Homomorphisms of signed planar graphs

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Abstract

Signed graphs are studied since the middle of the last century. Recently, the notion of homomorphism of signed graphs has been introduced since this notion captures a number of well known conjectures which can be reformulated using the definitions of signed homomorphism.

In this paper, we introduce and study the properties of some target graphs for signed homomorphism. Using these properties, we obtain upper bounds on the signed chromatic numbers of graphs with bounded acyclic chromatic number and of signed planar graphs with given girth.

Key words: Signed graphs, Homomorphisms, Discharging method.

1 Introduction

The class of signed graphs is a natural graph class where the edges are either positive or negative. They were first introduced to handle problems in social psychology: positive edges link friends whereas negative ones link enemies.

In the area of graph theory, they have been used as a way of extending classical results in graph coloring such as Hadwiger’s conjecture. Guenin [3] introduced
the notion of signed homomorphism for its relation with a well known conjecture of Seymour. In 2012, this notion has been further developed by Naserasr et al. [7] as this theory captures a number of well known conjectures which can be reformulated using the definitions of signed homomorphism. In this paper, we study signed homomorphisms for themselves.

A signified graph \((G, \Sigma)\) is a graph \(G\) with an assignment of positive (+1) and negative (−1) signs to its edges where \(\Sigma\) is the set of negative edges. In all the figures, negative edges are drawn with dashed edges. Figure 1(a) gives an example of signified graph.

Resigning a vertex \(v\) of a signified graph \((G, \Sigma)\) corresponds to give the opposite sign to the edges incident to \(v\). Given a signified graph \((G, \Sigma)\) and a set of vertices \(X \subseteq V(G)\), the graph obtained from \((G, \Sigma)\) by resigning every vertex of \(X\) is denoted by \((G, \Sigma^X)\).

Two signified graphs \((G, \Sigma_1)\) and \((G, \Sigma_2)\) are said to be equivalent if we can obtain \((G, \Sigma_1)\) from \((G, \Sigma_2)\) by resigning some vertices of \((G, \Sigma_2)\), i.e., \(\Sigma_2 = \Sigma_i^X\) for some \(X \subseteq V(G)\); in such a case, we use the notation \((G, \Sigma_1) \sim (G, \Sigma_2)\) (see Figure 1 for an example of equivalent signified graphs). Each equivalence class defined by the resigning process is called a signed graph and can be denoted by any member of its class. We might simply use \((G)\) for a signified/signed graph when its set of negative edges is clear from the context, while \(G\) refers to its underlying unsigned graph.

An m-edge-colored graph \(G\) is a graph where the vertices are linked by edges \(E(G)\) of \(m\) types. In other words, there is a partition \(E(G) = E_1(G) \cup \ldots \cup E_m(G)\) of the edges of \(G\), where \(E_j(G)\) contains all edges of type \(j\). Since signified graphs are defined with two types of edges (i.e. positive and negative edges), they correspond to 2-edge-colored graphs. In this paper, known and new results on 2-edge-colored graphs are stated in terms of signified graphs.

Given two graphs \((G, \Sigma)\) and \((H, \Lambda)\), \(\varphi\) is a signified homomorphism of \((G, \Sigma)\) to \((H, \Lambda)\) if \(\varphi : V(G) \to V(H)\) is a mapping such that every edge of \((G, \Sigma)\) is mapped to an edge of the same sign of \((H, \Lambda)\). Given two graphs \((G, \Sigma_1)\) and \((H, \Lambda_1)\), we say that there is a signed homomorphism \(\varphi\) of \((G, \Sigma_1)\) to \((H, \Lambda_1)\) if there exists \((G, \Sigma_2) \sim (G, \Sigma_1)\) and \((H, \Lambda_2) \sim (H, \Lambda_1)\) such that \(\varphi\) is a signed homomorphism of \((G, \Sigma_2)\) to \((H, \Lambda_2)\).

Lemma 1 If \((G, \Sigma)\) admits a signed homomorphism to \((H, \Lambda)\), then there exists \((G, \Sigma') \sim (G, \Sigma)\) such that \((G, \Sigma')\) admits a signed homomorphism to \((H, \Lambda)\).
Proof. Since \((G, \Sigma)\) admits a signed homomorphism to \((H, \Lambda)\), this implies that there exist \((G, \Sigma') \sim (G, \Sigma), (H, \Lambda') \sim (H, \Lambda)\) and a signed homomorphism \(\varphi\) of \((G, \Sigma')\) to \((H, \Lambda')\). Let \(X \subseteq V(H)\) be the subset of vertices of \(H\) such that \((H, \Lambda) = (H, \Lambda' | X)\). Now let \(Y = \{v \in V(G) \mid \varphi(v) \in X\}\). Let \((G, \Sigma') = (G, \Sigma' | Y)\); it is clear that \(\varphi\) is a signed homomorphism of \((G, \Sigma')\) to \((H, \Lambda)\). \(\square\)

As a consequence of the above lemma, when dealing with signed homomorphisms, we will not need to resign the target graph.

The \textit{signified chromatic number} \(\chi_2(G, \Sigma)\) of the graph \((G, \Sigma)\) is the minimum order (number of vertices) of a graph \((H, \Lambda)\) such that \((G, \Sigma)\) admits a signed homomorphism to \((H, \Lambda)\). Similarly, the \textit{signed chromatic number} \(\chi_s(G, \Sigma)\) of the graph \((G, \Sigma)\) is the minimum order of a graph \((H, \Lambda)\) such that \((G, \Sigma)\) admits a signed homomorphism to \((H, \Lambda)\); equivalently, \(\chi_s(G, \Sigma) = \min \{\chi_2(G, \Sigma') \mid (G, \Sigma') \sim (G, \Sigma)\}\).

The \textit{signified chromatic number} \(\chi_2(G)\) of a graph \(G\) is defined as \(\chi_2(G) = \max\{\chi_2(G, \Sigma) \mid \Sigma \subseteq E(G)\}\). The \textit{signified chromatic number} \(\chi_2(F)\) of a graph class \(F\) is defined as \(\chi_2(F) = \max\{\chi_2(G) \mid G \in F\}\). The \textit{signed chromatic numbers} of a graph and a graph class are defined similarly.

Another equivalent definition of the signed chromatic numbers can be given by defining the signified coloring. A \textit{signified coloring} of a signified graph \((G)\) is a proper vertex-coloring \(\varphi\) of \(G\) such that if there exist two edges \(uv\) and \(xy\) with \(\varphi(u) = \varphi(x)\) and \(\varphi(v) = \varphi(y)\), then these two edges have the same sign. Hence, the \textit{signified chromatic number} of the signified graph \((G)\) is the minimum number of colors needed for a signified coloring of \((G)\).

In this paper, we studied signified and signed homomorphisms of outerplanar and planar graphs of given girth. The paper is organized as follows. We introduce the notation in Section 2. Section 3 is devoted to introduce and study the properties of several families of target graphs, namely the \textit{Anti-twinned graph} \(AT(G, \Sigma)\), the \textit{signified Zielonka graph} \(ZS_k\), the \textit{signified Paley graph} \(SP_q\), and the \textit{signified Tromp Paley graph} \(Tr(SP_q)\). We study signified homomorphisms of planar graphs (resp. outerplanar graphs) in Section 4 and we provide lower and upper bounds on the signed chromatic number. We get upper bounds on the signed chromatic number of planar graphs (resp. outerplanar graphs) of given girth in Section 5. We finally conclude in Section 6.

2 Notations

For a vertex \(v\) of a signified graph \((G)\), \(d_{(G)}(v)\) denotes the degree of \(v\). The set of positive neighbors of \(v\) is denoted by \(N^+(G)(v)\) and the set of negative neighbors of \(v\) is denoted by \(N^-(G)(v)\). Thus, the set of neighbors of \(v\), denoted by \(N(G)(v)\), is \(N(G)(v) = N^+(G)(v) \cup N^-(G)(v)\). A vertex of degree \(k\) (resp. at least \(k\), at most \(k\)) is called a \(k\)-vertex (resp. \(\geq k\)-vertex, \(\leq k\)-vertex). If a vertex \(u\) is adjacent to a \(k\)-vertex (resp. \(\geq k\)-vertex, \(\leq k\)-vertex) \(v\), then \(v\) is a \(k\)-neighbor (resp. \(\geq k\)-neighbor, \(\leq k\)-neighbor) of \(u\). A path of length \(k\) (i.e. formed by \(k\) edges) is...
called a $k$-path. Given a planar graph $G$ with its embedding in the plane and a vertex $v$ of $G$, we say that a sequence $(u_1, u_2, \ldots, u_k)$ of neighbors of $v$ are consecutive if $u_1, u_2, \ldots, u_k$ appear consecutively around $v$ in $G$ (clockwise or counterclockwise).

### 3 Target graphs

In this section, our goal is not only to find target graphs that will give the required upper bounds of our results of Sections 4 and 5. We aim at describing several families of target graphs that may be useful for signified and signed homomorphisms and we determine their properties. To this end, we describe below the Anti-twinned graph construction, the signified Zielonka graph $ZS_k$, the signified Paley graph and the Tromp signified Paley graph.

#### 3.1 The Anti-twinned graph

Let $(G, \Sigma)$ be a signified graph and let $(G^0, \Sigma^0)$ and $(G^1, \Sigma^1)$ be two isomorphic copies of $(G, \Sigma)$. In the following, given a vertex $u \in V(G)$, we denote $u_i$ the corresponding vertex of $u$ in the isomorphic copy $(G^i, \Sigma^i)$ of $(G, \Sigma)$. We define the anti-twinned graph $AT(G, \Sigma) = (H, \Lambda)$ on $2|V(G)|$ vertices as follows:

- $V(H) = V(G^0) \cup V(G^1)$
- $E(H) = E(G^0) \cup E(G^1) \cup \{u_i v_{1-i} : uv \in E(G)\}$
- $\Lambda = \Sigma^0 \cup \Sigma^1 \cup \{u_i v_{1-i} : uv \in E(G) \setminus \Sigma\}$

Figure 2 illustrates the construction of $AT(G, \Sigma)$. We can observe that for every $u_i \in V(G^i)$, there is no edge between $u_i$ and $u_{1-i}$. By construction we have the following property:

$$\forall u_i \in AT(G, \Sigma) : N^+(u_i) = N^-(u_{1-i}) \text{ and } N^-(u_i) = N^+(u_{1-i})$$

Such pairs of vertices are called anti-twin vertices, and for any $u \in AT(G, \Sigma)$ we denote by $atw(u)$ the anti-twin vertex of $u$. Remark that $atw(atw(u)) = u$. This notion can be extended to sets in a standard way: for a given $W \subseteq V(G^i)$, $W = \{v_1, v_2, \ldots, v_k\}$, then $atw(W) = \{atw(v_1), atw(v_2), \ldots, atw(v_k)\}$.

We say that a signified graph is anti-twinned if it is the anti-twinned graph of some signified graph.

**Observation 2** A signified graph is anti-twinned if and only if each of its vertices has a unique anti-twin.
Lemma 3 A graph \((G, \Sigma)\) admits a signed homomorphism to \((H, \Lambda)\) if and only if \((G, \Sigma)\) admits a signified homomorphism to \(AT(H, \Lambda)\).

Proof. Let \(\varphi\) be a signed homomorphism of \((G, \Sigma)\) to \((H, \Lambda)\). This implies that \(\varphi\) is a signified homomorphism of \((G, \Sigma_1)\) to \((H, \Lambda)\), where \((G, \Sigma_1) \sim (G, \Sigma)\).

Let \(X \subseteq V(G)\) be the subset of vertices of \(G\) such that \((G, \Sigma) = (G, \Sigma_1^{(X)})\). By definition of the resigning process, only the edges of the edge-cut between \(V(G) \setminus X\) and \(X\) get their sign changed.

Let \(\varphi' : V(G) \to V(AT(H))\) be defined as follows:

\[
\varphi'(u) = \begin{cases} 
\text{atw}(\varphi(u)), & \text{if } u \in X, \\
\varphi(u), & \text{otherwise}.
\end{cases}
\]

By construction of \(AT(H, \Lambda)\), if \(u\) and \(v\) induce an edge of sign \(s\), then \(u\) and \(\text{atw}(v)\) induce an edge of sign \(-s\). Therefore, it is easy to see that \(\varphi'\) is a signified homomorphism of \((G, \Sigma)\) to \(AT(H, \Lambda)\). This proves the only if part.

For the if part, suppose that \((G, \Sigma)\) admits a signified homomorphism \(\psi\) to \(AT(H, \Lambda)\). The signified graph \(AT(H, \Lambda)\) is obtained from two isomorphic copies \((H^0, \Lambda^0)\) and \((H^1, \Lambda^1)\) of \((H, \Lambda)\). Let \(Y \subseteq V(G)\) be the subset of vertices of \(G\) such that, for all \(y \in Y\), \(\psi(y)\) is a vertex of \(H^1\). Let \((G, \Sigma_1) = (G, \Sigma^{(Y)})\).

Let \(\psi' : V(G) \to V(AT(H))\) be defined as follows:

\[
\psi'(u) = \begin{cases} 
\text{atw}(\psi(u)), & \text{if } u \in Y, \\
\psi(u), & \text{otherwise}.
\end{cases}
\]

It is easy to see that \(\psi'\) is a signified homomorphism of \((G, \Sigma_1)\) to \(AT(H, \Lambda)\) such that every vertex maps to a vertex of \(H^0\). Then \(\psi'\) is a signified homomorphism from \((G, \Sigma_1)\) to \((H, \Lambda)\) and thus \((G, \Sigma)\) admits a signed homomorphism to \((H, \Lambda)\).

Corollary 4 If \((G, \Sigma)\) admits a signed homomorphism to an anti-twinned graph \(T\), then we have:

1. \(\chi_s(G) \leq \frac{|V(T)|}{2}\).
2. \((G, \Sigma')\) admits a signified homomorphism to \(T\) for every \((G, \Sigma') \sim (G, \Sigma)\).

3.2 The signified Zielonka graph \(ZS_k\)

The Zielonka graph \(Z_k\) was introduced by Zielonka [12] in the theory of bounded timestamp systems. Alon and Marshall [1] adapted this construction to signified graphs to obtain the signified Zielonka graph \(ZS_k\). They used this graph to get bounds on the signified chromatic number of graphs that admits an acyclic \(k\)-coloring.

Let us describe the construction of the signified Zielonka graph \(ZS_k\). Every vertex is of the form \((i; \alpha_1, \alpha_2, \ldots, \alpha_k)\) where \(1 \leq i \leq k\), \(\alpha_j \in \{+1, -1\}\) for \(j \neq i\) and \(\alpha_i = 0\). There are clearly \(k \cdot 2^{k-1}\) vertices in this graph. For \(i \neq j\),
there is an edge between the vertices \((i; \alpha_1, \alpha_2, \ldots, \alpha_k)\) and \((j; \beta_1, \beta_2, \ldots, \beta_k)\) and the sign of this edge is given by the product \(\alpha_j \times \beta_i\).

Proposition 5 The graph \(ZS_k\) is anti-twinned.

Proof. By Observation 2, we have to show that every vertex has an anti-twin. We claim that the anti-twin of the vertex \(v = (i; \alpha_1, \alpha_2, \ldots, \alpha_k)\) is the vertex \(v' = (i; -\alpha_1, -\alpha_2, \ldots, -\alpha_k)\). Indeed, \(v\) and \(v'\) are not adjacent and it is easy to check that for every edge \(uv\), the edge \(uv'\) exists, and that \(uv\) and \(uv'\) have opposite signs.

3.3 The signified Paley graph \(SP_q\)

In the remaining, \(q\) is any prime power such that \(q \equiv 1 \pmod{4}\). There is a unique (up to isomorphism) finite field \(\mathbb{F}_q\) of order \(q\). Let \(g\) be a generator of the field \(\mathbb{F}_q^*\). For every \(v \in \mathbb{F}_q^*\), let \(sq : \mathbb{F}_q^* \rightarrow \{-1, +1\}\) be the function square defined as \(sq(v) = +1\) if \(v\) is a square and \(sq(v) = -1\) if \(v\) is a non-square. Note that \(sq(g^t) = (-1)^t\) since \(g\) is necessarily a non-square.

The Paley graph \(P_q\) is the undirected graph with vertex set \(V(P_q) = \mathbb{F}_q\) and edge set \(E(P_q) = \{xy \mid sq(y-x) = +1\}\). Since \(-1\) is a square in \(\mathbb{F}_q\), \(sq(x-y) = sq(y-x)\) and therefore the definition of an edge is consistent. We also know that a Paley graph is self-complementary [10] and edge-transitive.

A \(k\)-regular graph \(G\) with \(n\) vertices is said to be strongly regular if (1) every two adjacent vertices have \(\lambda\) common neighbors and (2) every two non-adjacent vertices have \(\mu\) common neighbors. Such a graph is said to be a strongly regular graph with parameters \((n, k, \lambda, \mu)\). Paley graphs \(P_q\) are known to be strongly regular graphs with parameters \((q, q-1, q-5, q-1)\).

For any prime power \(q \equiv 1 \pmod{4}\), we define the signified Paley graph \(SP_q = (K_q, \Sigma)\) as the complete graph on \(q\) vertices with \(V(SP_q) = \mathbb{F}_q\) and \(\Sigma = \{xy \mid sq(y-x) = -1\}\). That is, \(SP_q\) is obtained from the Paley graph \(P_q\) by replacing the non-edges by negative edges. Figure 3 represents the signified Paley graph \(SP_5\). Since \(P_q\) is edge-transitive and self-complementary, \(SP_q\) is clearly edge-transitive.
3.4 The Tromp signified Paley graph $\text{Tr}(SP_q)$

Given an oriented graph $\vec{G}$, Tromp [11] proposed a construction of an oriented graph $\text{Tr}(G)$ called Tromp graph. We adapt this construction to signified graphs as follows.

For a given graph $(G, \Sigma)$, let us denote $(G^+, \Sigma)$ the graph obtained from $(G, \Sigma)$ by adding a universal vertex positively linked to all the vertices of $(G, \Sigma)$.

Then, the Tromp signified graph $\text{Tr}(G, \Sigma)$ of $(G, \Sigma)$ is defined to be the anti-twinned graph of $(G^+, \Sigma)$, that is $\text{Tr}(G, \Sigma) \sim \text{AT}(G^+, \Sigma)$. Figure 4 illustrates the construction of $\text{Tr}(G, \Sigma)$.

By construction, $\text{Tr}(G, \Sigma)$ is obtained from two isomorphic copies $(G_0^+, \Sigma_0^0)$ and $(G_1^+, \Sigma_1^1)$ of $(G, \Sigma)$ plus 2 vertices $\infty_0$ and $\infty_1$.

In the remainder, we focus on the specific graph family obtained by applying the Tromp’s construction to the signified Paley graph $SP_q$.

We consider the Tromp signified Paley graph $\text{Tr}(SP_q)$ on $2q + 2$ vertices obtained from $SP_q$. In the remainder of this paper, the vertex set of $\text{Tr}(SP_q)$ is $V(\text{Tr}(SP_q)) = \{0_0, 1_0, \ldots, q - 1_0, \infty_0, 0_1, 1_1, \ldots, q - 1_1, \infty_1\}$. The vertex set of the isomorphic copy $SP_i^q$ of $SP_q$ ($i \in \{0, 1\}$) thus, for every $u_i \in \{0_i, 1_i, \ldots, q - 1_i, \infty_i\}$, we have $\text{atw}(u_i) = u_{i-1}$. In addition, for every $u \in V(\text{Tr}(SP_q))$, we have $|N_{\text{Tr}(SP_q)}^+(u)| = |N_{\text{Tr}(SP_q)}^-(u)| = q$.

Let $i, j \in \{0, 1\}$ and $u, v \in \mathbb{F}_q$. If $i = j$, then $u_i$ and $v_j$ are in the same isomorphic copy of $SP_q$ in $\text{Tr}(SP_q)$; in this case, the sign of the edge $u_iv_j$ is $\text{sq}(u - v)$ by definition of $SP_q$. If $i \neq j$, then $u_i$ and $v_j$ are in distinct isomorphic copies of $SP_q$ in $\text{Tr}(SP_q)$; in this case, the sign of the edge $u_iv_j$ is $-\text{sq}(u - v)$. Therefore, in both cases, the sign of the edge $u_iv_j$ is

$$\text{sq}(u - v) \times (-1)^{i+j}. \quad (1)$$

Let $i, j \in \{0, 1\}$ and $v \in \mathbb{F}_q$. If $i = j$, then the sign of the edge $\infty_iv_j$ is $+1$, while it is $-1$ when $i \neq j$. Therefore, in both case, the sign of the edge $\infty_iv_j$ is
The graph \( Tr(SP_q) \) has remarkable symmetry and some useful properties given below.

**Lemma 6** The signified graph \( Tr(SP_q) \) is vertex-transitive.

**Proof.** The mapping \( \gamma_1 : V(Tr(SP_q)) \to V(Tr(SP_q)) \) defined as \( \gamma_1(u_i) = u_{1-i} \) is clearly an automorphism of \( Tr(SP_q) \).

Recall first that \( SP_q \) is edge-transitive and so vertex-transitive. If \( \varphi \) is an automorphism of \( SP_q \), we can define the corresponding automorphism \( \gamma_2 \) of \( Tr(SP_q) \) as:

\[
\gamma_2 : u_i \mapsto \begin{cases} u_i & \text{if } u = \infty \\ (\varphi(u))_i & \text{if } u \neq \infty \end{cases}
\]

We eventually define the mapping \( \gamma_3 : V(Tr(SP_q)) \to V(Tr(SP_q)) \) as:

\[
\gamma_3 : u_i \mapsto \begin{cases} \infty_i & \text{if } u = 0 \\ 0_i & \text{if } u = \infty \\ (u^{-1})_i & \text{if } u \text{ is a non-zero square} \\ (u^{-1})_{1-i} & \text{if } u \text{ is a non square} \end{cases}
\]

Let \( S \) be the sign of the edge \( u_iv_j \). To prove that \( \gamma_3 \) is an automorphism of \( Tr(SP_q) \), we will show that \( \gamma_3 \) maps \( u_iv_j \) to an edge of sign \( S' = S \).

Let \( g \) be a generator of the field \( \mathbb{F}_q \). Any vertex \( v \) of \( SP_q \) is an element of \( \mathbb{F}_q \) and therefore \( v = g^t \) for some \( t \) when \( v \neq 0 \). Recall that \( sq(g^t) = (-1)^t \).

- When \( u, v \in V(Tr(SP_q)) \) are neither 0 nor \( \infty \), we have \( u_iv_j = (g^t)_i(g^t')_j \) for some \( t \) and \( t' \). If \( t \) is even, that is \( g^t \) is a square, then \( \gamma_3((g^t)_i) = (g^{-t})_i \); otherwise, \( t \) is odd and thus \( \gamma_3((g^t)_i) = (g^{-t})_{1-i} \). Let \( i' = i + t \mod 2 \) and \( j' = j + t' \mod 2 \). Clearly, if \( t \) is even (resp. \( t' \)) is even, then \( i' = i \) (resp. \( j' = j \)); otherwise we have \( i' = 1 - i \) (resp. \( j' = 1 - j \)). Therefore, the function \( \gamma_3 \) maps the edge \( (g^t)_i(g^t')_j \) to the edge \( (g^{-t})_{i'}(g^{-t'})_{j'} \). Now, let us check that the sign of these two edges is the same.

  By Equation (1), the sign of the edge \( u_iv_j \) is \( S = sq(g^t - g^{t'}) \times (-1)^{i+j} \) and the sign of the edge \( (g^{-t})_{i'}(g^{-t'})_{j'} \) is \( S' = sq(g^{-t} - g^{-t'}) \times (-1)^{i'+j'} \). We then have:
\[ S = \text{sq}(g^t - g^{t'}) \times (-1)^{i+j} \]
\[ = \text{sq}(g^t - g^{t'}) \times (-1)^{i+j} \]
\[ = \text{sq}((g^{-t} - g^{-t'}) \times g^{t+t'}) \times (-1)^{i+j} \]
\[ = \text{sq}(g^{-t} - g^{-t'}) \times \text{sq}(g^{t+t'}) \times (-1)^{i+j} \]
\[ = \text{sq}(g^{-t} - g^{-t'}) \times (-1)^{i+t+j+t'} \]
\[ = \text{sq}(g^{-t} - g^{-t'}) \times (-1)^{t'+j'} = S' \]

- If \( u = 0 \) and \( v = \infty \), it is clear that the edge \( u_iv_j \) maps to an edge of the same sign.
- Consider now the case \( u = 0 \) and \( v \not\in \{0, \infty\} \). Let \( v = g^t \) for some \( t \). The function \( \gamma_3 \) maps the edge \( 0_i(g^t)_j \) on \( \infty_i(g^{-t})_j' \), where \( j' = j + t \mod 2 \). By Equations 1 and 2, the sign of the edge \( 0_i(g^t)_j \) is \( S = \text{sq}(g^t) \times (-1)^{i+j} \) and the sign of the edge \( \infty_i(g^{-t})_j' \) is \( S' = (-1)^{i+j} \). We then have:

\[ S = \text{sq}(g^t) \times (-1)^{i+j} = (-1)^t \times (-1)^{i+j} = (-1)^{i+j+t} = (-1)^{i+j'} = S' \]

- The case \( u = \infty \) and \( v \not\in \{0, \infty\} \) is similar to the previous case.

Therefore, the function \( \gamma_3 \) maps any edge to an edge of the same sign.

Combining the automorphisms \( \gamma_1, \gamma_2 \) and \( \gamma_3 \) easily proves that \( \text{Tr}(\text{SP}_q) \) is vertex-transitive. \( \Box \)

We define an anti-automorphism of a signified graph \((G, \Sigma)\) as a permutation \( \rho \) of the vertex set \( V(G) \) such that \( uv \) is a positive (resp. negative) edge if and only if \( \rho(u)\rho(v) \) is a negative (resp. positive) edge.

**Lemma 7** The graph \( \text{Tr}(\text{SP}_q) \) admits an anti-automorphism.

**Proof.** Let \( n \) be any non-square of \( \mathbb{F}_q \). We define the mapping \( \gamma_n : V(\text{Tr}(\text{SP}_q)) \rightarrow V(\text{Tr}(\text{SP}_q)) \) as:

\[ \gamma_n : u_i \rightarrow \begin{cases} u_i & \text{if } u = \infty \\ (n \times u)_{1-i} & \text{if } u \not= \infty \end{cases} \]

Let us check that \( \gamma_n \) maps every edge \( u_iv_j \in E(\text{Tr}(\text{SP}_q)) \) to an edge of opposite sign.

Let \( u, v \not= \infty \). By definition, \( \gamma_n \) maps \( u_iv_j \) to \((n \times u)_{1-i}(n \times v)_{1-j} \). By Equations 1 and 2, the sign of the edge \( u_iv_j \) is \( S = \text{sq}(v-u) \times (-1)^{i+j} \) and the sign
of the edge \((n \times u)_{1-i}(n \times v)_{1-j}\) is \(S' = sq(n \times (v - u)) \times (-1)^{1-i+1-j}\).

\[
\begin{align*}
S' &= sq(n \times (v - u)) \times (-1)^{1-i+1-j} \\
&= sq(n) \times sq(v - u) \times (-1)^{1+j} \\
&= -sq(v - u) \times (-1)^{1+j} = -S
\end{align*}
\]

Now, let \(u = \infty\) and \(v \neq \infty\). The mapping \(\gamma_n\) maps \(\infty_i v_j\) to \(\infty_i (n \times v)_{1-j}\). By Equation 2, the sign of the edge \(\infty_i v_j\) is \(S = (-1)^{i+j}\) and the sign of the edge \(\infty_i (n \times v)_{1-j}\) is \(S' = (-1)^{i+1-j} = -S\).

**Lemma 8** If there exists an isomorphism \(\psi\) that maps the triangle \((u_i, v_j, w_k)\) to the triangle \((u'_i, v'_j, w'_k)\), then \(\psi\) can be extended to an automorphism of \(Tr(SP_q)\).

**Proof.** There exists four types of triangles depending on the sign of their edges.

- Let us first consider the triangles \((u_i, v_j, w_k)\) and \((u'_i, v'_j, w'_k)\) with 3 positive edges. To prove that \(\psi\) can be extended to an automorphism of \(Tr(SP_q)\), it suffices to prove that for every triangle \((u_i, v_j, w_k)\), there exists an automorphism \(\psi'\) that maps \((u_i, v_j, w_k)\) to \((0, 1, 0, \infty_0)\). Using the vertex transitivity of \(Tr(SP_q)\) (Lemma 6), there exists an automorphism \(\varphi\) that maps \(w_k\) to \(\infty_0\). Then, since all positive edges incident to \(\infty_0\) have their extremities in \(SP_q^0\), \(\varphi\) necessarily maps the edge \(u_i v_j\) to an edge \(u'_0 v'_0\) in \(SP_q^0\). Since \(SP_q\) is edge-transitive, we can finally map \(u'_0 v'_0\) to \(0, 1, 0, \infty_0\).

- Consider now the triangles \((u_j, v_j, w_k)\) and \((u'_j, v'_j, w'_k)\) with 3 negative edges. Let \(\overline{Tr}(SP_q)\) be the signified graph obtained from \(Tr(SP_q)\) by changing the sign of every edge. By Lemma 7, \(\overline{Tr}(SP_q)\) is isomorphic to \(Tr(SP_q)\). By the previous item, there exists an automorphism that maps \((u_j, v_j, w_k)\) to \((u'_j, v'_j, w'_k)\) in \(\overline{Tr}(SP_q)\), and thus in \(Tr(SP_q)\).

- Finally consider the triangles \((u_i, v_j, w_k)\) and \((u'_i, v'_j, w'_k)\) with one edge of sign \(S\) and 2 edges of sign \(-S\). Let \(u_i\) and \(u'_i\) be the vertex incident to the edges of sign \(-S\). Consider the triangles \((atw(u_i), v_j, w_k)\) and \((atw(u'_i), v'_j, w'_k)\); they have 3 edges of sign \(S\). By the two previous cases, there exists an automorphism \(\psi\) that maps \((atw(u_i), v_j, w_k)\) to \((atw(u'_i), v'_j, w'_k)\). Since \(\psi\) preserves anti-twinning, \(\psi\) also maps \(u_i\) to \(u'_i\).

\[
\Box
\]

### 3.5 Coloring properties of target graphs

A **signed vector** of size \(k\) is a \(k\)-tuple \(\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \{+1, -1\}^k\). For a given signed vector \(\alpha\), its **conjugate** is the \(k\)-tuple \(\overline{\alpha} = (-\alpha_1, -\alpha_2, \ldots, -\alpha_k)\).

Given a sequence of \(k\) distinct vertices \(X_k = (v_1, v_2, \ldots, v_k)\) of a signified graph \((G, \Sigma)\) that induces a clique, a vertex \(u \in V(G)\) is an \(\alpha\)-**successor** of \(X_k\) if, for every \(i \in \{1, 2, \ldots, k\}\), the sign of the edge \(uv_i\) is \(\alpha_i\). The set of \(\alpha\)-successors of \(X_k\) is denoted by \(S^\alpha(X_k)\).
Consider the signified graph $SP_3$ depicted in Figure 3. For example, given $\alpha = (+1, -1)$ and $X = (0, 3)$, the vertex 1 is an $\alpha$-successor of $X$, the vertex 2 is an $\bar{\alpha}$-successor of $X$, we have $S^\alpha(X) = \{1\}$ and $S^{\bar{\alpha}}(X) = \{2\}$.

A signified graph $(G)$ has property $P_{k,l}$ if $|S^\alpha(X_k)| \geq l$ for any sequence $X_k$ of $k$ distinct vertices inducing a clique of $G$ and for any signed-vector $\alpha$ of size $k$.

**Lemma 9** If $SP_q$ has property $P_{n-1,k}$, then $Tr(SP_q)$ has property $P_{n,k}$.

**Proof.** Suppose that $SP_q$ has property $P_{n-1,k}$ and let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ be a given signed vector. Let $X = (u_1, u_2, \ldots, u_{n-1}, w)$ be $n$ distinct vertices inducing a clique of $Tr(SP_q)$. We have to prove that $X$ admits $k$ $\alpha$-successors. By noticing that $S^\bar{\alpha}(X) = atw(S^\alpha(X))$, we restrict the proof to the case $\alpha_n = +1$. We define $X' = (v_1, v_2, \ldots, v_{n-1}, v)$ such that $v_i = u_i$ if $u_iw$ is a positive edge and $v_i = atw(u_i)$ if $u_iw$ is a negative edge. Hence, $X'$ is a set of $n$ distinct vertices of $Tr(SP_q)$ such that $\bigcup v_i \subseteq N^+(w)$. By Lemma 6, $Tr(SP_q)$ is vertex-transitive and thus $N^+(w) \cong K_q \cong SP_q$. Therefore the $(n-1)$ vertices $X'' = X' \setminus \{w\} = (v_1, v_2, \ldots, v_{n-1})$ form a subset of some $V(SP_q)$. Then by Property $P_{n-1,k}$ of $SP_q$, there exist $k$ $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-1})$-successors $x_1, x_2, \ldots, x_k$ of $X''$ in $SP_q$, with $\alpha'_i = \alpha_i$ (resp. $\alpha'_i = -\alpha_i$) if $v_i = u_i$ (resp. if $v_i = atw(u_i)$). The $x_i$'s are clearly positive neighbors of $w$ and hence, they are $(\alpha'_1, \alpha'_2, \ldots, \alpha'_{n-1}, \alpha_n)$-successors of $X'$. So $X$ has $k$ $\alpha$-successors. \qed

**Lemma 10** If $(G, \Sigma)$ is a signified graph and $Tr(G, \Sigma)$ has property $P_{n,k}$, then $AT(G, \Sigma)$ has property $P_{n,k-1}$.

**Proof.** Recall that $Tr(G, \Sigma)$ is built from two isomorphic copies of $(G, \Sigma)$ plus two vertices $\infty_0$ and $\infty_1$. The graph $AT(G, \Sigma)$ is obtained from $Tr(G, \Sigma)$ by removing both $\infty_0$ and $\infty_1$. Now suppose $Tr(G, \Sigma)$ has property $P_{n,k}$. This means that any $n$ distinct vertices inducing a clique in $Tr(G, \Sigma)$ has $k$ $\alpha$-successors for any signed $n$-vector $\alpha$. Let $X \subseteq V(Tr(G, \Sigma))$ be any sequence of $n$ distinct vertices inducing a clique in $Tr(SP_q)$ such that both $\infty_0$ and $\infty_1$ do not belong to $X$. Then, for any signed $n$-vector $\alpha$, the set of $k$ $\alpha$-successors $S^\alpha(X)$ cannot contains both $\infty_0$ and $\infty_1$. Then, it is clear that $X$ has at least $k-1$ $\alpha$-successors in $AT(G, \Sigma)$. \qed

**Lemma 11**

1. $SP_q$ has properties $P_{1,\frac{q-1}{2}}$ and $P_{2,\frac{q-3}{4}}$.
2. $Tr(SP_q)$ has properties $P_{1,q}$, $P_{2,\frac{q-1}{2}}$, and $P_{3,\frac{q-5}{4}}$.
3. $AT(SP_q)$ has properties $P_{1,q-1}$, $P_{2,\frac{q-3}{4}}$, and $P_{3,\text{max}(0,\frac{q-2}{4})}$.

**Proof.**

1. These properties follow from the fact that the signified Paley graph $SP_q$ is built from the Paley graph $P_q$ which is self-complementary, edge transitive and strongly regular with parameters $(q, \frac{q-1}{2}, \frac{q-5}{4}, \frac{q-1}{4})$.
2. $Tr(SP_q)$ has property $P_{1,q}$ since it is vertex transitive by Lemma 6 and the vertex $\infty$ has $q$ positive and $q$ negative neighbors. The other properties follow from (1) and Lemma 9.
These properties follow from (2) and Lemma 10.

4 Results on signified homomorphisms

This section is devoted to study signified homomorphisms of planar graphs and outerplanar graphs.

An acyclic $k$-coloring is a proper vertex-coloring such that each cycle has at least three colors. In other words, the graph induced by any two color classes is a forest.

In 1998, Alon and Marshall [1] proved the following (the signified graph $ZS_k$ has $k \cdot 2^{k-1}$ vertices and has been considered in Section 3.2):

**Theorem 12 ([1])** Let $(G, \Sigma)$ be such that $G$ admits an acyclic $k$-coloring. Then $(G, \Sigma)$ admits a signified homomorphism to $ZS_k$ and thus $\chi_2(G) \leq k \cdot 2^{k-1}$.

Note that Theorem 12 is actually tight as shown by Huemer et al. [4] in 2008.

The girth of a graph is the length of a shortest cycle. We denote by $P_g$ (resp. $O_g$) the class of planar graphs (resp. outerplanar graphs) with girth at least $g$ (note that $P_3$ is simply the class of planar graphs).

Borodin [2] proved that every planar graph admits an acyclic 5-coloring. We thus get the following from Theorem 12:

**Corollary 13** Every planar graph admits a signified homomorphism to $ZS_5$. We thus have $\chi_2(P_3) \leq 80$.

In this same context, Montejano et al. [6] obtained in 2010 the following results:

**Theorem 14 ([6])**

1. Every planar graph of girth at least 5 admits a signified homomorphism to $Tr(SP_5)$. We thus have $\chi_2(P_5) \leq 20$.
2. Every planar graph of girth at least 6 admits a signified homomorphism to $Tr(SP_6)$. We thus have $\chi_2(P_6) \leq 12$.

In this section, we get the following new results. We obtain a first result on outerplanar graph with girth at least 4 (see Theorem 15) and a second one on planar graph with girth at least 4 (see Theorem 16). The latter result gives a new upper bound on the signified chromatic number. We then construct a planar graph with signified chromatic number 20 (see Theorem 18). We finally give properties that must verify the target graphs for outerplanar and planar graphs.

**Theorem 15** Every outerplanar graph with girth at least 4 admits a signified homomorphism to $AT(K^*_4)$, where $K^*_4$ denotes the complete graph on 4 vertices with exactly one negative edge.

**Proof.** Assume by contradiction that there exists a counterexample to the result and let $(H, \Lambda)$ be a minimal counterexample in term of number of ver-
Suppose \((H)\) contains a vertex \(u\) of degree at most 1. By minimality of \((H)\), the graph \((H') = (H \setminus \{u\})\) admits a signified homomorphism to \(AT(K_4^*)\). Since every vertex of \(AT(K_4^*)\) is incident to a positive and a negative edge, we can extend the signified homomorphism to \((H)\), a contradiction.

Suppose that \((H)\) contains two adjacent vertices \(u\) and \(v\) of degree 2. By minimality of \((H)\), the graph \((H' = (H \setminus \{u, v\})\) admits a signified homomorphism to \(AT(K_4^*)\). One can check that for every pair of (non necessarily distinct) vertices \(x\) and \(y\) of \(AT(SP_{25})\), there exist the 8 possible signified 3-paths. We can therefore extend the signified homomorphism to \((H)\), a contradiction.

Pinlou and Sopena [9] showed that every outerplanar graph with girth at least \(k\) and minimum degree at least 2 contains a face of length \(l = k\) with at least \((l - 2)\) consecutive vertices of degree 2. Therefore, the counterexample \((H)\) is not an outerplanar graph of girth 4, a contradiction, that completes the proof. 

\[\Box\]

**Theorem 16** Every planar graph with girth at least 4 admits a signified homomorphism to \(AT(SP_{25})\). We thus have \(\chi_2(P_4) \leq 50\).

Let \(n_3(G)\) be the number of \(\geq 3\)-vertices in the graph \(G\). Let us define the partial order \(\preceq\). Given two graphs \(G_1\) and \(G_2\), we have \(G_1 \prec G_2\) if and only if one of the following conditions holds:

- \(n_3(G_1) < n_3(G_2)\).
- \(n_3(G_1) = n_3(G_2)\) and \(|V(G_1)| + |E(G_1)| < |V(G_2)| + |E(G_2)|\).

Note that the partial order \(\preceq\) is well-defined and is a partial linear extension of the minor poset.

Let \((H)\) be a signified graph that does not admit a homomorphism to the signified graph \(AT(SP_{25})\) and such that its underlying graph \(H\) is a triangle-free planar graph which is minimal with respect to \(\preceq\). In the following, \(H\) is given with its embedding in the plane. A weak 7-vertex \(u\) in \(H\) is a 7-vertex adjacent to four \(\geq 2\)-vertices \(v_1, \ldots, v_4\) and three \(\geq 3\)-vertices \(w_1, w_2, w_3\) such that \(v_1, w_1, v_2, w_2, v_3, w_3, \text{ and } v_4\) are consecutive.

**Lemma 17** The graph \(H\) does not contain the following configurations:

- \((C1)\) a \(\leq 1\)-vertex;
- \((C2)\) a \(k\)-vertex adjacent to \(k\) \(\geq 2\)-vertices for \(2 \leq k \leq 49\);
- \((C3)\) a \(k\)-vertex adjacent to \((k - 1)\) \(\geq 2\)-vertices for \(2 \leq k \leq 24\);
- \((C4)\) a \(k\)-vertex adjacent to \((k - 2)\) \(\geq 2\)-vertices for \(3 \leq k \leq 12\);
- \((C5)\) a \(3\)-vertex;
- \((C6)\) a \(k\)-vertex adjacent to \((k - 3)\) \(\geq 2\)-vertices for \(4 \leq k \leq 6\);
- \((C7)\) two vertices \(u\) and \(v\) linked by two distinct 2-paths, both paths having a 2-vertex as internal vertex;
- \((C8)\) a 4-face \(wxyz\) such that \(x\) is \(\geq 2\)-vertex, \(w\) and \(y\) are weak 7-vertices, and \(z\) is a \(k\)-vertex adjacent to \((k - 4)\) \(\geq 2\)-vertices for \(4 \leq k \leq 9\);
\[2 \leq k \leq 49\]
\[v'_{1}\]
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\[v_{1}\]
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\[v_{46}\]
\[v_{47}\]
\[v_{48}\]
\[v_{49}\]

**Fig. 5. Configurations C2–C4.**

**Proof.** The drawing conventions for a configuration \(C_k\) contained in a signed graph \((H)\) are the following. First note that, in Figures 5 and 6, we only draw the underlying graph \(H\) of \((H)\), i.e. we do not distinguish positive and negative edges. The neighbors of a white vertex in \(H\) are exactly its neighbors in \(C_k\), whereas a black vertex may have other neighbors in \(H\). Two or more black vertices in \(C_k\) may coincide in a single vertex in \(H\), provided they do not share a common white neighbor. Configurations C2 - C8 are depicted in Figures 5 and 6.

For each configuration, we suppose that \(H\) contains the configuration and we consider a signified triangle-free graph \((H')\) such that \(H' \prec H\). We only argue that \(H' \prec H\) for configuration C5. For every other configuration, we have that \(H'\) is a minor of \(H\) and thus \(H' \prec H\). Therefore, by minimality of \((H)\), \((H')\) admits a signified homomorphism \(f\) to \(AT(SP_{25})\). Then we modify and extend \(f\) to obtain a signified homomorphism of \((H)\) to \(AT(SP_{25})\), contradicting the fact that \((H)\) is a counterexample.

By Lemma 11, \(AT(SP_{25})\) satisfies properties \(P_{1,24}, P_{2,11},\) and \(P_{3,4}\).

**Proof of configuration C1:** Trivial.

**Proof of configuration C2:** Suppose that \((H)\) contains the configuration depicted in Figure 5(a) and \(f\) is a signified homomorphism of \((H') = (H) \setminus \{v, v_1, \cdots, v_k\}\) to \(AT(SP_{25})\). For every \(i\), if the edges \(vv_i\) and \(v_i v'_i\) have the same sign (resp. different signs), then \(v\) must get a color distinct from \(\text{atw}(f(v'_i))\) (resp. \(f(v'_i))\). So, each \(v'_i\) forbids at most one color for \(v\). Thus there remains an available color for \(v\). Then we extend \(f\) to the vertices \(v_i\) using property \(P_{2,11}\).

**Proof of configuration C3:** Suppose that \((H)\) contains the configuration depicted in Figure 5(b) and \(f\) is a signified homomorphism of \((H') = (H) \setminus \{v, v_2, \cdots, v_k\}\) to \(AT(SP_{25})\). As shown in the proof of Configuration C2, each \(v'_i\) forbids at most one color for \(v\). So, we have at most 23 forbidden colors for \(v\) and by property \(P_{2,24}\), there remains at least one available color for \(v\). Then we extend \(f\) to the vertices \(v_i\) \((2 \leq i \leq k)\) using property \(P_{2,11}\).

**Proof of configuration C4:** Suppose that \((H)\) contains the configuration depicted in Figure 5(c) and \(f\) is a signified homomorphism of \((H') = (H) \setminus \{v_3, \cdots, v_k\}\) to \(AT(SP_{25})\). As shown in the proof of Configuration C2, each \(v'_i\) forbids at most one color for \(v\). So, we have at most 10 forbidden colors for \(v\) and by property \(P_{2,11}\), there remains at least one available color in order to recolor \(v\). Then we extend \(f\) to the vertices \(v_i\) \((3 \leq i \leq k)\) using property...
Proof of configuration C5: Suppose that (H) contains the configuration depicted in Figure 6(a). Let (H') be the graph obtained from (H) by deleting the vertex v and by adding, for every 1 ≤ i < j ≤ 3, a new vertex v_{ij} and the edges v_i v_{ij} and v_{ij} v_j. Each of the 6 edges v_i v_{ij} gets the sign α_i of the edge v_i v in (H). As configuration C4 is forbidden in H, we have d_H(v_i) ≥ 3 for i ∈ {1, 2, 3}. We have H' ≺ H since n_3(H') < n_3(H). Clearly, H' is triangle free. Hence, there exists a signified homomorphism f of (H') to AT(SP_{25}). By P_{3,4}, we can find an α-successor u of (f(v_1), f(v_2), f(v_3)) in AT(SP_{25}) with α = (α_1, α_2, α_3). Now fix f(v) = u. Note that f restricted to V(H) is a homomorphism of (H) to AT(SP_{25}).

Proof of configuration C6: Suppose that (H) contains the configuration depicted in Figure 6(b) and f is a signified homomorphism of (H') = (H) \ {v_4, ..., v_k} to AT(SP_{25}). As shown in the proof of Configuration C2, each v'_i forbids at most one color for v. So, we have at most 3 forbidden colors for v and by property P_{3,4}, there remains at least one available color for v. Then we extend f to the vertices v_i (4 ≤ i ≤ k) using property P_{2,11}.

Proof of configuration C7: Suppose that (H) contains the configuration depicted in Figure 6(c).

If u and w have no common neighbor other than v_1 and v_2, then we consider the graph (H') obtained from (H) \ {v_1, v_2} by adding the edge uw.

If u and w have at least one other common neighbor v_3, then consider the graph (H') obtained from (H) \ {v_1, v_2} by adding a vertex v adjacent to u and w such that uw is negative and the sign of vw is the product of the signs of uv_3 and v_3w. Therefore, we have at least two 2-paths linking u and w, one whose both edges have the same sign and one whose edges have different sign.

In both cases, H' is triangle free and is a minor of H, so that (H') admits a signified homomorphism f to AT(SP_{25}). Also, in both cases, f(u) and f(w) form an edge in AT(SP_{25}) since f(u) ≠ f(v) and f(u) ≠ f(aw(v)). Thus, the coloring of (H) \ {v_1, v_2} induced by f can be extended to (H) using property P_{2,11}.
Table 7. Sets of the form \((0, 1, x_0)\) having exactly 4 \((+1,+1,+1)\)-successors in \(AT(SP_{25})\).

Proof of configuration C8: Suppose that \((H)\) contains the configuration depicted in Figure 6(d). By Corollary 4(2), \((H)\) admits a signified homomorphism to \(AT(SP_{25})\) if and only if every equivalent signature of \((H)\) admits a signified homomorphism to \(AT(SP_{25})\). So, by resigning a subset of vertices in \(\{a, b, c, c_1, c_2\}\), we can assume that the edges \(da, ab, bc, cc_1,\) and \(cc_2\) are positive. Consider a signified homomorphism \(f\) of \((H') = (H \setminus \{d\})\) to \(AT(SP_{25})\). The edge \(dc\) in \((H)\) has to be negative, since otherwise \(f\) would be extendable to \((H)\) by setting \(f(d) = f(b)\). Also, we must have \(f(c) = f(a)\), since otherwise we could color \(d\) using property \(P_{2,11}\). Now, we show in the remaining of the proof that we can modify \(f\) such that \(f(e) \neq f(a)\). Let us define the signed vector \(\alpha = (+1,+1,+1)\). We assume that \(f(b), f(c_1)\) and \(f(c_2)\) are distinct, since the case when they are not distinct is easier to handle.

For \(1 \leq i \leq 3\), let \(k_i\) denote the color that is forbidden for \(c\) by \(w_i\), that is, \(k_i = f(w_i)\) if the edges of the 2-path linking \(c\) and \(w_i\) have distinct signs and \(k_i = atw(f(w_i))\) otherwise. By property \(P_{3,4}\), the sequence \(X = (f(c_1), f(c_2), f(b))\) has at least 4 \(\alpha\)-successors. Assume that \(X\) has at least 5 \(\alpha\)-successors. Then we can give to \(c\) a color distinct from \(k_1, k_2, k_3\), and \(f(a)\) that leads to \(f(c) \neq f(a)\).

Assume now that \(X\) has exactly 4 \(\alpha\)-successors. The graph \(AT(SP_{25})\) contains two copies of \(SP_{25}\), namely \(SP_{25}^0\) and \(SP_{25}^1\). Suppose that \(X\) is not contained in one copy of \(SP_{25}\). We consider the graph \(Tr(SP_{25})\) obtained by adding the anti-twin vertices \(\infty_0\) and \(\infty_1\) to \(AT(SP_{25})\). By Lemma 11, \(Tr(SP_{25})\) satisfies \(P_{3,5}\), so \(X\) admits at least 5 \(\alpha\)-successors in \(Tr(SP_{25})\). Since \(X\) is not contained in one copy of \(SP_{25}\) of the subgraph \(AT(SP_{25})\), the extra vertices \(\infty_0\) and \(\infty_1\) are not \(\alpha\)-successors of \(X\). This means that \(X\) has at least 5 \(\alpha\)-successors in \(AT(SP_{25})\), contradicting the hypothesis. So, without loss of generality, \(X\) is necessarily contained in \(SP_{25}^0\).

We represent the field \(F_{25}\) by the numbers \(a + b\sqrt{2}\), where \(a\) and \(b\) are integers modulo 5. Without loss of generality, we can assume that \(f(c_1) = 0\) and \(f(c_2) = 1\) since \(SP_{25}\) is edge-transitive. Table 7 gives the sequences \(X\) having exactly 4 \(\alpha\)-successors together with their 4 \(\alpha\)-successors.

We are now ready to modify \(f\). We decolor the vertices \(a, b,\) and \(c\). By property

| \((0, 1, 2_0)\) | \(3_0, 4_0, (1 + 2\sqrt{2})_1, (1 + 3\sqrt{2})_1\) |
|\((0, 1, 3_0)\) | \(2_0, 4_0, (3 + \sqrt{2})_1, (3 + 4\sqrt{2})_1\) |
|\((0, 1, 4_0)\) | \(2_0, 3_0, (3\sqrt{2})_1, (3\sqrt{2})_1\) |
|\((0, 1, (3 + 2\sqrt{2})_0)\) | \((3 + \sqrt{2})_1, (3\sqrt{2})_1, (1 + 3\sqrt{2})_1, (3 + 4\sqrt{2})_1\) |
|\((0, 1, (3 + 3\sqrt{2})_0)\) | \((3 + \sqrt{2})_1, (2\sqrt{2})_1, (1 + 2\sqrt{2})_1, (3 + 4\sqrt{2})_1\) |
Forbidden by property \( \alpha \), the sequences (0, 1, \( \beta \)) and (0, 1, \( \beta' \)) must have exactly 4 \( \alpha \)-successors, so we can assume that \( \{ \beta, \beta' \} \subset \{ 2, 3, 4, 3 + 2\sqrt{2}, 3 + 3\sqrt{2} \} \). Let us set \( f(b) = \beta \).

By property \( P_{3,4} \), we can color \( a \) such that \( f(a) \) is distinct from the colors forbidden by \( u_1, u_2, u_3 \). Now, \( f \) is not extendable to \( c \) and \( d \) only if the 4 \( \alpha \)-successors of (0, 1, \( \beta \)) are \( k_1, k_2, k_3, \) and \( f(a) \). In particular, \( k_1, k_2, \) and \( k_3 \) have to be \( \alpha \)-successors of (0, 1, \( \beta \)). Similarly, we can set \( f(b) = \beta' \) and obtain that \( k_1, k_2, \) and \( k_3 \) have to be \( \alpha \)-successors of (0, 1, \( \beta' \)) as well. This is a contradiction, since we can observe that no two distinct sequences in Table 7 have three common \( \alpha \)-successors.

**Proof of Theorem 16.** Let \( (H) \) be a minimal counterexample which is minimal with respect to \( \preceq \). By Lemma 17, \( (H) \) does not contain Configurations C1 to C8. There remains to show that every triangle-free planar graph contains at least one these 8 configurations. This has been already done using a discharging procedure in the proof of Theorem 2 in [8], where slightly weaker configurations were used.

We now exhibit the following lower bound for the signified chromatic number of planar graphs:

**Theorem 18** There exist planar graphs with signified chromatic number 20.

**Proof.** Let \((G_1, \Sigma_1)\) be the 6-path \( abcdfe \) with \( \Sigma = \{bc, de\} \) and let \((G_2, \Sigma_2)\) be the 6-path \( abcdfe \) with \( \Sigma = \{ab, cd, ef\} \). Note that \( \chi_2(G_1, \Sigma_1) = \chi_2(G_2, \Sigma_2) = 4 \).

Let \((G_3)\) be the outerplanar graph obtained from \((G_1, \Sigma_1), (G_2, \Sigma_2)\) and a vertex \( u \) such that \( u \) is positively (resp. negatively) linked to the six vertices of \((G_1)\) (resp. \((G_2)\)). For any signified coloring of \((G_3)\), we need 4 colors for the vertices of \((G_1)\), 4 other colors for the vertices of \((G_2)\), and a ninth color for \( u \). We therefore have \( \chi_2(G_3) = 9 \).

Let \((G_4)\) be the graph obtained from two copies of \((G_3)\) plus a vertex \( v \) such that \( v \) is positively linked to 13 vertices of the first copy of \((G_3)\) and negatively linked to the 13 vertices of the second one. This graph is depicted in Figure 8. Once again, it is easy to check that \( \chi_2(G_4) = 19 \).

Finally, let \((G_5)\) be the graph obtained from 28 copies \((G_4)_0, (G_4)_1, \ldots, (G_4)_{27}\) of \((G_4)\) as follows: we glue on each of the 27 vertices of \((G_4)_0\) the vertex \( v \) of a copy \((G_4)\). Since \( \chi_2(G_4) = 19 \), we have \( \chi_2(G_5) \geq 19 \). Suppose, \( \chi_2(G_5) = 19 \). Therefore, there exists a graph \((H)\) on 19 vertices such that \((G_5)\) admits a signified homomorphism to \((H)\). Moreover, since we glued a copy of \((G_4)\) on each vertices of \((G_4)_0\), then each color (i.e. each vertex of \((H)\)) must have 9 distinct positive neighbors and 9 distinct negative neighbors. The subgraph of \((H)\) induced by the positive edges is a 9-regular graph on 19 vertices. Such a graph does not exist since, in every graph, the number of vertices of odd degree must be even. 

\( \square \)
Montejano et al. [6] proved that any outerplanar graph admits a signified homomorphism to $SP_9$, that gives $\chi_2(G) \leq 9$ whenever $G$ is an outerplanar graph. They also proved that this bound is tight. We prove here that $SP_9$ is the only suitable target graph on 9 vertices.

**Theorem 19** The only graph of order 9 to which every outerplanar graph admits a signified homomorphism is $SP_9$.

**Proof.** Let $(G_1)$, $(G_2)$ and $(G_3)$ be the graphs constructed in the proof of Theorem 18.

Note that since $\chi_2(G_1, \Sigma_1) = 4$, then for any signified 4-coloring of $(G_1, \Sigma_1)$, among the three positive edges $ab$, $cd$ and $ef$, two of them will use 4 distinct colors. We will later refer to this property in the remainder of this proof as Property 1. Using the same arguments as the previous paragraph, we have that, for any signified 4-coloring of $(G_2, \Sigma_2)$, among the three negative edges $ab$, $cd$ and $ef$, two of them will use 4 distinct colors. We will later refer to this property in the remainder of this proof as Property 2.

Let $(G'_4)$ be the outerplanar graph obtained from 14 copies $(G_3)_{0}, (G_3)_{1}, \ldots, (G_3)_{14}$ of $(G_3)$ as follows: we glue on each of the 13 vertices of $(G_3)_0$ the vertex $u$ of a copy $(G_3)_i$. Since $\chi_2(G) \leq 9$ whenever $G$ is outerplanar, we have $\chi_2(G'_4) = 9$. Therefore, there exists a graph $(H_9)$ on 9 vertices such that $(G'_4)$ admits a signified homomorphism to $(H_9)$. Each of the nine colors appears on the vertices of the copy $(G_3)_0$. Since we glued a copy of $(G_3)$ on each vertex of $(G_3)_0$, then each color $c$ (i.e. each vertex $c$ of $(H_9)$) must have 4 positive neighbors $c^+_1, c^+_2, c^+_3, c^+_4$ and 4 negative neighbors $c^-_1, c^-_2, c^-_3, c^-_4$. Moreover, by Property 1 (resp. Property 2), we must have a positive (resp. negative) matching in the subgraph induced by $c^+_1, c^+_2, c^+_3, c^+_4$ (resp. $c^-_1, c^-_2, c^-_3, c^-_4$).

Meringer [5] provided an efficient algorithm to generate regular graphs with a given number of vertices and vertex degree. In particular, there exist 16

Fig. 8. A planar graph with signified chromatic number 19
connected 4-regular graphs on 9 vertices (see Figure 9). Replacing edges by positive edges and non-edges by negative edges, these are the 16 signified graphs such that each vertices have 4 positive and 4 negative neighbors. It is then easy to check that, among these 16 graphs, there is only one graph with a positive (resp. negative) matching in the subgraph induced by the positive (resp. negative) neighbors of each vertex (see Figure 9(n)): this is $SP_9$.

We finally give the same kind of result as Theorem 19 for planar graphs:

**Theorem 20** If every planar graphs admits a signified homomorphism to an anti-twinned graph $H_{20}$ of order 20, then $H_{20}$ is isomorphic to $Tr(SP_9)$.

**Proof.** Consider the outerplanar graph $(G_4')$ of the proof of Theorem 19. Let
(\(G'_5\)) be the planar graph obtained from \((G'_4)\) plus a universal vertex positively linked to all the vertices of \((G'_4)\).

The subgraph of \((G'_5)\) induced by the vertices of \((G'_4)\) necessarily maps to the positive neighborhood of some vertex \(v\) of \((H_{20})\). Since \((H_{20})\) is anti-twinned, every vertex has exactly 9 positive neighbors. Therefore, by Theorem 19, the positive neighborhood of \(v\) is isomorphic to \(SP_9\). Then the subgraph of \((H_{20})\) induced by \(v\) and its positive neighborhood is \(SP_9^+\). Since \(SP_9^+\) is a clique of order 10, it does not contain a pair of anti-twin vertices and thus \((H_{20})\) is isomorphic to \(AT(SP_9^+)\). Then, by definition of Tromp signified Paley graphs, \((H_{20})\) is isomorphic to \(Tr(SP_9)\).

5 Results on signed homomorphisms

The upper bounds on signified chromatic number given in Theorems 12, 14, 15 and 16 are obtained by showing that the considered graph class admits a signified homomorphism to some target graph. As mentioned in Corollary 4(1), when the target graph of a signified homomorphism is an anti-twinned graph, this gives a bound on the signed chromatic number. The upper bounds on signed chromatic number given in this section are a direct consequence of these above-mentioned results.

Naserasr et al. [7] proved the following:

**Theorem 21 ([7])** Let \(G\) be a graph that admits an acyclic \(k\)-coloring. We have \(\chi_s(G) \leq \left\lceil \frac{k}{2} \right\rceil \cdot 2^k - 1\).

By Theorem 12, \((G, \Sigma)\) admits a signified homomorphism to \(ZS_k\) whenever \(G\) admits an acyclic \(k\)-coloring. By Proposition 5 and Corollary 4(1), we get the following new upper bound that improves Theorem 21.

**Theorem 22** Let \(G\) be a graph that admits an acyclic \(k\)-coloring. We have \(\chi_s(G) \leq k \cdot 2^{k-2}\).

Recall that Huemer et al. [4] proved that Theorem 12 is tight, i.e. there exists a planar graph \((G)\) such that \(\chi_2(G) = k \cdot 2^{k-1}\). By Lemma 3, we can deduce that \(\chi_s(G) \geq k \cdot 2^{k-2}\), showing that Theorem 22 is actually tight.

We also get the following upper bounds:

**Theorem 23**

\((1)\) \(\chi_s(O_4) \leq 4\).
\((2)\) \(\chi_s(P_3) \leq 40\).
\((3)\) \(\chi_s(P_4) \leq 25\).
\((4)\) \(\chi_s(P_5) \leq 10\).
\((5)\) \(\chi_s(P_6) \leq 6\).

**Proof.**

\((1)\) The result follows from Theorem 15 and Corollary 4(1).
\((2)\) By Theorem 12 and Borodin’s result [2], every planar graph admits a signified homomorphism to \(ZS_5\), a graph on 80 vertices, which is anti-
twinced by Proposition 5. The result then follows from Corollary 4(1).

(3) The result follows from Theorem 16 and Corollary 4(1).
(4) The result follows from Theorem 14(1) and Corollary 4(1).
(5) The result follows from Theorem 14(2) and Corollary 4(1).

Concerning lower bounds, Naserasr et al. [7] constructed a planar graph with signed chromatic number 10. Note that this result also follows from Theorem 18 and Corollary 4(1). If 10 is the tight bound for signed chromatic number of planar graphs, we then get the following from Theorem 20 and Lemma 3:

**Theorem 24** If every planar graphs admits a signed homomorphism to a graph $H_{10}$ of order 10, then $H_{10}$ is isomorphic to $SP_6^+$.  

We can easily construct a planar graph of girth 4 with signed chromatic number 6 (see Figure 10). Finally, for higher girths, note that every even cycle with exactly one negative edge needs 4 colors for any signed coloring.

6 Conclusion

One of our aims was to introduce and study some relevant target graphs for signified homomorphisms. We studied the anti-twinced graph $AT(G, \Sigma)$, the signified Zielonka graph $ZS_k$, the signified Paley graph $SP_q$, and the signified Tromp Paley graph $Tr(SP_q)$. Theorems 19, 20 and 24 suggest that such target graphs are indeed significant.

We proved that there exist planar graphs with signified chromatic number 20 and ask whether this bound is tight:

**Open Problem 25** Does every planar graph admit a signified homomorphism to $Tr(SP_9)$?

We have checked by computer that every 4-connected planar triangulation with at most 15 vertices admits a homomorphism to $Tr(SP_9)$. The restriction to 4-connected triangulations (i.e. triangulations without separating triangles) is justified by Lemma 8. For the $2^{25}$ non-equivalent signatures of each of the 6244 4-connected planar triangulations with 15 vertices, our computer check took 150 CPU-days. Checking 4-connected triangulations with more vertices would require too much computing power.

Finally, it would be nice to drop the condition “anti-twinced” in Theorem 20.

References


Fig. 10. A planar graph of girth 4 with signed chromatic number 6.


