KERNELIZATION

Polynomial time preprocessing - data reduction

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Journée Algo & Calcul - NUMEV
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What does computational theory tell us ?

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  + Karp's list of 21 NP-Complete problems [1972]

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Developing means for predicting the performance of algorithms and heuristics on real data and on real computers is a grand challenge in algorithms. . .]

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Effective Heuristics for NP-Hard Problems by R. M. Karp

Presented at the Michael Rabin Celebration, August 2011

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We begin by describing the evolution of successful heuristic algorithms for two problems arising in computational biology:
(1) The Colorful Subgraph Problem. [. . .]
(2) Global Genome Alignment. [. . .]

We then discuss a general approach for tuning the parameters and design choices within a given heuristic algorithmic strategy, assuming the availability of a training set of typical problem instances.

Polynomial time preprocessing

"*It has become clear, however, that far from being trivial and uninteresting, pre-processing has unexpected practical power for real world input distributions, and is mathematically a much deeper subject than has generally been understood.*"

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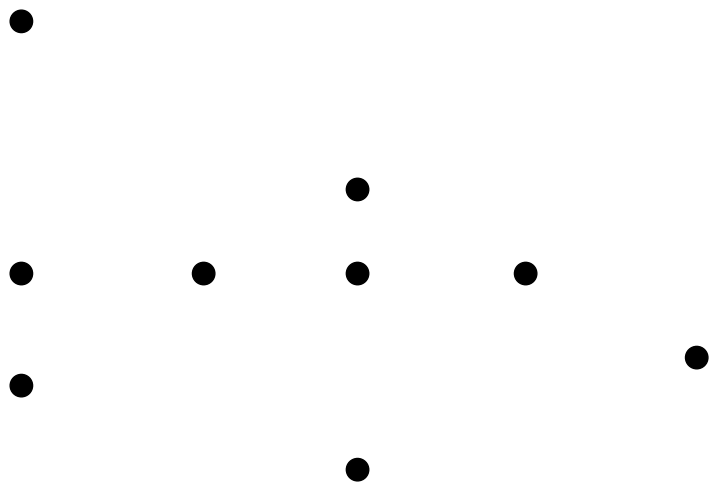
What can we expect from polynomial time preprocessing ?

- How small can a data-set be skrinked, while preserving the existence of a solution ?
- How to measure the efficiency of preprocessing ?
- Does every NP-Hard problem enjoy an efficient polynomial time preprocessing ?
- Can we provide a mathematical framework to explain the power and limitation of preprocessing ?

Kernelization (an example)



Is there a set of k lines covering the set S of points ?

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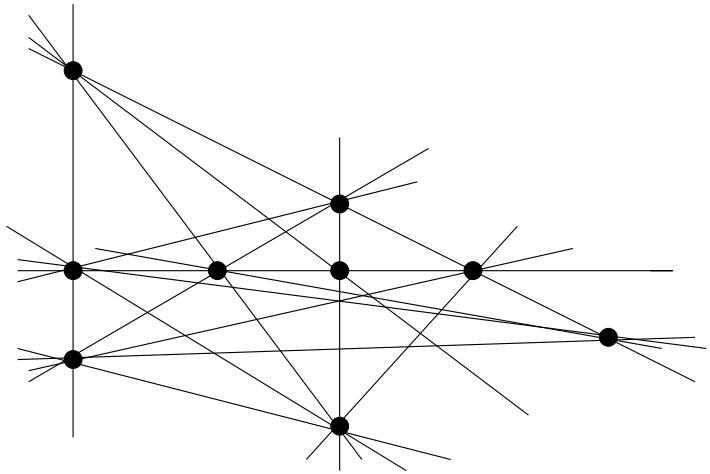

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Observation 1: We can restrict our attention to lines generated by the pairs of points of S

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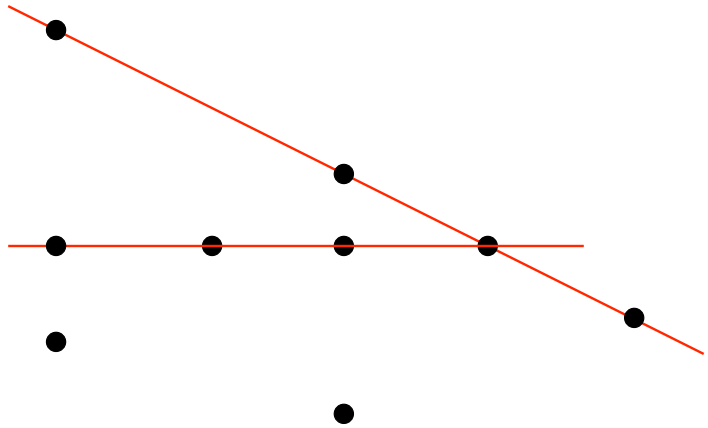

Is there a set of k lines covering the set S of points ?

Observation 2: If a line contains at least $k+1$ points, then any solution contains that line (e.g. $k = 3$)

$\Rightarrow$ [Degree rule] remove such a line and decrease k by 1

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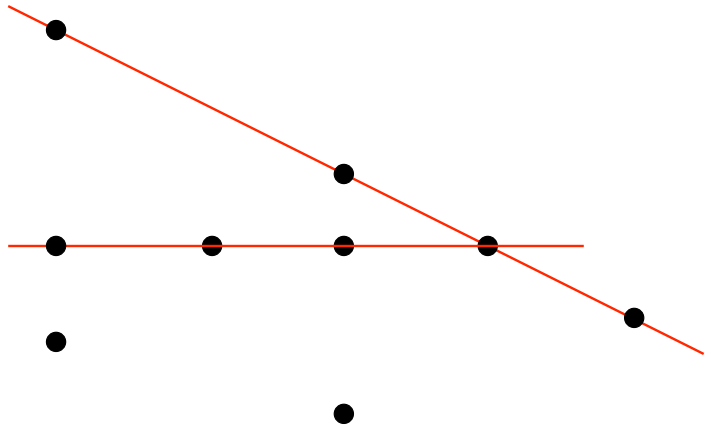

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$\Rightarrow$ the reduced instance contains at most k^2 points.

Kernelization (definition)

How to measure the size of the reduced instance ?

- need of a parameter independent from the data-size
- the parameter may express some structure of the instance (e.g. treewidth)
- the smaller the parameter is the better it is...
  (to be discussed later)

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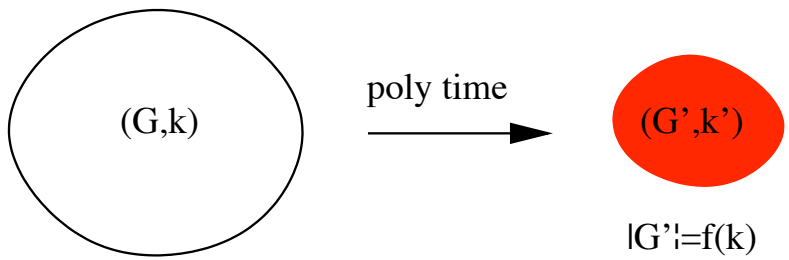

Given a parameterized instance $(\mathcal{I}, k)$ of a problem, a kernelization algorithm computes in polytime an equivalent instance $(\mathcal{I}', k')$ st.

$$k' = f(k) \qquad \text{and} \qquad |\mathcal{I}'| \leqslant g(k)$$

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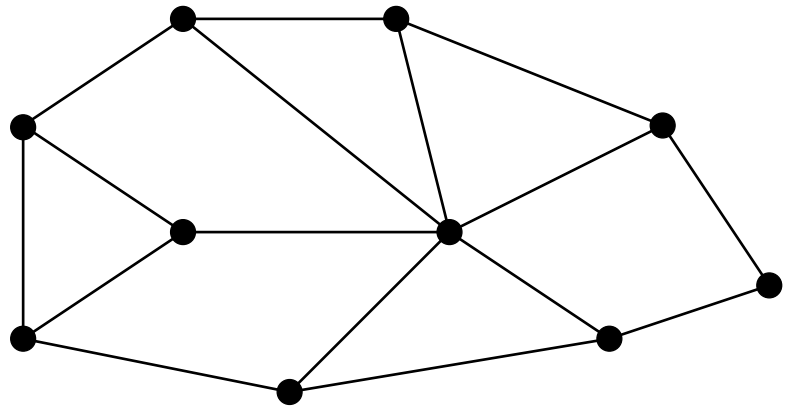
- a small kernel size $\longrightarrow$ linear ... cubic ... exponential ?
- a small parameter $\longrightarrow$ hierarchy of parameters
- efficient algorithm $\longrightarrow$ linear time ... cubic time
- equivalence of the solutions $\longrightarrow$ at least a certain guarantee

On kernel size (1): Vertex Cover

- Can we remove from G at most k edges to obtain an edge-free graph?



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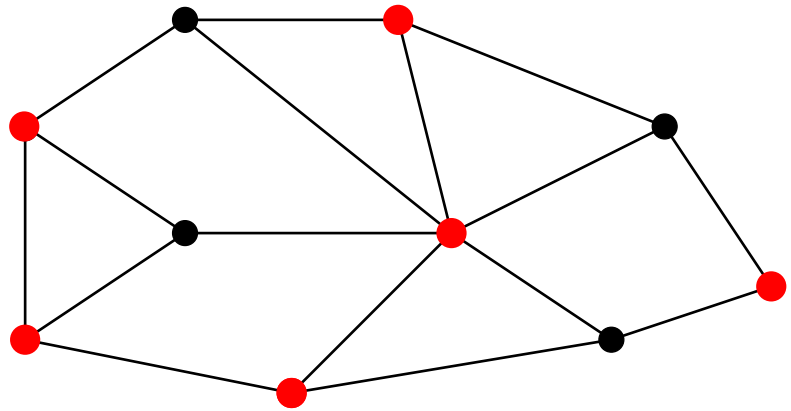

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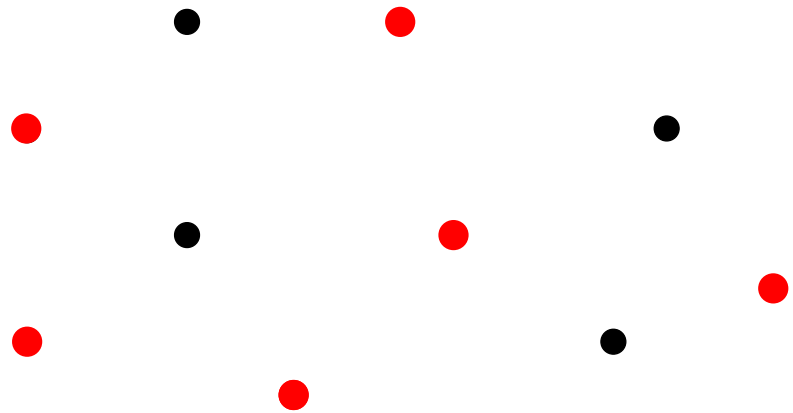

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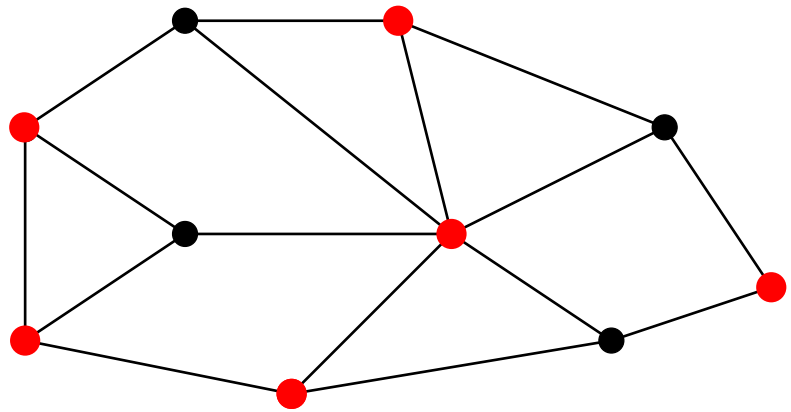

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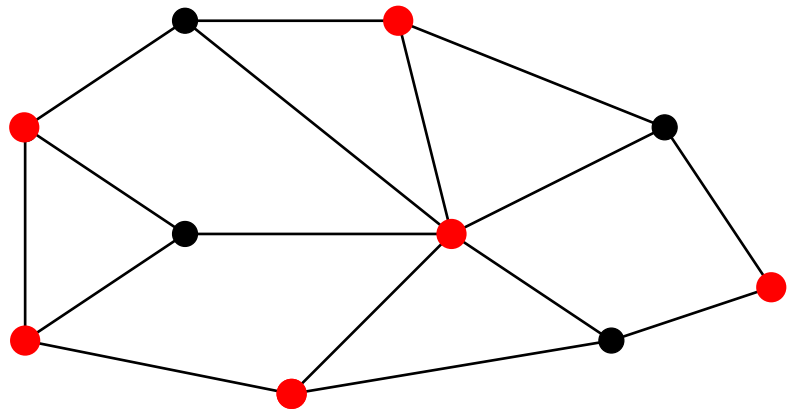

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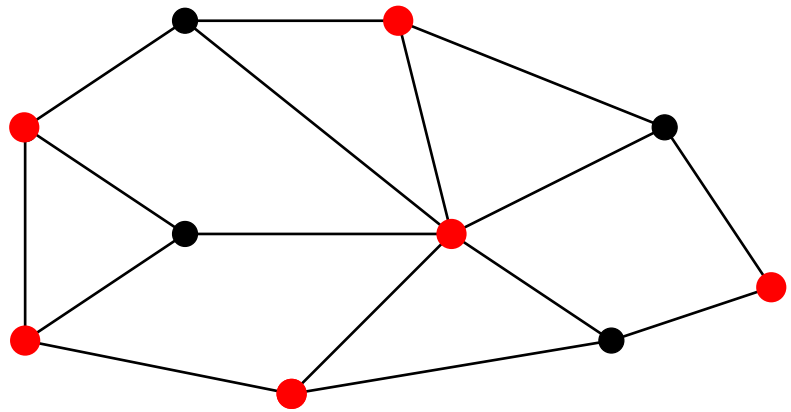

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- Vertex Cover parameterized by solution size has a $2k$ vertex kernel e.g. via linear programming
- Vertex Cover parameterized by solution size has no $O(k^{2-\epsilon})$ edge kernel unless $coNP \subseteq NP/poly$ [Dell, van Melkebeek]

On kernel size (2)

Theorem [Folkore]

A parameterized problem has a kernelization algorithm iff it can be solved in time $f(k).n^{O(1)}$ where k is the parameter and n the data-size (a.k.a the problem is Fixed Parameter Tractable).

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- Clique parameterized by the solution size is $W[1]$-complete
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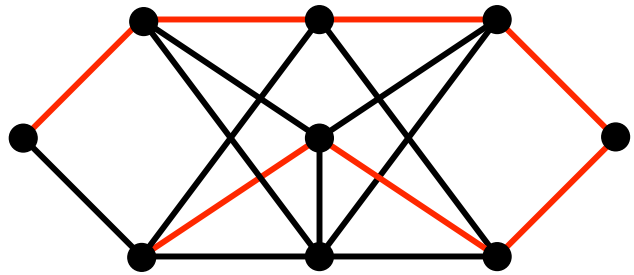
Corollary

Clique and Dominating Set parameterized by the solution size does not have a kernelization algorithm.

On kernel size (3): exponential lower bound

- A graph $G = (V, E)$ and a parameter $k \in \mathbb{N}$
- Does G contains a path of length k?

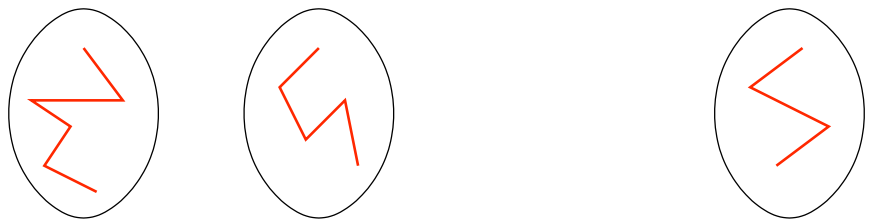


Longest Path is NP-Complete (cf. Hamiltonian Path) but can be solved in FPT time.

On kernel size (3): exponential lower bound

Hypothesis There exists a kernelization algorithm $\mathcal{A}$ for Longest Path which computes a polynomial size kernel.

- Let (G, k) be $(G_1, k) \oplus (G_2, k) \oplus \ \ldots \ \oplus (G_t, k)$



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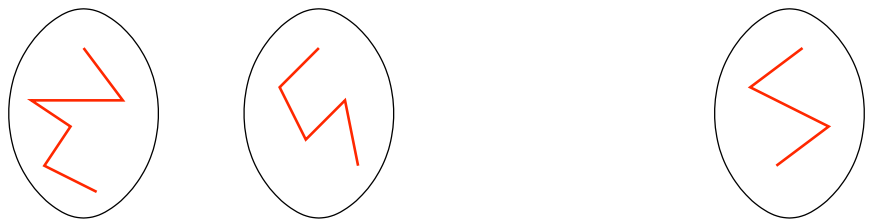

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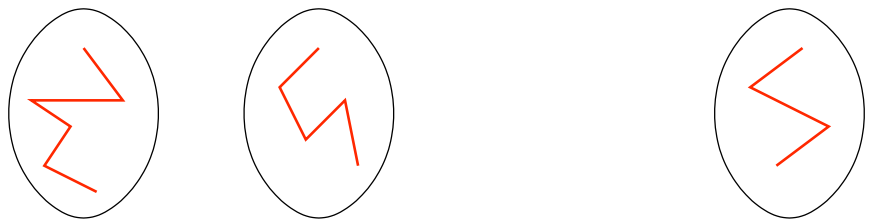

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Theorem [Fortnow, Santhaman + Bodlaender, Downey, Fellows, Hermelin] If a parameter problem is OR-composable and has a polynomial size kernel, then $PH = \Sigma_p^3$

On kernel size (4)

For a few years now a lower bound machinery is been developped:

- OR-composition
- Parameter preserving and polynomial transformation
- cross-composition
- weak-distillation
- ...

Meanwhile, recent results propose meta-theorem to prove polynomial upper-bounds

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Lemma Let k_1 and k_2 be two parameters such $k_1 \leqslant k_2$.
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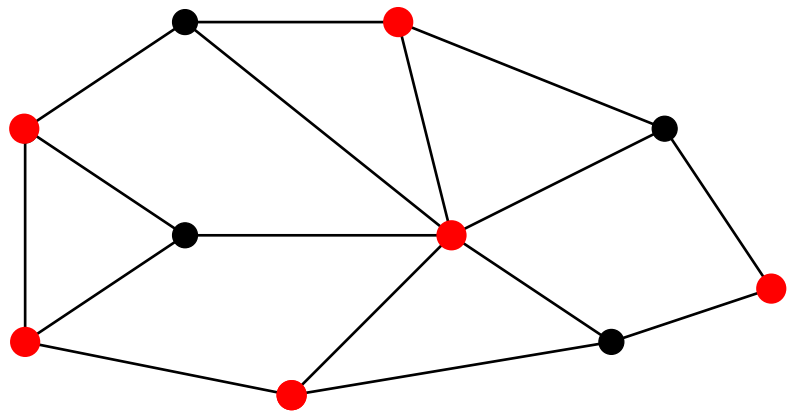
Vertex Cover $\geqslant$ Feedback Vertex set $\geqslant$ Treewidth



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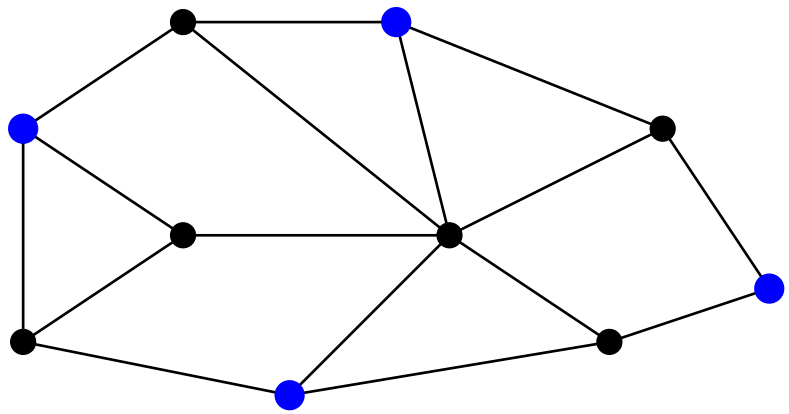

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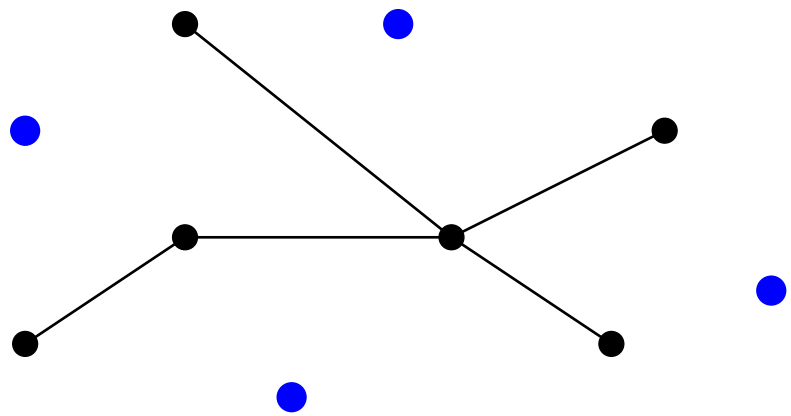

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Theorem Vertex Cover parameterized by

- Vertex Cover has a linear kernel
- Feedback Vertex Set has a cubic kernel
- Treewidth has no polynomial kernel

where the transition does take place ?

Complexity time of kernelization

How fast an optimal kernel can be computed?

- most effort has be put on optimizing kernel size, and only a few on computation time
- [Niedermeier et al.] have recently shown that a non-exhaustive application of the reduction rules may lead to a faster computation without altering too much the kernel size.

Which guarantee to preserve?

$\Rightarrow$ Approximative kernelization (α-fidely kernel, $\alpha \leqslant 1$)

1. transform in polytime an parameterized instance (x, k) into an instance (x', k') such that
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Theorem [Fellows, Shachnai]
Vertex Cover has a $2k(2-\alpha)/\alpha$-fidelity kernelization

Experimental studies

- Kernelization Algorithms for the Vertex Cover Problem: Theory and Experiments by F. Abu-Khzam, R. Collins, M. Fellows, M. Langston, W. Suters and C. Symons, 2004.

  > *Vertex Cover has many real-world applications, including many in the field of bioinformatics. It can be used in the construction of phylogenetic trees, in phenotype identification, and in analysis of microarray data, just to name a few.*
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- Partial Kernelization for Rank Aggregation: Theory and Experiments by N. Betzler, R. Bredereck and R. Niedermeier, 2010.

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- ANR Project AGAPE (2009-2013)
- KERNEL Project (Chercheur d'Avenir - Languedoc-Roussillon - 2012-2015)