Oriented cliques and colorings of graphs with low maximum degree

Janusz Dybizbański a, Pascal Ochem b,1, Alexandre Pinlou b,a,∗,1, Andrzej Szepietowski a

a Institute of Informatics Faculty of Mathematics, Physics and Informatics, University of Gdańsk, 80-308 Gdańsk, Poland
b LIRMM, University of Montpellier, CNRS, Montpellier, France

A R T I C L E   I N F O

Article history:
Received 16 August 2019
Received in revised form 13 January 2020
Accepted 13 January 2020
Available online 30 January 2020

Keywords:
Oriented graph
Homomorphism
Oriented clique
Bounded degree graph

A B S T R A C T

An oriented clique, or oclique, is an oriented graph G such that its oriented chromatic number \( \chi_o(G) \) equals its order \(|V(G)|\). We disprove a conjecture of Duffy, MacGillivray, and Sopena [Oriented colourings of graphs with maximum degree three and four, Duffy et al. (2019) by showing that for maximum degree 4, the maximum order of an oclique is equal to 12. For maximum degree 5, we prove that the maximum order of an oclique is between 16 and 18. In the same paper, Duffy et al. also proved that the oriented chromatic number of connected oriented graphs with maximum degree 3 and 4 is at most 9 and 69, respectively. We improve these results by showing that the oriented chromatic number of non-necessarily connected oriented graphs with maximum degree 3 (resp. 4) is at most 9 (resp. 26). The bound of 26 actually follows from a general result which determines properties for a target graph to be universal for graphs of bounded maximum degree. This generalization also allows us to get the upper bound of 90 (resp. 306, 1322) for the oriented chromatic number of graphs with maximum degree 5 (resp. 6, 7).

© 2020 Elsevier B.V. All rights reserved.

1. Introduction

Oriented graphs are directed graphs with neither loops nor opposite arcs. Unless otherwise specified, the term graph refers to oriented graph in the sequel.

For a graph G, we denote by V(G) its set of vertices and by A(G) its set of arcs. For two adjacent vertices u and v, we denote by \( uv \) the arc from u to v, or simply uv whenever its orientation is not relevant (therefore, uv = uv or uv = vu).

Given two graphs G and H, a homomorphism from G to H is a mapping \( \phi : V(G) \rightarrow V(H) \) that preserves the arcs, that is, \( \phi(x)\phi(y) \in A(H) \) whenever \( xy \in A(G) \).

An oriented k-coloring of G can be defined as a homomorphism from G to H, where H is a graph with k vertices. The existence of such a homomorphism from G to H is denoted by \( G \rightarrow H \). The vertices of H are called colors, and we say that G is H-colorable. The oriented chromatic number of a graph G, denoted by \( \chi_o(G) \), is defined as the smallest number of vertices of a graph H such that \( G \rightarrow H \). If \( \mathcal{F} \) is a family of oriented graphs, then \( \chi_o(\mathcal{F}) \) denotes the maximum of \( \chi_o(G) \) over all \( G \in \mathcal{F} \).

∗ Corresponding author.
E-mail addresses: jdybiz@inf.ug.edu.pl (J. Dybizbański), Pascal.Ochem@lirmm.fr (P. Ochem), Alexandre.Pinlou@lirmm.fr (A. Pinlou), matszp@inf.ug.edu.pl (A. Szepietowski).

1 The second and the third author were partially supported by the French ANR grant HOSIGRA under contract ANR-17-CE40-0022.

https://doi.org/10.1016/j.disc.2020.111829
0012-365X/© 2020 Elsevier B.V. All rights reserved.
The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors and the problem of bounding the oriented chromatic number has been investigated for various graph classes: outerplanar graphs (with given minimum girth) [18,20], 2-outerplanar graphs [9,16], planar graphs (with given minimum girth) [1–4,16,17,19], graphs with bounded maximum average degree [3,4], graphs with bounded degree [7,11,23], graphs with bounded treewidth [15,20,21], Halin graphs [8], and graph subdivisions [25]. A survey on the study of oriented colorings has been done by Sopena in 2001 and recently updated [22].

For bounded degree graphs, Kostochka et al. [11] proved as a general bound that graphs with maximum degree $\Delta$ have oriented chromatic number at most $2\Delta^2/3$. They also showed that, for every $\Delta$, there exists graphs with maximum degree $\Delta$ and oriented chromatic number at least $2^{\Delta/2}$. For low maximum degrees, specific results are only known for graphs with maximum degree 3 and 4. Sopena [20] proved that graphs with maximum degree 3 have an oriented chromatic number at most 16 and conjectured that any such connected graphs have an oriented chromatic number at most 7. The upper bound was later improved by Sopena and Vignal [23] to 11. Recently, Duffy et al. [7] proved that 9 colors are enough for connected graphs with maximum degree 4. They proved in the same paper that connected graphs with maximum degree 4 have oriented chromatic number at most 69. Lower bounds are given by Duffy et al. [7] who exhibit a graph with maximum degree 3 (resp. 4) and oriented chromatic number 7 (resp. 11). Note that the above-mentioned conjecture of Sopena is thus best possible. In each of the above cases, the lower bound is achieved by presenting an oclique. An oclique, or oriented clique, is an oriented graph $G$ such that $\chi_o(G) = |V(G)|$.

**Theorem 1** ([10]). An oriented graph is an oclique if and only if any two vertices are connected by a directed path of length 1 or 2.

In their paper, Duffy et al. proved the following upper bound on maximum size of an oclique with maximum degree $\Delta$.

**Theorem 2** ([7]). Every oclique with maximum degree $\Delta$ has at most $\left\lfloor \frac{(\Delta+1)^2+1}{2} \right\rfloor$ vertices.

The theorem gives the upper bound 8 (resp. 13, 18) for $\Delta = 3$ (resp. $\Delta = 4$, $\Delta = 5$). They improved the above general result for $\Delta = 3$ by showing that the largest number of vertices in a subcubic oclique is 7. They also prove that there exists an oclique of size 11 with maximum degree $\Delta = 4$. Moreover they conjectured that the maximum order of an oclique with maximum degree $\Delta = 4$ is 11.

In this paper, we first improve the known upper bounds for graphs with low maximum degree. Second, we consider ocliques of maximum degree 4 and 5, and disprove the above-mentioned conjecture of Duffy et al. [7].

We prove in Section 4 that the oriented chromatic number of graphs with maximum degree 3 is at most 9 ($\chi_o(G_3) \leq 9$), that is, we remove the condition of connectivity; see Theorem 8. In Section 5, we prove a general result which determines properties of a target graph to be universal for (non-necessarily connected) graphs of maximum degree $\Delta \geq 4$; see Theorem 11. As a consequence of this general result, we obtain that the oriented chromatic number of graphs with maximum degree 4 is at most 26 ($\chi_o(G_4) \leq 26$), substantially decreasing the bound of 69 due to Duffy et al. [7]. We also get that $\chi_o(G_5) \leq 90$, $\chi_o(G_6) \leq 306$, and $\chi_o(G_7) \leq 1322$.

In Section 6, we disprove the conjecture of Duffy et al. [7] by showing that the maximum order of an oclique maximum degree 4 equals 12 ($\chi_o(G_4) = 12$). More precisely, we exhibit an oclique of order 12 and maximum degree 4, and show that there is no such oclique of order at least 13. Similarly in Section 7, we exhibit an oclique of order 16 and maximum degree 5 ($\chi_o(G_5) \geq 16$).

The next two sections will be devoted to define the notation and to present the properties of the target graphs we use to prove our upper bounds.

2. Notation

In the remainder of this paper, we use the following notions. For a vertex $v$ of a graph $G$, we denote by $N^+_G(v)$ the set of outgoing neighbors of $v$, by $N^-_G(v)$ the set of incoming neighbors of $v$ and by $N_G(v) = N^+_G(v) \cup N^-_G(v)$ the set of neighbors of $v$ (subscripts are omitted when the considered graph is clearly identified from the context). The degree of a vertex $v$ (resp. in-degree, out-degree), denoted by $d(v)$ (resp. $d^-(v)$, $d^+(v)$), is the number of its neighbors $|N(v)|$ (resp. incoming neighbors $|N^+(v)|$, outgoing neighbors $|N^-(v)|$). Let $G_\Delta$ denote the family of oriented graphs with maximum degree $\Delta$. If two graphs $G$ and $H$ are isomorphic, we denote this by $G \cong H$.

3. Paley tournaments and Tromp digraphs

In this section, we describe the general construction of graphs that will be used to prove Theorems 8 and 11, and present some of their useful properties.

For a prime power $p \equiv 3 \pmod{4}$, the Paley tournament $QR_p$ is defined as the graph whose vertices are the elements of the field $\mathbb{F}_p$ and such that $uv$ is an arc if and only if $u - v$ is a non-zero quadratic residue of $\mathbb{F}_p$. Clearly $QR_p$ is vertex- and arc-transitive.

An orientation $n$-vector is a sequence $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{-1, 1\}^n$ of $n$ elements. An ordered $n$-clique is a sequence $S = (v_1, v_2, \ldots, v_n)$ of $n$ vertices that induce an $n$-clique of a graph $G$. The vertex $u$ is said to be an $\alpha$-successor of $S$ if for
any $i, 1 \leq i \leq n$, we have $\vec{uv}_i \in A(G)$ whenever $\alpha_i = 1$ and $\vec{vu}_i \in A(G)$ otherwise. We say that a graph $G$ has Property $P_{n,k}$ if, for every ordered $n$-clique $S$ of $G$ and any orientation $n$-vector $\alpha$, there exist $k$ distinct $\alpha$-successors of $S$. Such Properties $P_{n,k}$ have been extensively used in many papers dealing with oriented coloring.

**Proposition 3.** The Paley tournament $QR_p$ has Properties $P_1, p^{-1}$ and $P_2, p^{-3}$.

**Proof.** By the vertex-transitivity of $QR_p$, the in-degree of every vertex is equal to its out-degree. This implies that $QR_p$ has Property $P_1, p^{-1}$.

Let us prove that $QR_p$ has Property $P_2, p^{-3}$. To do so, by arc-transitivity of $QR_p$, we just have to show that there exist at least $\frac{p - 3}{4} \alpha$-successors of the sequence $S = \{0, 1\}$ for any of the four orientation vector $\alpha \in \{-1, 1\}^2$.

We first need to count the transitive triangles with arcs $\vec{xy}, \vec{yz}$, and $\vec{zx}$ in $QR_p$. There are $p$ choices for the source vertex $x$ of a transitive triangle. The number of transitive triangles such that $x = 0$ is equal to the number of arcs in $N^+(0)$, that is, $(\frac{p - 1}{2}) = \frac{p - 1}{8}(p - 3)$. Thus, $QR_p$ contains $\frac{p(p - 3)}{8}$ transitive triangles.

Considering $\alpha = (+1, +1)$, we can notice that $\{N^+(0) \cap N^+(1)\}$ is the number of transitive triangles such that $\vec{xy} = \vec{01}$. Since $QR_p$ is arc-transitive and contains $\frac{p(p - 1)}{2}$ arcs, $\{N^+(0) \cap N^+(1)\} = \frac{p(p - 1)(p - 3)}{8} = \frac{p - 3}{4}$. Similarly for $\alpha = (-1, -1)$ and $\alpha = (+1, -1)$, considering $\vec{yz} = \vec{01}$ gives $\{N^+(0) \cap N^-(1)\} = \frac{p - 3}{4}$ and considering $\vec{xz} = \vec{01}$ gives $\{N^+(0) \cap N^-(1)\} = \frac{p - 3}{4}$.

Finally for $\alpha = (-1, +1)$, we have $N^-(0) \cap N^+(1) = V\left(QR_p\right) \setminus \{0, 1, N^+(0) \cap N^+(1), N^-(0) \cap N^+(1), N^+(0) \cap N^-(1), N^-(0) \cap N^-(1)\}$, so that $\{N^-(0) \cap N^+(1)\} = p - 2 - \frac{3(p - 3)}{4} = \frac{p + 1}{4} > \frac{p - 3}{4}$. This proves $P_2, p^{-3}$. \qed

Paley tournaments will be used as basic brick to build new graphs as explained below. Tromp (unpublished manuscript) proposed the following construction. Let $G$ be a graph and let $G' \cong G$. The Tromp graph $Tr(G)$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$
- $\forall u \in V(G) : \vec{uu}, \infty'u', \infty' \in A(Tr(G))$
- $\forall u, v \in V(G) : \vec{uv}, \vec{vu}, \vec{uv'}, \vec{v'u} \in A(Tr(G))$

Fig. 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between $u$ and $u'$. Such pairs of vertices will be called anti-twin vertices, and we denote by $at(u) = u'$ the anti-twin vertex of $u$.

In the following, we apply Tromp’s construction to Paley tournaments $QR_p$ which produces graphs with interesting structural properties. First of all, Marshall [12] proved that any $Tr(QR_p)$ is vertex-transitive and arc-transitive. He also prove that any $Tr(QR_p)$ is triangle-transitive, meaning that, given two triangles $u_1u_2u_3$ and $v_1v_2v_3$ of $Tr(QR_p)$ with the same orientation, there exists an automorphism that maps $u_i$ to $v_i$. Second, it is possible to derive Properties $P_{n,k}$ for $Tr(QR_p)$ knowing those of $QR_p$ (see Proposition 4). The authors already studied properties of $Tr(QR_1)$ (see [16, Proposition 5]) and their results can be easily generalized to $Tr(QR_p)$:

**Proposition 4 ([16]).** If $QR_p$ has Property $P_{n-1,k}$, then $Tr(QR_p)$ has Property $P_{n,k}$.

Let us now introduce another type of properties. We say that a graph $G$ has Property $C_{n,k}$ if for every ordered $n$-clique $v_1, v_2, \ldots, v_n$ of $G$, we have $|\bigcup_{1 \leq i \leq n} N^+(v_i)| \geq k$ and $|\bigcup_{1 \leq i \leq n} N^-(v_i)| \geq k$.

**Remark 5.** Given two integers $n$ and $k$, a graph having Property $C_{n,k}$ has Property $C_{n',k'}$ for any $n'$ and $k'$ such that $n' \geq n$ and $k' \leq k$. 

![Fig. 1. The Tromp graph Tr(G).](image-url)
Proposition 6. The graph $Tr(QR_p)$ has Properties $C_2$, $2p+1$ and $C_3$, $2p+2$.

Proof. Recall that $Tr(QR_p)$ is built from two copies of $QR_p$, that will be named $QR_p$ and $QR'_p$ in the following (see Fig. 1). In this proof, the in- and out-neighborhood of a vertex $v$ of $Tr(QR_p)$ will be denoted by $N^-(v)$ and $N^+(v)$. The in- and out-neighborhood of a vertex $v$ in a subgraph $H$ of $Tr(QR_p)$ will be denoted by $N^H_T(v)$ and $N^{H^+}_T(v)$.

Given an ordered $n$-clique $v_1, v_2, \ldots, v_n$ of $Tr(QR_p)$, if $z \in \bigcup_{i \in [n]} N^+(v_i)$, then $\phi(z) = z' \in \bigcup_{i \in [n]} N^-(v_i)$. Thus $|\bigcup_{i \in [n]} N^+(v_i)| = |\bigcup_{i \in [n]} N^-(v_i)|$.

- Let us first consider Property $C_2$, $2p+1$. We have to prove that given two adjacent vertices $x$ and $y$ of $Tr(QR_p)$, we have $|N^+(x) \cup N^+(y)| \geq 3p+1$. Since $x$ and $y$ are adjacent, w.l.o.g. $x = 0$ and $y = \infty$ by arc-transitivity of $Tr(QR_p)$. Then $N^+(0) \cup N^+(\infty)$ contains:
  - $N^+(\infty) = \{0, 1', \ldots, (p-1)\}$ ($p$ vertices);
  - Note that $N^+(0) = N^+_0(0) \cup N^+_1(0) \cup N^+_{\infty}(0)$ ($p+1$ vertices).

- Let us now consider Property $C_3$, $2p+2$. We have to prove that given three vertices $x$, $y$, and $z$ of $Tr(QR_p)$ that form a triangle, we have $|N^+(x) \cup N^+(y) \cup N^+(z)| \geq 7p+3$. We have to consider two cases depending on whether $x, y, z$ form a transitive triangle or $x, y, z$ form a directed triangle. By triangle-transitivity of $Tr(QR_p)$, it suffices to consider the cases $x, y, z = 0, 1, \infty$ (transitive triangle) and $x, y, z = 0, 1', \infty$ (directed triangle).

Case $x, y, z = 0, 1, \infty$: Let $A = N^+(1) \setminus \{N^+(0) \cup N^+(\infty)\}$. We clearly have $|N^+(0) \cup N^+(1) \cup N^+(\infty)| = |N^+(0) \cup N^+(\infty)| + |A|$. Since we already know that $|N^+(0) \cup N^+(\infty)| = \frac{3p+1}{2}$ (see the previous point), let us focus on the set $A$. We have $N^+(1) = N^+_0(1) \cup N^+_1(1) \cup N^+_{\infty}(1)$. Since $N^+_0(1) \subset N^+(\infty)$ and $\infty \in N^+(\infty)$, we have $A = N^+_0(1) \setminus \{N^+(0) \cup N^+(\infty)\}$. Since vertex $\infty$ has no out-neighbor in $QR_p$, we have $A = N^+_0(1) \setminus N^+(0)$, that corresponds to the set of out-neighbors of $1$ in $QR_p$ which are not out-neighbors of $0$. Since $QR_p$ is a tournament, the vertices which are not out-neighbors of a given vertex are the in-neighbors of this vertex. Therefore, the set $A$ corresponds to the set of out-neighbors of $1$ in $QR_p$ which are in-neighbors of $0$ and thus $A = N^+_0(1) \cap N^+(0)$. This set has already been considered in the proof of Proposition 3 where we showed that $|A| = \frac{p+1}{4}$. Therefore, the set $N^+(0) \cup N^+(1) \cup N^+(\infty)$ contains $|N^+(0) \cup N^+(\infty)| + |A| \geq \frac{3p+1}{2} + \frac{p+1}{4} = \frac{7p+3}{4}$ vertices.

Case $x, y, z = 0, 1', \infty$: Note that $N^+(1') = N^-(1)$. Thus $N^+(0) \cup N^+(1') \cup N^+(\infty) = N^+(0) \cup N^-(1) \cup N^+(\infty)$. Using the same kind of arguments as previous case, we get that $|N^+(0) \cup N^-(1) \cup N^+(\infty)| \geq \frac{7p+3}{4}$. □

4. Upper bound of the oriented chromatic number of graphs with maximum degree 3

In this section, we consider graphs with maximum degree 3 and we prove that they all admit a homomorphism to the same target graph on nine vertices.

Duffy et al. [7] proved that every connected graph with maximum degree 3 has an oriented chromatic number at most 9. To achieve this bound, they use the Paley tournament $QR_7$ which has vertex set $V(QR_7) = \{0, 1, \ldots, 6\}$ and $\vec{u}v \in A(QR_7)$ whenever $v - u \equiv r \pmod{7}$ for $r \in \{1, 2, 4\}$ (see Fig. 2(a)). Here is a quick sketch of their proof. They first consider 2-degenerated graphs with maximum degree 3 (not necessarily connected) and prove the following:

Theorem 7 ([7]). Every 2-degenerate graph with maximum degree 3 which does not contain a 3-source adjacent to a 3-sink is $QR_7$-colorable.

Then, given a connected graph $G$ with maximum degree 3, they first consider the case where $G$ contains a 3-source. By removing all 3-sources from $G$, we obtain a graph $G'$ that is $QR_7$-colorable by Theorem 7. It is then easy to put back all the 3-sources and color them with a new color 7. Then, subsequently, they consider the case where $G$ does not contain 3-sources. Removing any arc $\vec{u}v$ from $G$ leads to a graph $G'$ which admits a $QR_7$-coloring $\phi$ by Theorem 7. To extend $\phi$ to $G$, it suffices to recolor $u$ and $v$ with two new colors so that $\phi(u) = 7$ and $\phi(v) = 8$. This gives that $G$ has an oriented chromatic number at most 9.

The condition of connectivity of $G$ is a necessary condition in their proof. Indeed, given a graph $G$ with maximum degree 3 which is not connected, we need to remove one arc $\vec{u}v$ from each 3-regular component $C_i$ of $G$ to get a 2-degenerate graph $G'$ which is $QR_7$-colorable by Theorem 7. However, to extend the coloring to $G$ using two new colors, say color 7 for the $u_i$'s and color 8 for the $v_i$'s, the colorings of each component must agree on the neighbors of each $u_i$'s and on the neighbors of each $v_i$'s, which is not necessarily the case. This potentially leads to different target graphs on
nine vertices for each component. Therefore, even if each component has an oriented chromatic number at most 9, the whole graph may have an oriented chromatic number strictly greater than 9.

In the following, we prove this is not the case by showing that it is possible to color each component with the same target graph $T_9$ on nine vertices whose construction is described below. This implies that the condition of connectivity is no more needed.

The oriented graph $T_9$ is obtained from $QR_7$ (see Fig. 2(a)) by adding two vertices labeled 7 and 8, and the arcs $\vec{07}$, $\vec{17}$, $\vec{73}$, $\vec{78}$, and $\vec{8i}$ for every $0 \leq i \leq 6$ (see Fig. 2(b) where the gray part stands for $QR_7$).

We prove the following:

Theorem 8. Every graph with maximum degree 3 admits a $T_9$-coloring and thus $\chi_o(G_3) \leq 9$.

Proof. It is sufficient to show that every connected graph $G$ with maximum degree 3 admits a $T_9$-coloring. We consider the following cases.

- We suppose that $G$ is 2-degenerate or $G$ contains a 3-source. Let $G'$ be the oriented graph obtained from $G$ by removing every 3-source. Since $G'$ is 2-degenerate and contains no 3-source, $G'$ admits a $QR_7$-coloring $\phi$ by Theorem 7. We extend $\phi$ to a $T_9$-coloring of $G$ by setting $\phi(u) = 8$ for every 3-source $u$ of $G$ (indeed, the vertex 8 of $T_9$ dominates all the vertices of $QR_7$).

- We suppose that $G$ is 3-regular and contains no 3-source. Notice that $G$ necessarily contains a vertex $v$ of out-degree two. Let $u$ denote the in-neighbor of $v$. Since $u$ is not a 3-source, it has an in-neighbor $u_1$. Let $u_2$ denote the neighbor of $u$ distinct from $u_1$ and $v$ (see Fig. 3). We consider the graph $G'$ obtained from $G$ by removing the arc $uv$. Since $G'$ is 2-degenerate and contains no 3-source, $G'$ admits a $QR_7$-coloring $\phi$ by Theorem 7.

  - If $G$ (or equivalently $G'$) contains the arc $u_1v_2$, then we necessarily have $\phi(u_1) \neq \phi(u_2)$. If $\phi(u_1)\phi(u_2) \in A(QR_7)$ (resp. $\phi(u_2)\phi(u_1) \in A(QR_7)$), we recolor $G'$ so that $\phi(u_1) = 1$ (resp. $\phi(u_1) = 0$) and $\phi(u_2) = 3$ by the arc-transitivity of $QR_7$. It can be easily checked that we can extend $\phi$ to a $T_9$-coloring of $G$ by setting $\phi(u) = 7$ and $\phi(v) = 8$.

  - If $G$ contains $u_1u_2$, then by the arc-transitivity of $QR_7$, we can assume that $\{\phi(u_1), \phi(u_2)\} \subseteq \{0, 1\}$. Again, $\phi$ can be extended to a $T_9$-coloring of $G$ by setting $\phi(u) = 7$ and $\phi(v) = 8$. □

5. Upper bound of the oriented chromatic number of graphs with maximum degree at least 4

In this section, we consider graphs with maximum degree at least 4.

Duffy et al. [7] recently proved that every connected graph with maximum degree 4 has an oriented chromatic number at most 69. To prove their result, they first consider the case of 3-degenerate graphs with maximum degree 4 and prove
that they admit a homomorphism to the Paley tournament $QR_{67}$ on 67 vertices. Then they show how to extend such a 67-coloring to connected graphs with maximum degree 4 using two more colors, leading to a 69-coloring.

We propose a general result which determines properties of a target graph to be universal for graphs of maximum degree $\Delta \geq 4$. As for graphs with maximum degree 3 (see Section 4), the condition of connectivity is not needed. In particular, our general result substantially decreases the bound of 69 colors for graphs with maximum 4 due to Duffy et al. [7] to 26 colors.

**Lemma 9.** Let $Tr(QR_p)$ be a graph with Property $P_{n,k}$. Let $\psi$ be a $Tr(QR_p)$-coloring of a graph $G$. If $u \in V(G)$ has degree at most $n$, then there exists at least $k$ possible colors for $u$ that leave unchanged $\psi(v)$ for $v \neq u$. Moreover, these $k$ colors induce a clique of $Tr(QR_p)$.

**Proof.** Let us show that the proof can be reduced to the case where $\psi(N(u))$ induces a clique of $Tr(QR_p)$. Let $u$ denote the $i$th neighbor of $u$. If $\psi(u_i) = \psi(u_j)$ for some $i < j$, then $G$ contains either $\overline{u_i u_j}$ and $\overline{u_i u_j}$ or $\overline{u_i u_j}$ and $\overline{u_i u_j}$. Thus, the coloring constraints on $u$ due to the arcs $u_i u_j$ and $u_i u_j$ are the same. Similarly, if $\psi(u_i) = \psi(u_j)$, then w.l.o.g. $G$ contains $\overline{u_i u_j}$ and $\overline{u_i u_j}$. Thus, the coloring constraints on $u$ due to the arcs $\overline{u_i u_j}$ and $\overline{u_i u_j}$ are the same. In both cases, the coloring constraints on $u$ are unchanged by removing the arc $u_i u_j$.

So we can assume that $\psi(u_i) \notin \{\psi(u_j), \psi(u_j)\}$ for every $1 \leq i < j \leq d(u)$. Thus, $\psi(N(u))$ induces a clique of $Tr(QR_p)$ of size $d(u)$ and the result holds by definition of the Property $P_{n,k}$. □

**Theorem 10.** Let $\Delta \geq 3$. If $Tr(QR_p)$ has Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,n-\frac{n-1}{2}}$, where $n = |Tr(QR_p)| = 2p + 2$, then every $(\Delta - 1)$-degenerate graph with maximum degree $\Delta$ admits a $Tr(QR_p)$-coloring.

**Proof.** Let $G$ be a minimal counter-example to Theorem 10. By definition, $G$ contains a $k$-vertex $u$ with $k \leq \Delta - 1$. Let $v_1, v_2, \ldots, v_k$ be the neighbors of $u$.

Suppose first that the neighborhood of $u$ contains an arc and assume w.l.o.g. that this arc is $v_1 v_2$. The graph $G'$ obtained from $G$ by removing the arc $v_1 v_2$ admits a $Tr(QR_p)$-coloring $\phi$ by minimality of $G$. The degree of $v_1$ is at most $\Delta - 1$ in $G'$, so by Lemma 9 and Property $P_{\Delta-1,\Delta-2}$, there exists a set $S$ of $\Delta - 2$ available colors for $v_1$. Since $S$ is a clique, $S$ cannot contain both a color $c$ and $\phi(c)$. Thus, we can assign to $v_1$ a color in $S$ that is distinct from $\phi(v_1)$ and $\phi(v_1)$ for every $1 \leq i \leq k$ because there are at most $\Delta - 3$ values of $i$ and $|S| \geq \Delta - 2$. Note that we necessarily have $\{\phi(v_2), \phi(v_2)\} \cap S = \emptyset$ since $v_1 v_2$ is an arc of $G'$.

Now $\phi$ is such that $\phi(v_1) \notin \{\phi(v_1), \phi(v_1)\}$ for every $1 \leq i \leq k$. Using arguments along the lines of the proof of Lemma 9, $\phi$ can be extended to $G$ by recoloring $u$ using Property $P_{\Delta-1,\Delta-2}$.

Assume now that the neighborhood of $u$ contains no arc. The graph $G'$ obtained from $G$ by removing the vertex $u$ admits a $Tr(QR_p)$-coloring $\phi$ by minimality of $G$. The degree of $v_1$ in $G'$ is at most $\Delta - 1$. By Lemma 9 and Property $P_{\Delta-1,\Delta-2}$, we have $\Delta - 2$ available colors for $v_1$ and these $\Delta - 2$ colors induce a clique. By Property $C_{\Delta-2,n-\frac{n-1}{2}}$, given these $\Delta - 2$ possible colors for $v_1$, there are at least $n - \frac{n-1}{2}$ choices of colors for $u$. Thus $v_1$ forbids at most $\frac{n-1}{2}$ colors for $u$.

Since the neighborhood of $u$ contains no arc, the previous arguments hold for each $v_1$ independently. That is, every $v_i$ forbids at most $\frac{n-1}{2}$ colors for $u$. Therefore, at most $k \frac{n-1}{2} \leq n - 1$ colors are forbidden. So there exists at least one available color for $u$ and $\phi$ can be extended to a $Tr(QR_p)$-coloring of $G$, a contradiction. □

To achieve our bounds on oriented chromatic number, we construct the graph $Tr^*(QR_p)$ on $2p + 4$ vertices by adding two vertices $t_0$ and $t_1$ such that $t_0$ is a twin vertex of vertex 0 (i.e., a vertex with the same neighborhood as vertex 0) and $t_1$ is a twin vertex of vertex 1. We finally add the arc $t_0 t_1$.

**Theorem 11.** Every graph with maximum degree $\Delta \geq 4$ admits a $Tr^*(QR_p)$-coloring where $Tr(QR_p)$ is a Tromp graph with Properties $P_{\Delta-1,\Delta-2}$ and $C_{\Delta-2,2p+2-\frac{2p+1}{2}}$.

**Proof.** Since $Tr(QR_p)$ is a subgraph of $Tr^*(QR_p)$, it remains to prove that every connected $\Delta$-regular graph $H$ admits a $Tr^*(QR_p)$-coloring.

Let $H = H - \{u v\}$ where $u v$ is any arc of $H$. By Theorem 10, $H'$ admits a $Tr(QR_p)$-coloring $\phi$. By vertex-transitivity of $Tr(QR_p)$, we may assume that $\phi(v) = 0$. By Property $P_{\Delta-1,\Delta-2}$ of $Tr(QR_p)$, we have $\Delta - 2$ available colors $c_1, c_2, \ldots, c_{\Delta-2}$ for $u$. Note that, given $1 \leq i < j \leq \Delta - 2$, we necessarily have $c_i \neq \phi(c_j)$. We thus recolor $u$ with one of the $c_i$'s so that $\phi(u) \notin \{0, 0\}$ since $\Delta \geq 4$. Therefore, $\phi(u) \phi(v)$ is an arc of $Tr(QR_p)$.

If $\phi(u) \phi(v)$ is an arc of $Tr(QR_p)$, then $\phi$ is a $Tr(QR_p)$-coloring of $H$ and thus a $Tr^*(QR_p)$-coloring of $H$. Therefore, $\phi(v) \phi(v)$ is an arc of $Tr(QR_p)$. By arc-transitivity of $Tr(QR_p)$, we may assume that $\phi(u) = 1$. We recolor $u$ and $v$ so that $\phi(u) = t_1$ and $\phi(v) = t_0$. Since each $t_i$ is a twin vertex of vertex $i$ in $Tr^*(QR_p)$ and $t_0 t_1$ is an arc $Tr^*(QR_p)$, it is easy to verify that $\phi$ is now a $Tr^*(QR_p)$-coloring of $H$. □

The Properties $P_{n,k}$ of Paley tournaments $QR_n$ can be expressed by a formula for $n \leq 2$ (see Proposition 3). For higher values of $n$, there is no known formula. The Properties $P_{n,k}$ of $QR_p$ that we have computed for small values of $p$ and
the program that we used can be found at http://www.lirmm.fr/~ochem/target/. For our purpose, we are interested in
Properties of the form $P_{n,n}$.

**Proposition 12.** The smallest Paley tournament with Property $P_{2,2}$ (resp. $P_{3,3}$, $P_{4,4}$, $P_{5,5}$) is $QR_{11}$ (resp. $QR_{43}$, $QR_{151}$, $QR_{659}$).

Determining more of such properties would be possible with more computing power and time. Note that Paley
tournament $QR_{151}$ has in fact Property $P_{5,6}$ and there exist no smaller Paley tournament with Property $P_{5,5}$.

As a corollary of Propositions 4 and 12, we get the following.

**Corollary 13.** The Tromp graph $Tr(QR_{11})$ (resp. $Tr(QR_{43})$, $Tr(QR_{151})$, $Tr(QR_{659})$) has Property $P_{3,2}$ (resp. $P_{4,3}$, $P_{5,4}$, $P_{6,5}$).

As corollaries of Remark 5, Proposition 6, Theorem 11 and Corollary 13, we obtain the following four results. We give
a proof of the last one (Corollary 17), and the other three corollaries follow the same arguments.

**Corollary 14.** Every graph $G \in \mathcal{G}_4$ admits a $Tr^*(QR_{11})$-coloring. Thus, $\chi_0(\mathcal{G}_4) \leq 26$.

**Corollary 15.** Every graph $G \in \mathcal{G}_5$ admits a $Tr^*(QR_{43})$-coloring. Thus, $\chi_0(\mathcal{G}_5) \leq 90$.

**Corollary 16.** Every graph $G \in \mathcal{G}_6$ admits a $Tr^*(QR_{151})$-coloring. Thus, $\chi_0(\mathcal{G}_6) \leq 306$.

**Corollary 17.** Every graph $G \in \mathcal{G}_7$ admits a $Tr^*(QR_{659})$-coloring. Thus, $\chi_0(\mathcal{G}_7) \leq 1322$.

**Proof.** Let $\Delta = 7$. By Proposition 6, the graph $Tr(QR_{659})$ has Property $C_{5,110} = C_{4,1154}$. By Remark 5, it thus has Property $C_{5,1101} = C_{7,2p+1} = C_{4,1154}$. By Corollary 13, it also has Property $P_{6,5} = P_{\Delta-1,4-2}$. The graph $Tr(QR_{659})$ verifies the hypothesis of
Theorem 11, and thus every graph with maximum degree 7 admits a $Tr^*(QR_{659})$-coloring. \qed

**6. Oclique with maximum degree 4**

**Lemma 18.** There is an oclique of order 12 with maximum degree $\Delta = 4$.

**Proof.** Consider the oriented graph presented on Fig. 4. The graph consists of two directed cycles: $(v_0, v_1, v_2, v_3, v_4, v_5)$ and $(u_5, u_4, u_3, u_2, u_1, u_0)$. These two cycles are connected by the arcs $v_iu_i$ and $u_{i+3}mod 6$ for every $0 \leq i \leq 5$. The graph is vertex-transitive. By Theorem 1, it is to find a directed 2-path between $v_0$ and every vertex not adjacent to $v_0$: $v_0 \rightarrow v_1 \rightarrow v_2, v_0 \rightarrow u_0 \rightarrow v_3, v_4 \rightarrow v_5 \rightarrow v_0, v_0 \rightarrow v_1 \rightarrow u_1, u_2 \rightarrow v_5 \rightarrow v_0, u_4 \rightarrow u_3 \rightarrow v_0, u_0 \rightarrow u_0 \rightarrow u_5$. \qed

**Lemma 19.** Suppose that $G$ is an oclique of maximum degree $\Delta \geq 2$ and order

$$n = \left\lfloor \frac{(\Delta + 1)^2 + 1}{2} \right\rfloor.$$

Then:

- Every two vertices are connected by only one directed path of length at most 2.
- For every vertex $v$:
  - either $\deg_{in}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $\deg_{out}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$,
  - or $\deg_{out}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $\deg_{in}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$.
The maximum order of an oclique with maximum degree $\Delta = 4$ is 12.

---

**Proof.** The number of directed paths of length 1 is at most $\Delta \frac{n}{2}$. Each vertex $v \in V(G)$ is the midpoint of $deg_{in}(v) \cdot deg_{out}(v)$ directed paths of length 2, so we have
\[
\sum_{v \in V(G)} deg_{in}(v) \cdot deg_{out}(v)
\]
directed paths of length 2 in the graph.

First, suppose that $\Delta$ is even. Then $n = \frac{(\Delta + 1)^2 + 1}{2}$ and the sum $(\ast)$ achieves the maximum value of $\frac{\Delta^2 n}{4}$ if and only if $deg_{in}(v) = deg_{out}(v) = \frac{\Delta}{2}$ for every vertex $v$. Hence, we have at most $\frac{\Delta n}{2} + \frac{\Delta^2 n}{4}$ directed paths of length at most 2, and this maximal value is equal to $\binom{n}{2}$, that is the number of all pairs of vertices. Hence, if every pair of vertices is connected by a path of length 1 or two, then every vertex has in-degree and out-degree equal to $\frac{\Delta}{2}$, and every two vertices are connected by only one directed path of length at most 2.

If $\Delta$ is odd, then $n = \frac{(\Delta + 1)^2}{2}$ and the sum $(\ast)$ achieves the maximum value of $\frac{(\Delta^2 - 1) n}{4}$ if and only if for every vertex $v$,

- either $deg_{in}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $deg_{out}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$,
- or $deg_{out}(v) = \left\lfloor \frac{\Delta}{2} \right\rfloor$ and $deg_{in}(v) = \left\lceil \frac{\Delta}{2} \right\rceil$.

Hence, there are at most $\Delta \frac{n}{2} + \frac{(\Delta^2 - 1) n}{4}$ directed paths of length at most 2. Again, this maximal value is equal to $\binom{n}{2}$. □

**Corollary 20.** The underlying graph of oclique on $n = \left\lfloor \frac{(\Delta + 1)^2 + 1}{2} \right\rfloor$ vertices and maximum degree $\Delta$ is triangle free.

Now, we shall prove that there is no oclique with maximum degree $\Delta = 4$ and order $n = \left\lfloor \frac{(\Delta + 1)^2 + 1}{2} \right\rfloor = 13$. In [13] authors described a method to generate regular graphs of given girth (length of shortest cycle). Using this generator we can generate 4-regular graphs on 13 vertices with girth at least 4 (triangle-free).\(^2\) Only one of these graphs has diameter 2, namely, the graph $G^{13} = (\mathbb{Z}_{13}, E)$, where $(u, v) \in E \iff (v - u) \in \{-1, 1, -5, 5\}$, see Fig. 5. All arithmetic in $G^{13}$ is made modulo 13. Hence, only an orientation of $G^{13}$ can be an oclique with $\Delta = 4$ and $n = 13$.

**Lemma 21.** No orientation of $G^{13}$ is an oclique.

**Proof.** Suppose, for a contradiction, that there is an orientation of $G^{13}$ which is an oclique. By Lemma 19, every vertex of this orientation has in-degree and out-degree equal to 2, and every two vertices are connected by only one directed path of length at most 2. Vertex 0 is connected to vertex 2 only by one path of length at most 2, namely $0 \to 1 \to 2$. Without loss of generality assume that those edges are oriented: $0 \to 1 \to 2$. Furthermore, there is only one path going from 1 to 3, namely the one going through vertex 2. Hence, there is the arc $2 \to 3$ and so on, we show that there is the cycle $0 \to 1 \to 2 \to \cdots \to 12 \to 0$. Edges which are not oriented so far also form the cycle $0, 5, 10, 2, 7, 12, 4, 9, 1, 6, 11, 3, 8$.

Similarly as above also this cycle must be oriented in one direction. Hence, either every edge of the cycle goes from $x$ to $x + 5$, or every edges of the cycle goes from $x + 5$ to $x$. However, these two cases are isomorphic via the mapping $x \to 8 - x$, so we only consider the former orientation. Then we have two directed paths between 0 and 6, namely via 1 or via 5. A contradiction. □

Thus, we have proved the following theorem:

**Theorem 22.** The maximum order of an oclique with maximum degree $\Delta = 4$ is 12.

\(^2\) There are 31 such graphs which can be found at http://www.mathe2.uni-bayreuth.de/markus/reggraphs.html.
Corollary 23. \( \chi_o(G_4) \geq 12 \).

7. Oclique with maximum degree 5

Lemma 24. There is an oclique with maximum degree 5 and order 16.

Proof. Consider the graph presented in Fig. 6. The graph is constructed from the oriented 4-regular oclique \( G \) on 12 vertices presented in Fig. 4 by adding four vertices \( x_1, x_2, x_3, x_4 \). These vertices form the cycle \( x_1 \rightarrow x_2 \rightarrow x_3 \rightarrow x_2 \rightarrow x_1 \) and there are arcs:

\[ x_2 \rightarrow v_0, x_2 \rightarrow v_2, x_2 \rightarrow v_4, \]
\[ x_4 \rightarrow v_1, x_4 \rightarrow v_3, x_4 \rightarrow v_5, \]
\[ u_0 \rightarrow v_1, u_2 \rightarrow v_1, u_4 \rightarrow v_1, \]
\[ u_1 \rightarrow v_3, u_3 \rightarrow v_3, u_5 \rightarrow v_3. \]

We will show that any two vertices in \( G \) are connected by a path of length at most 2. Since new vertices form the oriented cycle \( C_4 \), there are paths between these vertices. It is easy to find all other directed paths. For example, paths connecting \( x_2 \) and all other vertices are as follows:

\[ x_2 \rightarrow v_1 \text{ for } i \in \{0, 2, 4\}, \]
\[ x_2 \rightarrow v_4 \rightarrow v_{i+1} \text{ mod } 6 \text{ for } i \in \{1, 3, 5\}, \]
\[ v_1 \rightarrow u_i \rightarrow v_1 \text{ for } i \in \{0, 2, 4\}, \]
\[ u_i \rightarrow x_2 \rightarrow x_2 \text{ for } i \in \{1, 3, 5\}. \]

From Lemma 24 and Theorem 2 we have the following:

Corollary 25. The maximum order of oclique with maximum degree 5 is between 16 and 18.

Corollary 26. \( \chi_o(G_5) \geq 16 \).

8. Discussion

Our attempts to construct an oclique based on the construction of the oriented 4-regular oclique on 12 vertices (see Fig. 4) lead to a linear estimation in terms of \( \Delta \). However, we can obtain a lower bound confirming that the maximum order of oclique is quadratic.

Lemma 27. For every \( \Delta \), there exists an oclique of maximum degree \( \Delta \) and order \( \frac{\Delta^2}{7} + O(\Delta) \).
Proof. The postage stamp problem\(^3\) asks, given integers \(h\) and \(k\), for a set of integers \(V = \{v_1, v_2, \ldots, v_l\}\) that maximizes the smallest integer \(PSP(h, k)\) that cannot be written as the sum of at most \(h\) (not necessarily distinct) elements of \(V\). In the context of cliques, we are interested in the case of \(h = 2\) stamps, which corresponds to directed paths of length at most 2. Using a set \(V\) corresponding to the value of \(PSP(2, k)\), we construct the circulant oriented graph \(G_h\) on \(n = 2PSP(2, k) - 1\) vertices \(g_0, \ldots, g_{n-1}\) and the arcs \(g_i \rightarrow g_{i+j}\) such that \(0 \leq i < n\), \(1 \leq j \leq k\), and indices are taken modulo \(n\). By the properties of \(V\), there exists a directed path of length at most 2 in \(G_h\) from \(g_0\) to every vertex \(g_i\) with \(1 \leq i \leq PSP(2, k) - 1\). Now since \(G_h\) is circular, \(G_h\) is an oclique. Also, \(G_h\) is \(\Delta\)-regular with \(\Delta = 2k\). We use the bound \(PSP(2, k) \geq \frac{2}{n}k^2 + O(k)\) \(^{24}\) to obtain \(|V(G_h)| = 2PSP(2, k) - 1 \geq 2 \times \frac{2}{n}k^2 + O(k) = \frac{k^2}{n} + O(\Delta)\).

Finally, Duffy proves in a recent preprint \(^6\) that the oriented chromatic number of a connected cubic graph is at most 8.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgment

We thank Kevin P. Costello for telling us that the key notion in Lemma 27 is called the postage stamp problem in the literature.

References


\(^6\) J. Dybizbański, P. Ochem, A. Pinlou et al. / Discrete Mathematics 343 (2020) 111829


\(^13\) M. Meringer, Fast generation of regular graphs and construction of cages, J. Graph Theory 30 (2) (1999) 137–146.


\(^16\) P. Ochem, A. Pinlou, Oriented coloring of triangle-free planar graphs and 2-outplanar graphs, Graphs Combin. 30 (2) (2014) 439–453.


\(^20\) É. Sopena, The chromatic number of oriented graphs, J. Graph Theory 25 (1997) 191–205.


\(^3\) https://en.wikipedia.org/wiki/Postage_stamp_problem.