Oriented vertex and arc colorings of partial 2-trees

Pascal Ochem\textsuperscript{1} Alexandre Pinlou\textsuperscript{2}

LaBRI, Université Bordeaux 1
351 Cours de la Libération
33405 Talence Cedex, France

1 Introduction

We consider finite simple oriented graphs, that is digraphs with no opposite arcs. For an oriented graph $G$, we denote by $V(G)$ its set of vertices and by $A(G)$ its set of arcs. The number of vertices of $G$ is the order of $G$. The girth of a graph $G$ is the size of a smallest cycle in $G$. We denote by $T_g$ the class of partial 2-trees (also known as series-parallel graphs) with girth at least $g$.

The notion of oriented vertex-coloring was introduced by Courcelle \cite{2} as follows: an oriented $k$-vertex-coloring of an oriented graph $G$ is a mapping $\varphi$ from $V(G)$ to a set of $k$ colors such that (i) $\varphi(u) \neq \varphi(v)$ whenever $uv \in A(G)$ and (ii) $\varphi(v) \neq \varphi(x)$ whenever $uv, vx \in A(G)$ and $\varphi(u) = \varphi(y)$. The oriented chromatic number of $G$, denoted by $\chi_o(G)$, is defined as the smallest $k$ such that $G$ admits an oriented $k$-vertex-coloring. The oriented chromatic number $\chi_o(\mathcal{F})$ of a class of oriented graphs $\mathcal{F}$ is defined as the maximum of $\chi_o(G)$ taken over all graphs $G$ in $\mathcal{F}$.

Let $G$ and $H$ be two oriented graphs. A homomorphism from $G$ to $H$ is a mapping $\varphi$ from $V(G)$ to $V(H)$ that preserves the arcs: $\varphi(u)\varphi(v) \in A(H)$ whenever

\textsuperscript{1} Email:Pascal.Ochem@labri.fr
\textsuperscript{2} Email: Alexandre.Pinlou@labri.fr
An oriented \( k \)-vertex-coloring of an oriented graph \( G \) can be equivalently defined as a homomorphism \( \varphi \) from \( G \) to \( H \), where \( H \) is an oriented graph of order \( k \). The oriented chromatic number of \( G \) can then be viewed as the smallest order of an oriented graph \( H \) such that \( G \) admits a homomorphism to \( H \). Links between colorings and homomorphisms are presented in more details in the monograph [3] by Hell and Nešetřil.

Oriented vertex-colorings have been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes (see e.g. [1,8,9]). Concerning partial 2-trees, Sopena proved [9] that their oriented chromatic number is at most 7 (this bound was shown to be tight). Pinlou and Sopena [8] obtained tight bounds for the oriented chromatic number of outerplanar graphs with given girth (outerplanar graphs form a strict subclass of partial 2-trees). Moreover, they proved that \( \chi_o(T_g) = 7 \) for every \( g, 3 \leq g \leq 4 \). In this paper, we complete the characterization of the oriented chromatic numbers of partial 2-trees with given girth:

**Theorem 1.1**

1. \( \chi_o(T_g) = 6 \) for every girth \( g, 5 \leq g \leq 6 \);
2. \( \chi_o(T_g) = 5 \) for every girth \( g, g \geq 7 \);

One can define oriented arc-colorings of oriented graphs in a natural way by saying that, as in the undirected case, an oriented arc-coloring of an oriented graph \( G \) is an oriented vertex-coloring of its line digraph \( LD(G) \) (recall that \( LD(G) \) is given by \( V(LD(G)) = A(G) \) and \( ab \in A(LD(G)) \) whenever \( a = uv \) and \( b = vw \)). Therefore, an oriented arc-coloring \( \varphi \) of \( G \) must satisfy (i) \( \varphi(uv) \neq \varphi(vw) \) whenever \( uv \) and \( vw \) are two consecutive arcs in \( G \), and (ii) \( \varphi(vw) \neq \varphi(xy) \) whenever \( uv, vw, xy, yz \in A(G) \) with \( \varphi(uv) = \varphi(yz) \). The oriented chromatic index of \( G \), denoted by \( \chi'_o(G) \), is defined as the smallest order of an oriented graph \( H \) such that \( LD(G) \) admits a homomorphism to \( H \). The oriented chromatic index \( \chi'_o(\mathcal{F}) \) of a class of oriented graphs \( \mathcal{F} \) is defined as the maximum of \( \chi'_o(G) \) taken over all graphs \( G \) in \( \mathcal{F} \).

The oriented chromatic index of oriented graphs was recently studied and several upper and lower bounds are known (see [6,7,8]).

Upper bounds for the oriented chromatic index can be easily derived from oriented chromatic number:

**Claim 1.2** [6] For every oriented graph \( G \), \( \chi'_o(G) \leq \chi_o(G) \).

Our second result gives estimates of the oriented chromatic indexes of partial 2-trees with girth 4, 5 and 6, and a characterization for all other girths:
Theorem 1.3

(1) \( \chi'_o(T_3) = 7 \);
(2) \( 6 \leq \chi'_o(T_4) \leq 7 \);
(3) \( 5 \leq \chi'_o(T_g) \leq 6 \) for every girth \( g \), \( 5 \leq g \leq 6 \);
(4) \( \chi'_o(T_g) = 5 \) for every girth \( g \), \( 7 \leq g \leq 17 \);
(5) \( \chi'_o(T_g) = 4 \) for every girth \( g \), \( g \geq 18 \);

In the rest of the paper, we will use the following notation. A vertex of degree \( k \) will be called a \( k \)-vertex. We denote by \( \delta(G) \) the minimum degree of the graph \( G \).

A \( k \)-path in a graph \( G \) is a path \( P = [u, v_1, v_2, \ldots, v_{k-1}, w] \) of length \( k \) (i.e. a path with \( k \) arcs); the vertices \( u \) and \( w \) are the endpoints of \( P \). Note that a 1-path is an arc. A \( (k, d) \)-path is a \( k \)-path such that all internal vertices \( v_i \) have degree \( d \).

A 2-vertex contraction is the contraction of an edge incident to a 2-vertex.

2 Sketches of proof

The proofs of Theorems 1.1 and 1.3 use some structural properties on partial 2-trees with given girth and on graph classes closed under 2-vertex contraction. These properties are given in the two following lemmas.

**Lemma 2.1** Let \( \mathcal{C} \) be a graph class closed under 2-vertex contraction such that every non-empty graph \( G \in \mathcal{C} \) with girth at least \( g \) contains either a 1-vertex or a \( (k,2) \)-path, for some \( k \geq 2 \). Then, for every \( n \geq 0 \), every non-empty graph \( G' \in \mathcal{C} \) with girth at least \( g + n \left\lceil \frac{g}{k-1} \right\rceil \) contains either a 1-vertex or a \( (k+n,2) \)-path.

For a graph \( G \) with girth at least \( g \) and a vertex \( v \in V(G) \), we denote:

\[ D^G_g(v) = |\{ u \in V(G), \ d(u) \geq 3 \text{ such that there exists a unique path of 2-vertices linking } u \text{ and } v \text{ or } u \text{ and } v \text{ are the endpoints of at least a } \left( \left\lceil \frac{g}{k} \right\rceil + 2 \right) \text{-path} \}|. \]

**Lemma 2.2** Let \( G \) be a partial 2-tree with girth \( g \) such that \( \delta(G) \geq 2 \). Then, either there exists a \( \left( \left\lceil \frac{g}{2} \right\rceil + 1,2 \right) \)-path, or there exists a \( \geq 3 \)-vertex \( v \) such that \( D^G_g(v) \leq 2 \).

Note that this lemma generalizes Lemma 2 p. 305 of Lih et al. [4] which characterizes partial 2-trees with girth 3.

**Upper bounds**

Thanks to the above lemmas, the upper bounds of Theorems 1.1 and 1.3 are obtained by showing that the considered partial 2-trees admit a homomorphism to one of the tournaments \( T_4, T_5, T_6, \) and \( T_7 \) depicted on Fig. 1.
Lower bounds

Finally, to get the lower bounds of Theorems 1.1 and 1.3, we construct partial 2-trees with the required girth which need the specified number of colors. More fully:

• The graph $G_6$ depicted in Fig. 2(b) is a partial 2-tree with girth 6 such that
\(\chi_o(G_6) = 6\). Therefore, \(\chi_o(T_g) \geq 6\) for every \(g \leq 6\).

- Nešetřil et al. [5] constructed for every \(g \geq 3\), an oriented outerplanar graph with girth \(g\) which has oriented chromatic number 5. Therefore, \(\chi_o(T_g) \geq 5\) for every \(g \geq 7\).

- The first three assumptions of Theorem 1.3 directly follow from Claim 1.2, Theorem 1.1(1) and some results of Pinlou and Sopena [8], namely \(\chi_o(T_3) = 7\), \(\chi'_o(O_4) = 6\), and \(\chi'_o(O_6) = 5\).

- The graph \(G_{17}\) depicted in Fig. 3(c) is a partial 2-tree with girth 17 such that \(\chi'_o(G_{17}) = 5\). Therefore, \(\chi'_o(T_g) \geq 5\) for every \(g \leq 17\).

- It not difficult to check that, for every \(g \geq 3\), the partial 2-tree \(G\) obtained from two vertex-disjoint circuits, the first one of size \(g\) and the second one of size \(k \geq g\) with \(k \not\equiv 0 \pmod{3}\) has girth \(g\) and \(\chi'_o(G) = 4\). Therefore \(\chi'_o(T_g) \geq 4\) for every \(g \geq 18\).

References


