THE DOMINATION NUMBER OF GRIDS

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Abstract. In this paper, we conclude the calculation of the domination number of all \((n, m)\) grid graphs. Indeed, we prove Chang’s conjecture saying that for every \(16 \leq n \leq m\), \(\gamma(G_{n,m}) = \left\lceil \frac{(n+2)(m+2)}{5} \right\rceil - 4\).

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1. Introduction. A dominating set in a graph \(G\) is a subset of vertices \(S\) such that every vertex in \(V(G) \setminus S\) is a neighbor of some vertex of \(S\). The domination number of \(G\) is the minimum size of a dominating set of \(G\); we denote it by \(\gamma(G)\). This paper is devoted to the calculation of the domination number of complete grids.

The notation \(i\) denotes the set \(\{1, 2, \ldots, i\}\). If \(w\) is a word in the alphabet \(A\), \(w[i]\) is the \(i\)th letter of \(w\), and for every \(a\) in \(A\), \(|w[a]|\) denotes the number of occurrences of \(a\) in \(w\) (i.e., \(|\{i \in \{1, \ldots, |w|\} : w[i] = a\}|\)). For a vertex \(v\), \(N[v]\) denotes the closed neighborhood of \(v\) (i.e., the set of neighbors of \(v\) and \(v\) itself). For a subset of vertices \(S\) of a vertex set \(V\) of a graph, we denote by \(N[S] = \bigcup_{v \in S} N[v]\). Note that \(D\) is a dominating set of \(G\) if and only if \(N[D] = V(G)\). Let \(G_{n,m}\) be the complete \((n, m)\) grid; i.e., the vertex set of \(G_{n,m}\) is \(V_{n,m} := [n] \times [m]\), and two vertices \((i, j)\) and \((k, l)\) are adjacent if \(|k-i| + |l-j| = 1\). The couple \((1, 1)\) denotes the bottom-left vertex of the grid and the couple \((i, j)\) denotes the vertex of the \(i\)th column and the \(j\)th row. We will always assume in this paper that \(n \leq m\). Let us illustrate our purpose by an example of a dominating set of the complete grid \(G_{24,24}\) on Figure 1.

The first results on the domination number of grids were obtained about 30 years ago with the exact values of \(\gamma(G_2,n)\), \(\gamma(G_{3,n})\), and \(\gamma(G_{4,n})\) found by Jacobson and Kinch [8] in 1983. In 1993, Chang and Clark [3] found those of \(\gamma(G_{5,n})\) and \(\gamma(G_{6,n})\). These results were obtained analytically. Chang [2] devoted his Ph.D. thesis to study the domination number of grids; he conjectured that this invariant behaves well provided that \(n\) is large enough. Specifically, Chang conjectured the following.

Conjecture 1 (Chang [2]). For every \(16 \leq n \leq m\),

\[\gamma(G_{n,m}) = \left\lceil \frac{(n+2)(m+2)}{5} \right\rceil - 4.\]
Observe that, for instance, this formula would give 131 for the domination number of the grid in Figure 1. To motivate his bound, Chang proposed some constructions of dominating sets achieving the upper bound.

**Lemma 1** (Chang [2]). *For every* $8 \leq n \leq m$,* 

$$\gamma(G_{n,m}) \leq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4.$$ 

In the meantime, some algorithms based on dynamic programming were designed to compute a lower bound of $\gamma(G_{n,m})$. There were numerous intermediate results found for $\gamma(G_{n,m})$ for small values of $n$ and $m$ (see [4, 9, 10] for details). In 1986, Hare, Hedetniemi, and Hare [9] gave a polynomial time algorithm to compute $\gamma(G_{n,m})$ when $n$ is fixed. Nevertheless, this algorithm is not usable in practice when $n$ is over 20. Fisher [6] developed the idea of searching for periodicity in the dynamic programming algorithms, and using this technique, he found the exact values of $\gamma(G_{n,m})$ for all $n \leq 21$. We recall these values for the sake of completeness.

**Theorem 2** (Fisher [6]). *For all* $n \leq m$ *and* $n \leq 21$, *we have*
\[
\gamma(G_{n,m}) = \begin{cases} 
\frac{m}{3} & \text{if } n = 1 \\
\frac{m+1}{2} & \text{if } n = 2 \\
\frac{3m+1}{4} & \text{if } n = 3 \\
m+1 & \text{if } n = 4 \text{ and } m = 5, 6, 9 \\
m & \text{if } n = 4 \text{ and } m \not= 5, 6, 9 \\
\frac{6m+4}{5} - 1 & \text{if } n = 5 \text{ and } m = 7 \\
\frac{6m+4}{5} & \text{if } n = 5 \text{ and } m \not= 7 \\
\frac{10m+4}{7} & \text{if } n = 6 \\
\frac{5m+1}{3} & \text{if } n = 7 \\
\frac{15m+7}{8} & \text{if } n = 8 \\
\frac{23m+10}{11} - 1 & \text{if } n = 9 \\
\frac{30m+15}{13} & \text{if } n = 10 \text{ and } m \equiv_{13} 10 \text{ or } m = 13, 16 \\
\frac{30m+15}{13} & \text{if } n = 10 \text{ and } m \not\equiv_{13} 10 \text{ and } m \not= 13, 16 \\
\frac{38m+22}{15} - 1 & \text{if } n = 11 \text{ and } m = 11, 18, 20, 22, 33 \\
\frac{38m+22}{15} & \text{if } n = 11 \text{ and } m \not= 11, 18, 20, 22, 33 \\
\frac{80m+38}{29} & \text{if } n = 12 \\
\frac{98m+54}{33} - 1 & \text{if } n = 13 \text{ and } m \equiv_{33} 13, 16, 18, 19 \\
\frac{98m+54}{33} & \text{if } n = 13 \text{ and } m \not\equiv_{33} 13, 16, 18, 19 \\
\frac{35m+20}{11} - 1 & \text{if } n = 14 \text{ and } m \equiv_{22} 7 \\
\frac{35m+20}{11} & \text{if } n = 14 \text{ and } m \not\equiv_{22} 7 \\
\frac{44m+28}{13} - 1 & \text{if } n = 15 \text{ and } m \equiv_{26} 5 \\
\frac{44m+28}{13} & \text{if } n = 15 \text{ and } m \not\equiv_{26} 5 \\
\frac{(n+2)(m+2)}{5} - 4 & \text{if } n \geq 16 
\end{cases}
\]

Note that these values are obtained by specific formulas for every \(1 \leq n \leq 15\) and by the formula of Conjecture 1 for every \(16 \leq n \leq 21\). This proves Chang’s conjecture for all values \(16 \leq n \leq 21\).
In 2004, Conjecture 1 was confirmed up to an additive constant.

**Theorem 3 (Guichard [7]).** For every $16 \leq n \leq m$,

$$
\gamma(G_{n,m}) \geq \left\lceil \frac{(n+2)(m+2)}{5} \right\rceil - 9.
$$

In this paper, we prove Chang’s conjecture, hence finishing the computation of $\gamma(G_{n,m})$. We adapt Guichard’s ideas to improve the additive constant from $-9$ to $-4$ when $24 \leq n \leq m$. Cases $n = 22$ and $n = 23$ can be proved in a couple of hours using Fisher’s method (described in [6]) on a modern computer. They can also be proved by a slight improvement of the technique presented in the next section.

## 2. Values of $\gamma(G_{n,m})$ when $24 \leq n \leq m$. Our method follows the idea of Guichard [7]. A slight improvement is enough to give the exact bound.

A vertex of the grid $G_{n,m}$ dominates at most 5 vertices (its 4 neighbors and itself). It is then clear that $\gamma(G_{n,m}) \geq \frac{n \cdot m}{5}$. The previous inequality would become an equality if there would be a dominating set $D$ such that every vertex of $G_{n,m}$ is dominated only once, and all vertices of $D$ are of degree 4 (i.e., dominates exactly 5 vertices); in this case, we would have $5 \cdot |D| - n \cdot m = 0$. This is clearly impossible (e.g., to dominate the corners of the grid, we need vertices of degree at most 3). Therefore, our goal is to find a dominating set $D$ of $G_{n,m}$ such that the difference $5 \cdot |D| - n \cdot m$ is the smallest.

Let $S$ be a subset of $V(G_{n,m})$. The loss of $S$ is $\ell(S) = 5 \cdot |S| - |N[S]|$.

**Proposition 4.** The following properties of the loss function are straightforward:

(i) For every $S$, $\ell(S) \geq 0$ (positivity).
(ii) If $S_1 \cap S_2 = \emptyset$, then $\ell(S_1 \cup S_2) = \ell(S_1) + \ell(S_2) + |N[S_1] \cap N[S_2]|$.
(iii) If $S' \subseteq S$, then $\ell(S') \leq \ell(S)$ (increasing function).
(iv) If $S_1 \cap S_2 = \emptyset$, then $\ell(S_1 \cup S_2) \geq \ell(S_1) + \ell(S_2)$ (superadditivity).

Let us denote by $\ell_{n,m}$ the minimum of $\ell(D)$ when $D$ is a dominating set of $G_{n,m}$.

**Lemma 5.** $\gamma(G_{n,m}) = \frac{n \cdot m - \ell_{n,m}}{5}$.

**Proof.** If $D$ is a dominating set of $G_{n,m}$, then $\ell(D) = 5 \cdot |D| - |N[D]| = 5 \cdot |D| - n \cdot m$. Hence, by minimality of $\ell_{n,m}$ and $\gamma(G_{n,m})$, we have $\ell_{n,m} = 5 \cdot \gamma(G_{n,m}) - n \cdot m$. Our aim is to get a lower bound for $\ell_{n,m}$. As the reader can observe in Figure 1, the loss is concentrated on the border of the grid. We now analyze more carefully the loss generated by the border of thickness 10.

We define the border $B_{n,m} \subseteq V_{n,m}$ of $G_{n,m}$ as the set of vertices $(i, j)$ where $i \leq 10$, or $j \leq 10$, or $i > n - 10$, or $j > m - 10$. Given a subset $S \subseteq V$, let $I(S)$ be the internal vertices of $S$, i.e., $I(S) = \{v \in S : N[v] \subseteq S\}$. These sets are illustrated in Figure 2. We will compute $b_{n,m} = \min_D \ell(D)$, where $D$ is a subset of $B_{n,m}$ and dominates $I(B_{n,m})$, i.e., $D \subseteq B_{n,m}$ and $I(B_{n,m}) \subseteq N[D]$. Observe that this lower bound $b_{n,m}$ is a lower bound of $\ell_{n,m}$. Indeed, for every dominating set $D$ of $G_{n,m}$, the set $D' = D \cap B_{n,m}$ dominates $I(B_{n,m})$; hence $b_{n,m} \leq \ell(D') \leq \ell(D)$. In the remainder, we will compute $b_{n,m}$, and we will show that $b_{n,m} = \ell_{n,m}$.

In the following, we split the border $B_{n,m}$ into four parts, $O_{m-12}$, $P_{n-12}$, $Q_{m-12}$, $R_{n-12}$, which are defined just below.

For $p \geq 12$, let $P_p \subseteq B_{n,m}$ defined as follows: $P_p = ([10] \times \{12\}) \cup ([11] \times \{11\}) \cup ([p] \times [10])$. We define the input vertices of $P_p$ as $[10] \times \{12\}$ and the output vertices of $P_p$ as $[p] \times [10]$. The set $P_{19}$, illustrated for $p = 19$ in Figure 3, corresponds to the set of black and gray vertices. The input vertices are the gray circles, and the output vertices are the gray squares. Recall that in our drawing conventions, the vertex $(1, 1)$ is the
bottom-left vertex, and hence the vertex \((i, j)\) is in the \(i\)th column from the left and in the \(j\)th row from the bottom.

For \(n, m \in \mathbb{N}^*\), let \(f_{n,m} : [n] \times [m] \to [m] \times [n] \) be the bijection such that \(f_{n,m}(i, j) = (j, n - i + 1)\). It is clear that the set \(B_{n,m}\) is the disjoint union of the following four sets depicted in Figure 4: \(P_{n-12}\), \(Q_{m-12} = f_{n,m}(P_{m-12})\), \(R_{n-12} = f_{m,n} \circ f_{n,m}(P_{n-12})\), and \(O_{m-12} = f_{n,m}^{-1}(P_{m-12})\). Similar to \(P_{n-12}\), the sets \(O_{n-12}\), \(Q_{n-12}\), and \(R_{m-12}\) have input
and output vertices. For instance, the output vertices of $Q_{m-12}$ correspond in Figure 3 to the white squares. Every set playing a symmetric role, we now focus on $P_{n-12}$.

Given a subset $S$ of $V_{n,m}$, let the labeling $\phi_S: V_{n,m} \rightarrow \{0, 1, 2\}$ be such that

$$\phi_S(i, j) = \begin{cases} 
0 & \text{if } (i, j) \in S, \\
1 & \text{if } (i, j) \in N[S] \setminus S, \\
2 & \text{otherwise}.
\end{cases}$$

Note that $\phi_S$ is such that any two adjacent vertices in $G_{n,m}$ cannot be labeled 0 and 2.

Given $p \geq 12$ and a set $S \subseteq P_p$, the input word (respectively, output word) of $S$ for $P_p$, denoted by $w^\text{in}(S)$ (respectively, $w^\text{out}_p(S)$), is the 10-letter word in the alphabet $\{0, 1, 2\}$ obtained by reading $\phi_S$ on the input vertices (respectively, output vertices) of $P_p$. More precisely, its ith letter is $\phi_S(i, 12)$ (respectively, $\phi_S(p, i)$). This notion extends to $O_p$, $Q_p$, and $R_p$ in a natural way: For example, the output word of $S \subseteq O_p$ for $O_p$ is $w^\text{out}_p(f_{n,m}(S))$.

According to the definition of $\phi$, the input and output words belong to the set $W$ of 10-letter words in $\{0, 1, 2\}$, which avoid 02 and 20. The number of $k$-digit trinary numbers without 02 or 20 is given by the following formula [6]:

$$\frac{(1 + \sqrt{2})^{k+1} + (1 - \sqrt{2})^{k+1}}{2}.$$

(1)

The size of $W$ is therefore $|W| = 8119$. 

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FIG. 4. The sets $O_{m-12}$, $P_{n-12}$, $Q_{m-12}$, and $R_{n-12}$. 

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Given two words $w, w' \in W$, we define $D_p^{w, w'}$ as the family of subsets $D$ of $P_p$ such that
- $D$ dominates $I(P_p)$,
- $w$ is the input word $w^{\text{in}}(D)$,
- $w'$ is the output word $w^{\text{out}}(D)$.

To complete our calculation, we will need to know, for two given words $w, w' \in W$, the minimum loss over all losses $\ell(D)$, where $D \in D_p^{w, w'}$. For this purpose, we introduce the $(8119, 8119)$ square matrix $C_p$. For $w, w' \in W$, let $C_p[w, w'] = \min_{D \in D_p^{w, w'}} \ell(D)$, where the minimum of the empty set is $+\infty$.

Let $w, w' \in W$ be two given words. Because of the symmetry of $P_{12}$ with respect to the first diagonal (bottom-left to top-right) of the grid, if a vertex set $D$ belongs to $D_p^{w, w'}$, then $D' = \{(j, i) | (i, j) \in D\}$ belongs to $D_p^{w', w}$. Moreover, it is clear that, again because of the symmetry, $\ell(D) = \ell(D')$. Therefore, we have $C_{12}[w, w'] = C_{12}[w', w]$, and thus $C_{12}$ is a symmetric matrix. Despite the size of $C_{12}$ and the size of $P_{12}$ (141 vertices), it is possible to compute $C_{12}$ in less than one hour by computer. Indeed, we choose a sequence of two consecutive parts of the border, say, $\{0, 1, 2\}$, the minimal loss is easily deduced from $C_p$. Given two words $w, w'$, we define for every labeling $f$ of $P_{n;m}$, the output word of $w$ is $w^f = \{f(x, y) | (x, y) \in P_{n;m}\}$, where the order is the lexical order. For every labeling $f$, the minimal loss $\ell(f) = \min_{D \in D_p^{w, w'}} \ell(D)$. Moreover, it is clear that, again because of the symmetry, $\ell(f) = \ell(f')$. For this purpose, we introduce $C_p[w, w'] = \min_{D \in D_p^{w, w'}} \ell(D)$, where the minimum of the empty set is $+\infty$.

Given two words $w, w' \in W$, let $\ell(w, w') = |\{i \in [10] : w[i] \neq 2 \text{ and } w'[i] = 0\}| + |\{i \in [10] : w'[i] \neq 2 \text{ and } w[i] = 0\}|$.

**Lemma 6.** Let $D$ be a dominating set of $G_{n,m}$, and let us denote $D_p = D \cap P_{n;12}$ and $D_Q = D \cap Q_{n;12}$. Then $\ell(D \cap (P_{n;12} \cup Q_{n;12})) = \ell(D_p) + \ell(D_Q) + \ell(w, w')$, where $w = w^{\text{in}}(D_p)$ and $w' = w^{\text{out}}(f_{n,m}(D_Q))$. Moreover, $w$ and $w'$ are compatible.

**Proof.** By Proposition 4(ii), $\ell(D \cap (P_{n;12} \cup Q_{n;12})) = \ell(D_p) + \ell(D_Q) + |N[D_p] \cap N[D_Q]|$. It suffices then to note that $\ell(w, w') = |N[D_p] \cap N[D_Q]|$ to get $\ell(D \cap (P_{n;12} \cup Q_{n;12})) = \ell(D_p) + \ell(D_Q) + \ell(w, w')$.

In what remains, we prove that $w$ and $w'$ are compatible. If those two words were not compatible, there would exist an index $i \in [9]$ such that $w^{\text{out}}(f_{n,m}(D_Q))[i] + w^{\text{in}}(D_p)[i] > 2$. Thus at least one of these two letters should be a 2, and the other one should not be 0.

Suppose that $w^{\text{out}}(f_{n,m}(D_Q))[i] = 2$, and note that this means that the vertex $(i, 13)$ is not dominated by a vertex in $D_Q$. Since $D$ is a dominating set of $G_{n,m}$, every
Note that its unique neighbor in \( Q_{m-12} \) is the output word of \( w \). Let \( w_P \) be the input word of \( \Delta P \), where for all \( \pi \), \( \pi \in [2,0] \) and \( \pi \in [2,0] \), the value \( \ell(w, w') \):

\[
L[w, w'] = \begin{cases} 
+\infty & \text{if } w \text{ and } w' \text{ are not compatible,} \\
\ell(w, w') & \text{otherwise.}
\end{cases}
\]

Thus, \( b_{n,m} \) is symmetric since \( \ell(w, w') = \ell(w', w) \).

Let \( \otimes \) be matrix multiplication in \((\min, +)\) algebra; i.e., \( C = A \otimes B \) is the matrix where for all \( i, j \), \( C[i,j] = \min_a A[i,k] + B[k,j] \).

Let \( M_p = L \otimes C_p \) for \( p \geq 12 \).

By construction, \( M_{n-12}[w, w'] \) corresponds to the minimum possible loss \( \ell(D \cap P_{n-12}) \) of a dominating set \( D \subseteq V_{n,m} \) that dominates \( I(P_{n-12}) \) and such that \( w \) is the output word of \( Q_{m-12} \) and \( w' \) is the output word of \( P_{n-12} \).

**Lemma 7.** For all \( 24 \leq n \leq m \), we have

\[
b_{n,m} \geq \min_{w_1, w_2, w_3, w_4 \in W} M_{n-12}[w_1, w_2] + M_{m-12}[w_2, w_3] + M_{n-12}[w_3, w_4] + M_{m-12}[w_4, w_1].
\]

**Proof.** Consider a set \( D \subseteq B_{n,m} \) that dominates \( I(B_{n,m}) \) and achieves \( \ell(D) = b_{n,m} \).

Let \( D_P = D \cap P_{n-12}, D_Q = D \cap Q_{m-12}, D_R = D \cap R_{m-12}, \) and \( D_O = D \cap O_{m-12} \). Let \( w_P (w_Q, w_R, \) and \( w_O \), respectively) be the input word of \( P_{n-12} (Q_{m-12}, R_{m-12}, \) and \( O_{m-12} \). By definition of \( C_p \), the loss of \( D_P \) is at least \( C_{n-12}[w_P, w'_P] \). Similarly, we have \( \ell(D_Q) \geq C_{m-12}[w_Q, w'_Q], \ell(D_R) \geq C_{n-12}[w_R, w'_R], \) and \( \ell(D_O) \geq C_{m-12}[w_O, w'_O] \). By the definition of the loss,

\[
\ell(D) = b_{n,m} = 5 \cdot |D| - |N[D]| = \ell(D_O) + \ell(D_P) + \ell(D_Q) + \ell(D_R) + L[w'_O, w_P] + L[w'_P, w_Q] + L[w'_Q, w_R] + L[w'_R, w_O]
\]

by Lemma 6 and since \( N[D_P] \cap N[D_R] = N[D_Q] \cap N[D_O] = \emptyset \)

\(
\geq C_{m-12}[w_O, w'_O] + C_{n-12}[w_P, w'_P] + C_{m-12}[w_Q, w'_Q] + C_{n-12}[w_R, w'_R] + L[w'_O, w_P] + L[w'_P, w_Q] + L[w'_Q, w_R] + L[w'_R, w_O]
\)

\(
\geq M_{m-12}[w_O, w'_O] + M_{n-12}[w_P, w'_P] + M_{m-12}[w_Q, w'_Q] + M_{n-12}[w_R, w'_R] + M_{n-12}[w_O, w'_O]
\)

since \( w'_O \) and \( w_P \) (respectively, \( w'_P \) and \( w_Q, w'_Q \) and \( w_R, w'_R \) and \( w_O \)) are compatibles. \( \square \)
According to Lemma 7, to bound $b_{n,m}$ it would be thus interesting to know $M_p$ for $p > 12$. It is why we introduce the following $(8119, 8119)$ square matrix $T$.

**Lemma 8.** There exists a matrix $T$ such that $C_{p+1} = C_p \otimes T$ for all $p \geq 12$. This matrix is defined as follows:

$$T[w, w'] = \begin{cases} 
+\infty & \text{if } \exists i \in [10] \text{ s.t. } w[i] = 0 \text{ and } w'[i] = 2 \\
+\infty & \text{if } \exists i \in [9] \text{ s.t. } w[i] = 2 \text{ and } w'[i] \neq 0 \\
+\infty & \text{if } \exists i \in \{2, \ldots, 9\} \text{ s.t. } w'[i] = 1, w[i] \neq 0, w'[i - 1] \neq 0 \\
& \text{and } w'[i + 1] \neq 0 \\
+\infty & \text{if } w'[1] = 1, w[1] \neq 0 \text{ and } w'[2] \neq 0 \\
+\infty & \text{if } w'[10] = 1, w[10] \neq 0 \text{ and } w'[9] \neq 0 \\
3 \cdot |w'|_0 - |w|_2 - |w'|_1 + |w|_0 - 1 & \text{if } w'[10] = 0 \\
3 \cdot |w'|_0 - |w|_2 - |w'|_1 + |w|_0 & \text{otherwise},
\end{cases}$$

Proof. Consider a set $S' \subseteq P_{p+1}$ dominating $I(P_{p+1})$, and let $S = S' \cap P_p$. Let $w = w_{p+1}^\text{out}(S)$ and $w' = w_{p+1}^\text{out}(S')$. Let $\Delta(S, S') = \ell(S') - \ell(S)$. By definition of the loss, $\Delta(S, S') = 5 \cdot |S'| \cdot |S| - |N(S') \cap N(S)|$. Let us compute $\Delta(S, S')$ in terms of the number of occurrences of 0's, 1's, and 2's in the words $w$ and $w'$. The set $S' \setminus S$ corresponds to the vertices $\{(p + 1, i) | i \in [10], w'[i] = 0\}$. The set $N(S') \cap N(S)$ corresponds to the vertices dominated by $S'$ but not dominated by $S$; these vertices clearly belong to the columns $p$, $p + 1$, and $p + 2$. Since $S'$ dominates $I(P_{p+1})$, those in the column $p$ are the vertices $\{(p, i) | i \in [10], w[i] = 2\}$. Those in the column $p + 1$ are the vertices $\{(p + 1, i) | i \in [10], w'[i] \neq 2, w[i] \neq 0\}$ and possibly the vertex $(p + 1, 11)$ when $w'[10] = 0$. Finally, those in the column $p + 2$ are the vertices $\{(p + 2, i) | i \in [10], w'[i] = 0\}$. We then get

$$\Delta(S, S') = \begin{cases} 
3 \cdot |w'|_0 - |w|_2 - |w'|_1 + |w|_0 - 1 & \text{if } w'[10] = 0, \\
3 \cdot |w'|_0 - |w|_2 - |w'|_1 + |w|_0 & \text{otherwise},
\end{cases}$$

where $|w|_n$ denotes the number of occurrences of the letter $n$ in the word $w$.

Thus $\Delta(S, S')$ depends only on the output words of $S$ and $S'$, and we can denote this value by $\Delta(w, w')$. Note, however, that there exist pairs of words $(w, w')$ that could not be the output words of $S$ and $S'$; there are three cases:

- **Case 1.** $w[i] = 0$ and $w'[i] = 2$ since the vertex $(p + 1, i)$ would be dominated by $(p, i)$ contradicting its label 2.
- **Case 2.** $w[i] = 2$ and $w'[i] \neq 0$ for $i \in [9]$ since $(p, i)$ would not be dominated, contradicting the fact that $S'$ dominates $I(P_{p+1})$.
- **Case 3.** $w'[i] = 1$ and $w'[i - 1] \neq 0, w'[i + 1] \neq 0, w[i] \neq 0$ since $(p + 1, i)$ would be dominated according to its label but none of its neighbors belong to $S'$.

For these forbidden cases, we set $\Delta(w, w') = +\infty$.

By definition, $C_{p+1}[w, w']$ is the minimum loss $\ell(S')$ of a set $S' \subseteq P_{p+1}$ that dominates $I(P_{p+1})$, with $w_i$ as input word and $w'$ as output word. It is clear that $S = S' \cap P_p$ dominates $I(P_p)$ and has $w_i$ as input word. Let $w$ be its output word and note that $C_{p+1}[w, w] = \ell(S') = \ell(S) + \Delta(w, w')$. The minimality of $\ell(S')$ implies the minimality of $\ell(S)$ over the sets $X \in D_p^{w,w'}$. Indeed, any set $X \in D_p^{w,w'}$ could be turned into a set $X' \in D_{p+1}^{w,w'}$ by adding vertices of the $p + 1$th column accordingly to $w$. Thus

$$C_{p+1}[w, w] = C_p[w, w'] + \Delta(w, w'),$$

which implies that

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\[ C_{p+1}[w, w'] \geq \min_{w} C_p[w, w'] + \Delta(w, w'). \]

On the other hand, for every word \( w_o \in W \) such that \( C_p[w, w_o] \neq +\infty \) and \( \Delta(w, w') \neq +\infty \), there is a set \( S \in D_{p+1}^{w, w'} \), with \( \ell(S) = C_p[w, w_o] \), that can be turned into a set \( S' \in D_{p+1}^{w, w'} \) with \( \ell(S') = C_p[w, w_o] + \Delta(w, w') \). Thus

\[ C_{p+1}[w, w] \leq \min_{w_o} C_p[w, w_o] + \Delta(w, w'). \]

This concludes the proof of the lemma. \( \square \)

By the definition of \( M_p \), we have also \( M_{p+1} = M_p \otimes T \). Note that \( T \) is a sparse matrix: About 95.5\% of its 8119\(^2\) entries are +\infty. Thus the multiplication by \( T \) in the \((\min, +)\) algebra can be done in a reasonable amount of time by an algorithm using \( O(n^3) \) additions and comparisons. Note that the best known algorithm for multiplication of the \((n, n)\) matrix in the \((\min, +)\) algebra is due to Chan \cite{Chan1}: He provided a \( O(\frac{n^2}{\log n}) \)-time algorithm. However, for simplicity reasons, we implemented the algorithm in \( O(n^3) \).

Fact 9. The computations give us that \( M_{126} = M_{125} + 1 \). Thus, since \( (A + c) \otimes B = (A \otimes B) + c \) for any matrices \( A, B \) and any integer \( c \), we have that \( M_{125+k} = M_{125} + k \) for every \( k \in \mathbb{N} \).

Let us define \( M'_p = \min_{k \in \mathbb{N}} (M_{p+k} - k) \). Then, for all \( q \geq p \), \( M_q \geq M'_p + (q - p) \). By Fact 9, \( M'_p = \min_{k \in \{0, \dots, 125-p\}} (M_{p+k} - k) \).

Fact 10. By computing \( M'_{12} \), and \( A' = M'_{12} \otimes M'_{12} \), we obtain that \( \min_{w_1, w_3} (A' + A^T)[w_1, w_3] = 76 \) (where \( A^T \) is the transpose of \( A \)).

This implies that

\[
\begin{align*}
\min_{w_1, w_2, w_3, w_4} & M'_{12}[w_1, w_2] + M'_{12}[w_2, w_3] + M'_{12}[w_3, w_4] + M'_{12}[w_4, w_1] \\
& = 76
\end{align*}
\]

\[
\begin{align*}
\min & M'_{12}[w_1, w_2] + M'_{12}[w_2, w_3] + M'_{12}[w_3, w_4] + M'_{12}[w_4, w_1] \\
& = 76
\end{align*}
\]

Theorem 11. If \( 24 \leq n \leq m \), then

\[ \gamma(G_{n,m}) = \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4. \]

Proof. By Chang’s construction \cite{Chan2}, \( \gamma(G_{n,m}) \leq \left\lfloor \frac{(n+2)(m+2)}{5} \right\rfloor - 4 \). Let us now compute a lower bound for the loss of a dominating set of \( G_{n,m} \).

\[
\ell_{n,m} \geq b_{n,m}
\]

\[
\begin{align*}
& \geq \min_{w_1, w_2, w_3, w_4} M_{n-12}[w_1, w_2] + M_{m-12}[w_2, w_3] + M_{n-12}[w_3, w_4] + M_{m-12}[w_4, w_1] \\
& \text{by Lemma 7}
\end{align*}
\]

\[
\begin{align*}
& \geq \min_{w_1, w_2, w_3, w_4} M'_{12}[w_1, w_2] + (n-12-12) + M'_{12}[w_2, w_3] + (m-12-12) \\
& + M'_{12}[w_3, w_4] + (n-12-12) + M'_{12}[w_4, w_1] + (m-12-12) \\
& \geq 2(n+m-48) + \min_{w_1, w_2, w_3, w_4} M'_{12}[w_1, w_2] + M'_{12}[w_2, w_3] \\
& + M'_{12}[w_3, w_4] + M'_{12}[w_4, w_1] \\
& \geq 2(n+m-48) + 76 \\
& \geq 2(n+m) - 20.
\end{align*}
\]
Thus by Lemma 5, we have

\[
\gamma(G_{n,m}) \geq \left(\frac{n + 2(m + 2)}{5}\right) - 4
\]

Our proofs and computations in this paper led us to get a lower bound on the loss \(b_{n,m}\), namely, \(b_{n,m} \geq 2(n + m) - 20\). This lower bound appeared to be the best possible and allowed us to conclude the proof of our main theorem. This lower bound is highly related to the shape of the border \(B_{n,m}\) (thickness of 10 with four extra vertices in the corners). This shape was obtained after several unsuccessful attempts using other different shapes. Indeed, a border of thickness less than 10 or a border of thickness 10 without the extra four vertices in the corner did not give a big enough lower bound to conclude.

REFERENCES
