval
- $I_1 \cap I_2$ is a common interval
- $I_1 \setminus I_2$ and $I_2 \setminus I_1$ are common intervals
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Let $\mathcal{I}$ be a partition of $[1 \ldots n]$ into common intervals of the permutation $\sigma$, then we denote by $\sigma/\mathcal{I}$ the \textit{quotient permutation} defined on $[1 \ldots |\mathcal{I}|]$.
Common intervals (2)

The family of common intervals is *weakly partitive*:

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Common intervals (3)

**Theorem** Let $\sigma$ be a permutation on $[1, \ldots, n]$ and $\mathcal{I}$ be the partition into maximal common intervals of $\sigma$, then either

1. $\sigma/\mathcal{I} = 1_{|\mathcal{I}|}$ - the identity on $[1 \ldots |\mathcal{I}|]$
2. $\sigma/\mathcal{I} = \overline{1}_{|\mathcal{I}|}$ - the reverse identity on $[1 \ldots |\mathcal{I}|]$
3. $\sigma/\mathcal{I}$ is prime - or simple (does not have non-trivial common interval)
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**Theorem** (see e.g. [Bergeron et al.’08])
The common interval tree can be computed in $O(n)$ time.
Common intervals (4)

A permutation $\sigma$ is **separable** if it does not contain the pattern $3 \, 1 \, 4 \, 2$.
A permutation $\sigma$ is **separable** if it does not contain the pattern $3\ 1\ 4\ 2$

- $3\ 1\ 4\ 2$ corresponds to the $P_4$

\[
\begin{array}{cccccccc}
7 & 1 & 6 & 5 & 4 & 3 & 2 & 9 & 8 & 11 & 10
\end{array}
\]
Common intervals (4)

A permutation $\sigma$ is **separable** if it does not contain the pattern $3 \ 1 \ 4 \ 2$

- $3 \ 1 \ 4 \ 2$ corresponds to the $P_4$

- a permutation is separable iff its common **interval tree** does not contain prime nodes

- a permutation is separable iff the permutation graph $G(\sigma, 1)$ is a **cograph** ($P_4$-free graph)
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Ehrenfeucht et al’s modular decomposition algorithm

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![Graph](image-url)
A module in a tournament is a set $S$ such that for every $x \notin S$

- either $\forall y \in S, \ x \rightarrow y$ or $\forall y \in S, \ y \rightarrow x$
A tournament is **transitive** if there exists a permutation $\sigma$ of $V(T)$ with no **backward** arcs

**Theorem:** Let $T$ be a tournament and $\mathcal{M}(T)$ be the modular partition into maximal strong modules, then

1. either $T/\mathcal{M}(T)$ is **transitive** - contains no backward arc
2. or $T/\mathcal{M}(T)$ is prime
A tournament is transitive if there exists a permutation $\sigma$ of $V(T)$ with no backward arcs.

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Modular decomposition algorithm for tournaments

A factoring permutation of a tournament $T$ (or a graph) is a permutation $\sigma$ of its vertices such that every (strong) module of $T$ is an interval of $\sigma$
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Factoring permutation via a partition refinement algorithm in linear time [de Mongolfier’03]
Modular decomposition algorithm for tournaments

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![Diagram of modular decomposition tree from a factoring permutation in linear time [Capelle’97]]
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- Factoring permutation via a partition refinement algorithm in linear time [de Mongolfier’03]

![Diagram showing modular decomposition and factoring permutation]
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- Factoring permutation via a partition refinement algorithm in linear time [de Mongolfier’03]

- Modular decomposition tree from a factoring permutation in linear time [Capelle’97]
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Modular decomposition of tournaments

Kernelization algorithm for FAST
FAST: Feedback Arc Set in Tournament

- A tournament $T$ and an integer $k$
- Find a set of at most $k$ arcs whose reversal transform $T$ into a transitive tournament
FAST: Feedback Arc Set in Tournament

- A tournament $T$ and an integer $k$
- Find a permutation $\sigma$ of the vertices with at most $k$ backward edges
FAST: Feedback Arc Set in Tournament

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- NP-Complete [Alon’06] [Charbit et al.’07]
- FTP [Raman, Saurabh’06] [Alon et al.’09]
- $(1 + \epsilon)$-approximation scheme [Kenyon-Mathieu, Schudy’07]
Obs.: A tournament is \textit{transitive} iff there is no (directed) triangle

\begin{itemize}
  \item \textbf{Rule 1} \textit{[irrelevant vertex]} If a vertex \( v \) is not contained in any triangle, then delete \( v \).
  \item \textbf{Rule 2} \textit{[sunflower]} If there is an arc belonging to more than \( k \) distinct triangles, then reverse it and decrease \( k \) by 1.
\end{itemize}

The span \( s(→uv) \) of a backward arc of a reduced tournament is \( ⩽ 2k + 2 \).
**FAST (2)**

**Obs.:** A tournament is transitive iff there is no (directed) triangle.
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**FAST (2)**

**Obs.:** A tournament is **transitive** iff there is no (directed) triangle

**Rule 1** [irrelevant vertex] If a vertex $v$ is not contained in any triangle, then delete $v$

A reduced tournament contains no source nor sink

**Rule 2** [sunflower] If there is an arc belonging to more that $k$ distinct triangles, then reverse it and decrease $k$ by 1

The span $s(uv)$ of a backward arc of a reduced tournament is $\leq 2k + 2$
Rule 3 [acyclic module] Let $M$ be a maximal acyclic module. If there are at most $p = |M|$ arcs from $N^+(M)$ to $N^-(M)$, then reverse all these arcs and decrease $k$ by $p$. 
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\[
\sum t_i^2 \leq \sum s(\overrightarrow{uv}) \leq k(2k + 2)
\]
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Theorem [Bessy et al.’09]: Every instance $(T, k)$ of $k$-FAST can be reduced in polynomial time to an equivalent instance $(T, k')$ such that

\[|T| \leq 2k + \sum t_i = O(k\sqrt{k}) \text{ and } k' \leq k\]
Kernelization algorithm

Given a parameterized instance \((I, k)\) of a problem, a kernelization algorithm computes in \textit{polytime} an \textit{equivalent} instance \((I', k')\) st.

\[ k' = f(k) \quad \text{and} \quad |I'| \leq g(k) \]
Given a parameterized instance \((\mathcal{I}, k)\) of a problem, a **kernelization algorithm** computes in polytime an equivalent instance \((\mathcal{I}', k')\) st.

\[ k' = f(k) \quad \text{and} \quad |\mathcal{I}'| \leq g(k) \]

- we described a \(O(k\sqrt{k})\)-vertex kernel for FAST based on modular decomposition
- best known result: \(O(k)\)-vertex kernel ([Bessy et al.'09], [P., Perez, Thomassé’11])
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k' = f(k) \quad \text{and} \quad |\mathcal{I}'| \leq g(k)
\]

Other modular decomposition kernelizations:

- \(O(k^2)\)-vertex kernel for **CLUSTER EDITING**
- \(O(k^3)\)-vertex kernel for **COGRAPH EDITING**
- also for **MIN FLIP CONSENSUS TREE, CLOSEST 3-LEAF POWER**...
Some conclusions

- Modular decomposition plays an important role in the context of many graph classes
  - permutation graphs, interval graphs, comparability graphs . . .
  - even perfect graph

- Many examples of partitive (weakly partitive) families are known in various contexts
  - modules in undirected graphs, digraphs, hypergraphs
  - common interval of permutations

- Various generalizations
  - bimodular decomposition (module adapted to bipartite graphs)
  - bipartitive families: eg. split decomposition of graphs - $O((n + \alpha(n,m))$ circle graph recognition

- Crossing families, union-difference families of sets . . .

- Clique-width (cographs are clique-width 2 graphs), rankwidth
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  - common interval of permutations
Some conclusions

- Modular decomposition plays an important role in the context of many graph classes
  - permutation graphs, interval graphs, comparability graphs . . .
  - even perfect graph

- Many examples of partitive (weakly partitive) families are known in various contexts
  - modules in undirected graphs, digraphs, hypergraphs
  - common interval of permutations

- Various generalizations
  - bimodular decomposition (module adapted to bipartite graphs)
  - bipartitive families: eg. split decomposition of graphs - $O(n + \alpha(n, m).m)$ circle graph recognition
  - crossing families, union-difference families of sets . . .
  - clique-width (cographs are clique-widtht 2 graphs), rankwidth
To learn / read more

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- F. de Montgolfier, "Décomposition modulaire de graphes, théorie, extensions et algorithmes", Phd Thesis (in French), 2003

- B.M. Bui Xuan, "Tree-representation of set families in graph decompositions and efficient algorithms", Phd Thesis, 2008
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  attend I. Todonca’s talk !!!
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- Common intervals and sorting by reversal:

  attend M. Bouvel’s talk!!!