In this paper, we consider simple undirected graphs without loops. A proper $k$-coloring of the vertices of a graph $G = (V, E)$ is an assignment of colors from 1 to $k$ such that no two adjacent vertices have the same color. The chromatic number of $G$, denoted $\chi(G)$, is the smallest integer $k$ so that $G$ has a proper $k$-coloring. In 1969, Kramer and Kramer introduced the notion of 2-distance $k$-coloring [23] which is a proper $k$-coloring such that no pair of vertices at distance 2 have the same color. The 2-distance chromatic number of $G$, denoted $\chi^2(G)$, is the smallest integer $k$ so that $G$ has a 2-distance $k$-coloring. An example of 2-distance coloring is given in Figure 1a.

![Figure 1: A graph $G$ with $\chi^2(G) = 6$ and $\chi(G) = 3$](image)

Note that for any graph $G$ with maximum degree $\Delta$, $\Delta + 1 \leq \chi^2(G) \leq \Delta^2 + 1$. The lower bound is trivial since, in a 2-distance coloring, every neighbor of a vertex with degree $\Delta$, and $v$ itself must have a different color. As for the upper bound, a greedy algorithm shows that $\chi^2(G) \leq \Delta(G)^2 + 1$. Moreover, this bound is tight for some graphs, for example, Moore graphs of type $(\Delta, 2)$, which are graphs where all vertices have degree $\Delta$, are at distance at most two from each other, and the total number of vertices is $\Delta^2 + 1$. The Moore graphs of type $(3, 2)$ and of type $(7, 2)$ are the Petersen graph and the Hoffman-Singleton graph respectively.

In this paper, we focus on planar graphs which are graphs that can be drawn in the plane without crossing the edges. One motivation to study this class of graphs is the following famous conjecture stating an upper bound which is linear in $\Delta$:

**Conjecture 1 (Wegner, 1977 [31])** Let $G$ be a planar graph with maximum degree $\Delta$. Then,

$$\chi^2(G) \leq \begin{cases} 7, & \text{if } \Delta \leq 3, \\ \Delta + 5, & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3\Delta}{2} \rfloor + 1, & \text{if } \Delta \geq 8. \end{cases}$$

Wegner showed that the upper bounds of the conjecture are tight. For instance when $\Delta \geq 8$ consider the Wegner graph obtained as follows: take a triangle $xyz$, multiply each edge $\frac{3}{2}\Delta$ times, subdivide them once, then add an edge between $x$ and $y$. The case $\Delta \leq 3$ of Conjecture 1 was proved independently by Thomassen [29] and by Hartke et al. [17]. For $\Delta \geq 8$, Havet et al. [18] proved that the bound is $\frac{3}{2}\Delta(1 + o(1))$, where $o(1)$ is as $\Delta \to \infty$.

The coefficient before $\Delta$ becomes 1 when the girth, the length of a shortest cycle, of the graph increases. Extensive researches have been done in this case, and many results have taken the following form: every planar graph $G$ of girth $g \geq g_0$ and $\Delta(G) \geq \Delta_0$ satisfies $\chi^2(G) \leq \Delta + c(g_0, \Delta_0)$, where $c(g_0, \Delta_0)$ is a constant depending only on $g_0$ and $\Delta_0$. Table 1 shows all known such results on the 2-distance chromatic number of planar graphs with fixed girth, up to our own knowledge.
Table 1: The latest results with a coefficient 1 before $\Delta$ in the upper bound of $\chi^2$.

For example, the result from line "7" and column "$\Delta + 1$" from Table 1 reads as follows: "every planar graph $G$ of girth at least 7 and of $\Delta$ at least 16 satisfies $\chi^2(G) \leq \Delta + 1$." The crossed out cases in the first column correspond to the fact that, for $g_0 \leq 6$, there are planar graphs $G$ with $\chi^2(G) = \Delta + 2$ for arbitrarily large $\Delta$[16]. The lack of results for $g \geq 4$ is due to the fact that the Wegner graph without $xy$ has girth 4, and $\chi^2 = \lceil \frac{3\Delta}{2} \rceil - 1$ for all $\Delta$.

The "2-distance" condition in 2-distance colorings requires that vertices at distance at most two have different colors. In other words, all neighbors of the same vertex must have different colors. Recently, this condition was generalized and the notion of $r$-hued coloring was introduced [26]. Let $r, k \geq 1$ be two integers. An $r$-hued $k$-coloring of the vertices of $G$ is a proper $k$-coloring of the vertices, such that all vertices are $r$-hued. A vertex is $r$-hued if the number of colors in its neighborhood $N_G(v) = \{x \mid xv \in E\}$ is at least $\min\{d_G(v), r\}$. The $r$-hued chromatic number of $G$, denoted $\chi_r(G)$, is the smallest integer $k$ so that $G$ has an $r$-hued $k$-coloring. It is indeed a generalization of 2-distance colorings which correspond to the case $r \geq \Delta$, as all vertices in the same neighborhood will have different colors. More generally, its link to proper coloring and 2-distance coloring resides in the following equation:

$$\chi(G) = \chi_1(G) \leq \chi_2(G) \leq \cdots \leq \chi_{\Delta}(G) = \chi_{\Delta+1}(G) = \cdots = \chi^2(G) \quad (1)$$

Examples of $r$-hued colorings are given in Figure 1b and Figure 1c.

Similar to the 2-distance chromatic number, the $r$-hued chromatic number is linear in $r$ when it comes to planar graphs. In 2014, Song et al. proposed a generalization of Conjecture 1:

**Conjecture 2 (Song et al., 2014 [27])** Let $G$ be a planar graph. Then,

$$\chi_r(G) \leq \begin{cases} 
  r + 3, & \text{if } 1 \leq r \leq 2, \\
  r + 5, & \text{if } 3 \leq r \leq 7, \\
  \lceil \frac{3r}{2} \rceil + 1, & \text{if } r \geq 8.
\end{cases}$$

Note that Conjecture 2 implies Conjecture 1 except for the case $r = 3$. Moreover, the only extremal known examples reaching the upper bounds of Conjecture 2 are the same as for Conjecture 1. It is less clear what would be the expected upper bound when $r < \Delta$. In 2018, Song and Lai [28] proved that, if $r \geq 8$, then every planar graph $G$ verifies $\chi_r(G) \leq 2r + 16$. Similar to 2-distance coloring, the coefficient before $r$ in this upper bound becomes 1 for graphs with a higher girth. Table 2 shows all known results of the following form: let $r$ and $r_0$ be integers such that $r \geq r_0$, all planar graph $G$ of girth $g(G) \geq g_0$ satisfies $\chi_r(G) \leq r + c(g_0, r_0)$, where $c(g_0, r_0)$ is a constant depending only on $g_0$ and $r_0$. The result from the "9th" line and "$r + 1$" column reads "for $r \geq 8$, all planar graph $G$ of girth at least 9 satisfies $\chi_r(G) \leq r + 1$".

Since $r + 1$ is a trivial lower bound for $\chi_r$, we study the class of planar graphs verifying $\chi_r = r + 1$ and show the following:
\[ g(G) \geq 8, \quad \chi_r(G) = r + 1 \quad \text{for} \quad r \geq 9. \]

Our proof uses the discharging method and exploits planarity arguments.

For \( r \geq \Delta \), Theorem 3 gives an improvement of a result on 2-distance coloring published in [19] (see Table 1):

**Corollary 4** If \( G \) is a planar graph with \( g(G) \geq 8 \) and \( \Delta(G) \geq 9 \), then \( \chi_2(G) = \Delta(G) + 1 \).

### References


