Conflict Packing
Linear kernels for FAST and Dense-RTI

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Joint work with
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Shonan, May 10, 2013
"Observation": **global** reduction rules lead to better kernel size than **local** reduction rules

- Integer linear programming for **Vertex Cover**
- Crown decomposition for **Vertex Cover**, **Cluster Editing**
- Matching (or expansion lemma) for **Feedback Vertex Set**
- Region decomposition, protrusion decomposition . . .

**This talk**: conflict packing and matching based reduction rules
Feedback Arc Set in Tournaments (FAST)

▶ Input: A tournament $T = (V, A)$ and a parameter $k$
▶ Question: Can $T$ be transformed into an acyclic tournament by at most $k$ arc reversals?
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→ find a vertex ordering $\pi$ with at most $k$ backward arcs
Feedback Arc Set in Tournaments (FAST)

- NP-Complete [Alon’06] [Charbit et al.’07]
- $(1 + \epsilon)$-approximation scheme [Kenyon-Mathieu, Schudy’07]
- FTP [Raman, Saurabh’06] [Alon et al.’09]
- $O(k^2)$ vertex kernel [Dom et al.’06], [Alon et al.’09]
- $O(k\sqrt{k})$ vertex kernel [Bessy et al.’10]
- $O(k)$ vertex kernel [Bessy et al.’10]
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Our result

A simpler proof of an $O(k)$ vertex kernel for $k$-FAST
Dense Rooted Triplet Inconsistency (dense RTI)

- **Input:** A dense set of rooted triplets \( \mathcal{R} \) on a set \( L \) of leaves and a parameter \( k \)
- **Question:** Does \( \mathcal{R} \) contain a subset \( \mathcal{R}' \) of consistent rooted triplets such that \( |\mathcal{R}'| \geq |\mathcal{R}| - k \)?
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$\mathcal{R}$ is **dense** if it contains a rooted triplet for every $\{a, b, c\} \in L^3$
Known Results

- **NP-Complete** [Birka et al.’08]
- dual parameterization is known as **Maximum Rooted Triplet Consistency** [Birka et al.’08]
- **FPT** [Guillemot, Berry’07]
- subexponential **FPT** [Guillemot, Mnich’10]
- **$O(k^2)$ kernel** [Guillemot, Mnich’10]
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Our result

An $O(k)$ kernel for $k$-Dense RTI

Open: No constant approximation is known
Kernels for **FAST**

An elementary quadratic kernel
A matching based linear kernel

Linear kernel for **DENSE-RTI**
**Theorem:** A tournament is acyclic (or transitive) iff it contains no (directed) triangle

**Rule 1** [irrelevant vertex] If a vertex $v$ is not contained in any triangle, then delete $v$

> A reduced tournament contains no source nor sink

**Rule 2** [sunflower] If there is an arc belonging to more than $k$ distinct triangles, then reverse it and decrease $k$ by 1

**Observation:** Rule 1 + Rule 2 yield a quadratic kernel
We denote by $T_\sigma = (V, A, \sigma)$ a tournament equipped with an ordering $\sigma$ on $V$. 
A linear kernel for fast

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Let $\overrightarrow{uv}$ be a backward arc of $T_\sigma$ and let $w \in \text{span}(\overrightarrow{uv})$ a vertex not incident to any backward arc, then

$$c(\overrightarrow{uv}) = \{u, w, v\}$$

is a $\overrightarrow{uv}$-certificate.

Let $F$ be a set of backward arcs, a $F$-certificate is a set

$$c(F) = \{c(\overrightarrow{uv}) \mid \overrightarrow{uv} \in F\}$$

of arc-disjoint certificates.
Let $\mathcal{P} = \{V_1, \ldots, V_l\}$ be an $\sigma$-partition of a $T_\sigma = (V, A, \sigma)$.

- Denote the external arcs by $A_E = \{\overrightarrow{uv} \mid u \in V_i, v \in V_j, i \neq j\}$
- Denote the internal arcs by $A_I = A \setminus A_E$
A linear kernel for fast

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- Denote the internal arcs by $A_I = A \setminus A_E$

A $\sigma$-partition is safe if the backward arcs of $A_E$ can be certified within $A_E$

Rule 4 [Bessy et al’09] (safe partition) Let $\mathcal{P}$ be a safe $\sigma$-partition of $T_\sigma$. Then reverse every external backward arc and decrement $k$ accordingly.
How to compute in polynomial time a safe partition?

Prove that the irrelevant vertex rule and the safe partition rule yields a $4k$-kernel
A linear kernel for fast

- How to compute in polynomial time a safe partition?
- Prove that the irrelevant vertex rule and the safe partition rule yields a $4k$-kernel

A conflict packing is a maximal set $C$ of arc-disjoint certificates. We denote by $V(C)$ the vertices covered by $C$.

**Lemma 1:** If $C$ is a conflict packing of a positive instance $(T, k)$ of FAST, then $|V(C)| \leq 3k$
Lemma 2: If \( C \) is a conflict packing of \( (T, k) \), then \( \exists \sigma \) such that
if \( \overrightarrow{uv} \) is backward arc of \( T_\sigma \), then \( \{u, v\} \subseteq V(C) \)
Lemma 2: If $C$ is a conflict packing of $(T, k)$, then $\exists \sigma$ such that if $\overrightarrow{uv}$ is backward arc of $T_\sigma$, then $\{u, v\} \subseteq V(C)$.

Let $C$ be a conflict packing of $T$. Then $T' = T - V(C)$ is transitive. Let $\pi$ be a transitive ordering of $V \setminus V(C)$. 
Lemma 2: If $\mathcal{C}$ is a conflict packing of $(T, k)$, then $\exists \sigma$ such that if $\overrightarrow{uv}$ is backward arc of $T_\sigma$, then $\{u, v\} \subseteq V(C)$

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- Moreover for every vertex $x \in V(C)$, $T' + \{x\}$ is transitive.
- So $x$ has a unique insertion location in $\pi$
Lemma 2: If \( C \) is a conflict packing of \((T, k)\), then \( \exists \sigma \) such that if \( \overrightarrow{uv} \) is backward arc of \( T_\sigma \), then \( \{u, v\} \subseteq V(C) \)

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- Moreover for every vertex \( x \in V(C) \), \( T' + \{x\} \) is transitive.

- So \( x \) has a unique insertion location in \( \pi \)

- The ordering \( \sigma \) is obtained by inserting every vertex of \( V(C) \) at its location in \( \pi \) (vertices at the same location are arbitrarily ordered)
Let us denote by $A_B$ the set of backward arcs of $T_{\sigma}$.
Let $C$ be a conflict packing, let us denote by $F = V \setminus V(C)$ the set of free vertices
A linear kernel for Fast

Let us denote by $A_B$ the set of backward arcs of $T_\sigma$.
Let $C$ be a conflict packing, let us denote by $F = V \setminus V(C)$ the set of free vertices.

Let $B = (A_B, F, E)$ be the bipartite graph such that $(\overrightarrow{uv}, w) \in E \iff \{u, v, w\}$ is a conflict / triangle in $T$. 

Let $S$ be a vertex cover of $B$ and set $S_F = F \cap S$.

Let $P$ be the $\sigma$-partition containing a part $\{v\}$ for every vertex $v \notin S_F$ for every maximal consecutive (in $\sigma$) subset of $V(C) \cup S_F$. 

![Graph depiction](attachment:image.png)
A linear kernel for FAST

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A linear kernel for FAST

Lemma: If \((T, k)\) is a positive instance of \(k\)-FAST of size at least \(4k\) and without any irrelevant vertex, then \(P\) is a safe \(\sigma\)-partition containing at least one external backward arc.
A linear kernel for fast

Lemma: If \((T, k)\) is a positive instance of \(k\)-FAST of size at least \(4k\) and without any irrelevant vertex, then \(\mathcal{P}\) is a safe \(\sigma\)-partition containing at least one external backward arc.

- A maximum matching \(\mathcal{M}\) in \(B\) is a conflict packing
- \(|\mathcal{M}| = |S| \leq k\), where \(S\) is a vertex cover
- as \(|V(C)| \leq 3k\), \(L \setminus S_F \neq \emptyset\) and \(\mathcal{P}\) is non-trivial

(red vertices are incident to backward arc, black and blue not)
A linear kernel for Fast

**Lemma**: If \((T, k)\) is a positive instance of \(k\)-FAST of size at least \(4k\) and without any irrelevant vertex, then \(\mathcal{P}\) is a safe \(\sigma\)-partition containing at least one external backward arc.

Claim: the **external** backward arcs of \(A_B\) can be externally certified.

- By König-Ergevary Lemma, there is a matching \(\mathcal{M}_E\) between \(A_B\) and \(F \setminus S_F\) saturating \(F \setminus S_F\).
- \(\mathcal{M}_E\) provides the external certificate
A linear kernel for fast

Theorem [P., Perez, Thomassé]
FAST parameterized by the solution size $k$ has a $4k$ vertex kernel.
A linear kernel for dense-RTI

Let \( R \) be a set of rooted triplets on \( L \).

Question: Can we remove at most \( k \) rooted triplets from \( R \) to obtain a consistent set of rooted triplets?
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Let $\mathcal{R}$ be a set of rooted triplets on $L$.

**Question**: Can we remove at most $k$ rooted triplets from $\mathcal{R}$ to obtain a consistent set of rooted triplets?

Or equiv. : can we find a binary tree hosting all but at most $k$ rooted triplets of $\mathcal{R}$?

- we need to define what is a conflict and a certificate
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Or equiv.: can we find a binary tree hosting all but at most $k$ rooted triplets of $\mathcal{R}$?

- we need to define what is a conflict and a certificate
- Instead of an ordering $\sigma$ (as in FAST), we seek for a tree
- instead of a $\sigma$-partition, we need to define a tree-partition
A linear kernel for dense-RTI

Let $\mathcal{R}$ be a dense set of rooted triplets on $L$. A conflict is a subset $L'$ of leaves such that $\mathcal{R}/L'$ is inconsistent. (i.e. no tree can host the rooted triplets of $\mathcal{R}/L'$)
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Lemma [Guillemot and Mnich'10]: A set $\mathcal{R}$ of rooted triplets is consistent

- iff $\mathcal{R}$ contains no conflict on four leaves
- iff $\mathcal{R}$ contains no conflict of the form

{ab|c, cd|b, bd|a} or {ab|c, cd|b, ad|b}
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→ it is enough to focus on subsets of four leaves

(we used triangles in FAST)
**A linear kernel for dense-RTI**

A tree-embedded instance $R_T = (\mathcal{R}, L, T)$ is formed by set of triplets $\mathcal{R}$ and a binary tree $T$ on leaf set $L$.

$$span_T(L') = \{ \ell \in L \mid \ell \text{ is a leaf of } T_{LCA(L')} \}$$
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A tree-embedded instance $R_T = (\mathcal{R}, L, T)$ is formed by set of triplets $\mathcal{R}$ and a binary tree $T$ on leaf set $L$.

$$\text{span}_T(L') = \{ \ell \in L \mid \ell \text{ is a leaf of } T_{LCA(L')} \}$$

Lemma: Let $R_T = (\mathcal{R}, L, T)$ be an embedded instance. Let $t = bc|a$ be the unique inconsistent rooted triplet in $\mathcal{R}/\{a, b, c, d\}$. Then

$$\{a, b, c, d\} \text{ is a conflict iff } d \in \text{span}(\{a, b, c\})$$
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If $d \in \text{span}(\{a, b, c\})$ does not belong to any inconsistent rooted triplet with $T$, then $c_T(t) = \{a, b, c, d\}$ is a certificate of $t$. 
A linear kernel for dense-RTI

\( \mathcal{P} = \{ T_1, \ldots T_r \} \) is a tree-partition of \( T \) if \( T \) contains a set of nodes (or leaves) \( x_1, \ldots x_r \) such that

- for every \( i \in [r] \), \( T_i \) is the subtree of \( T \) rooted at \( x_i \)
- \( L \) is partitioned by the sets of leaves of the \( T_i \)'s
A linear kernel for dense-RTI

\( \mathcal{P} = \{ T_1, \ldots, T_r \} \) is a **tree-partition** of \( T \) if \( T \) contains a set of nodes (or leaves) \( x_1, \ldots, x_r \) such that

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Observe that a tree-partition clearly distinguish the sets \( \mathcal{R}_E \) of external triplets and \( \mathcal{R}_I \) of internal triplets
A linear kernel for dense-RTI

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Observe that a tree-partition clearly distinguish the sets $\mathcal{R}_E$ of external triplets and $\mathcal{R}_I$ of internal triplets.

A tree-partition is safe if it is possible to externally certify the set of inconstant rooted triplets of $\mathcal{R}_E$. 
A linear kernel for dense-RTI

Rule 1 [irrelevant leaf] If a leaf $\ell \in L$ does not belong to any conflict, then remove $\ell$.

Rule 2 [Safe partition] Let $\mathcal{P}_T$ be a tree-partition of $R_T = (\mathcal{R}, L, T)$. Then edit every inconsistent rooted triplet of $\mathcal{R}_E$ with respect to $T$ and decrease $k$ accordingly.
A linear kernel for dense-RTI

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Rule 2 [Safe partition] Let \( P_T \) be a tree-partition of \( R_T = (\mathcal{R}, L, T) \). Then edit every inconsistent rooted triplet of \( \mathcal{R}_E \) with respect to \( T \) and decrease \( k \) accordingly.

- How to compute in polynomial time a safe tree-partition?

- Prove that these two rules yield a 5\( k \)-leaf kernel.
A linear kernel for dense-RTI

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Rule 2 [Safe partition] Let $P_T$ be a tree-partition of $R_T = (R, L, T)$. Then edit every inconsistent rooted triplet of $R_E$ with respect to $T$ and decrease $k$ accordingly.

- How to compute in polynomial time a safe tree-partition? Use a greedy conflict packing (as in FAST).

- Prove that these two rules yield a $5k$-leaf kernel. Use a matching argument to show that the external inconsistent triplets can be externally certified.
A linear kernel for dense-RTI

Assuming \((R = (R, L), k)\) is a YES-instance of size at least \(5k\):

- a conflict packing \(C\) covers at most \(4k\) leaves of \(L\)
- Compute in polynomial time a binary tree \(T\) such that every inconsistent rooted triplet \(t\) w.r.t. \(T\) satisfies \(L(t) \subseteq L(C)\).
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- a conflict packing \(C\) covers at most \(4k\) leaves of \(L\)
- Compute in polynomial time a binary tree \(T\) such that every inconsistent rooted triplet \(t\) w.r.t. \(T\) satisfies \(L(t) \subseteq L(C)\).
- Build the bipartite graph \(B\) between inconsistent rooted triplets and free leaves (non covered by \(C\)) such that every edge represents a conflict.
- Use a vertex cover of \(B\) to construct a safe tree-partition.
A linear kernel for dense-RTI

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- Build the bipartite graph \( B \) between inconsistent rooted triplets and free leaves (non covered by \( C \)) such that every edge represents a conflict.

- Use a vertex cover of \( B \) to construct a safe tree-partition.

**Theorem [P., Perez, Thomassé]**
Dense-RTI paramaterized by the solution size \( k \) has a \( 5k \) leaf kernel.
We described

- a linear kernel for Feedback Arc Set in Tournaments
- a linear kernel for Rooted Triplet Inconsistency

Conflict packing also yields:

- a linear kernel for Betweenness in Tournaments
- a quadratic kernel for Feedback Arc Set in Bipartite Tournaments

**Question:** parameterized local search for dense-RTI?

Thank you...