An oriented coloring of planar graphs with girth at least five

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ABSTRACT

An oriented k-coloring of an oriented graph G is a homomorphism from G to an oriented graph H of order k. We prove that every oriented graph with a maximum average degree less than 10 and girth at least 5 has an oriented chromatic number at most 16. This implies that every oriented planar graph with girth at least 5 has an oriented chromatic number at most 16, that improves the previous known bound of 19 due to Borodin et al. [O.V. Borodin, A.V. Kostochka, J. Nešetřil, A. Raspaud, É. Sopena, On the maximum average degree and the oriented chromatic number of a graph, Discrete Math. 206 (1999) 77–89].

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1. Introduction

Oriented graphs are directed graphs without loops or opposite arcs. For an oriented graph G, we denote by V(G) the vertices and by A(G) its set of arcs. For two adjacent vertices u and v, we denote by u→v the arc from u to v or simply uv whenever its orientation is not relevant (therefore, uv = u→v or u→v = v→u). The number of vertices of G is the order of G.

An oriented k-coloring of an oriented graph G is a mapping ϕ from V(G) to a set of k colors such that (1) ϕ(u) ≠ ϕ(v) whenever u→v is an arc in G, and (2) ϕ(u) ≠ ϕ(x) whenever u→v and u→x are two arcs in G with ϕ(v) = ϕ(w). In other words, an oriented k-coloring of G is a partition of the vertices of G into k stable sets S1, S2, ..., Sk such that all the arcs between any pair of stable sets Si and Sj have the same direction (either from Si to Sj, or from Sj to Si). The oriented chromatic number of an oriented graph, denoted by χ(ω)(G), is defined as the smallest k such that G admits an oriented k-coloring.

Let G and H be two oriented graphs. A homomorphism from G to H is a mapping ϕ : V(G) → V(H) that preserves the arcs: ϕ(x)ϕ(y) ∈ A(H) whenever xy ∈ A(G).

An oriented k-coloring of G can be equivalently defined as a homomorphism from G to H, where H is an oriented graph of order k. The existence of such a homomorphism from G to H is denoted by G → H. The vertices of H are called colors, and we say that G is H-colorable. The oriented chromatic number of G can then be defined as the smallest order of an oriented graph H such that G → H. Links between colorings and homomorphisms are presented in more details in the recent monograph [6] by Hell and Nešetřil.

The notion of oriented coloring introduced by Courcelle [5] has been studied by several authors in the last decade and the problem of bounding the oriented chromatic number has been investigated for various graph classes: outerplanar graphs (with given girth) [13,15], planar graphs (with given girth) [1–4,10,14], graphs with bounded maximum average degree [3,4], graphs with bounded degree [7], graphs with bounded treewidth [11,15,16], and graph subdivisions [18].

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A very challenging question in this area is to determine the oriented chromatic number of planar graphs. Raspaud and Sopena [14] proved in 1994 that their oriented chromatic number is at most 80. Recently, Marshall [8] proved that there exist planar graphs with an oriented chromatic number at least 17. The gap between the lower and the upper bound is very large, but it seems very hard to reduce.

In this paper, we focus on the oriented chromatic number of graphs with bounded maximum average degree. The average degree of a graph $G$, denoted by ad$(G)$, is defined as twice the number of edges over the number of vertices ($\text{ad}(G) = \frac{2|E(G)|}{|V(G)|}$). The maximum average degree of $G$, denoted by mad$(G)$, is then defined as the maximum of the average degrees taken over all subgraphs of $G$:

$$\text{mad}(G) = \max_{H \subseteq G} \text{ad}(H).$$

The girth of a graph $G$ is the length of a shortest cycle of $G$.

Borodin et al. [3,4] gave bounds of the oriented chromatic number of graphs with bounded maximum average degree:

**Theorem 1** ([3,4]). Let $G$ be a graph.

1. If $\text{mad}(G) < \frac{12}{5}$ and $G$ has girth at least 5, then $\chi_o(G) \leq 5$ [3].
2. If $\text{mad}(G) < \frac{11}{4}$ and $G$ has girth at least 5, then $\chi_o(G) \leq 7$ [4].
3. If $\text{mad}(G) < 3$, then $\chi_o(G) \leq 11$ [4].
4. If $\text{mad}(G) < \frac{10}{3}$, then $\chi_o(G) \leq 19$ [4].

We focus here on the class of graphs with maximum average degree less than $\frac{10}{3}$ and girth at least 5. The main result of this paper is given by the following theorem:

**Theorem 2.** Let $G$ be a graph with $\text{mad}(G) < \frac{10}{3}$ and girth at least 5. Then $\chi_o(G) \leq 16$.

Actually, we prove a stronger result: we show that every oriented graph $G$ with $\text{mad}(G) < \frac{10}{3}$ and girth at least 5 admits a homomorphism to $T_{16}$, where $T_{16}$ is the Tromp graph of order 16 whose construction is described in Section 2.

When considering planar graphs, the maximum average degree and the girth are linked by the following well-known relation:

**Claim 3** ([4]). Let $G$ be a planar graph with girth $g$. Then, $\text{mad}(G) < 2 + \frac{4}{g-2}$.

In particular, by means of Theorem 1(4), we get as a corollary that every planar graph with girth at least 5 has an oriented chromatic number at most 19. Theorem 2 improves this bound and gives that every planar graph with girth at least 5 has an oriented chromatic number at most 16.

The best current knowledge for the upper bounds of the oriented chromatic number of planar graphs is then the following:

**Theorem 4** ([1–4,11,14]). Let $G$ be a planar graph.

1. If $G$ has girth at least 12, then $\chi_o(G) \leq 5$ [3].
2. If $G$ has girth at least 11, then $\chi_o(G) \leq 6$ [11].
3. If $G$ has girth at least 7, then $\chi_o(G) \leq 7$ [1].
4. If $G$ has girth at least 6, then $\chi_o(G) \leq 11$ [4].
5. If $G$ has girth at least 5, then $\chi_o(G) \leq 16$.
6. If $G$ has girth at least 4, then $\chi_o(G) \leq 47$ [2].
7. If $G$ has no restriction of girth, then $\chi_o(G) \leq 80$ [14].

Note that among the bounds of the previous theorem, only the one for girth 12 is tight.

In the remainder, we use the following notations. For a vertex $v$ of a graph $G$, we denote by $d_v^-(G)$ its indegree, by $d_v^+(G)$ its outdegree, and by $d_v(G)$ its degree (subscripts are omitted when the considered graph is clearly identified from the context). We denote by $N_v^+(G)$ the set of outgoing neighbors of $v$, by $N_v^-(G)$ the set of incoming neighbors of $v$ and by $N_v(G) = N_v^+(G) \cup N_v^-(G)$ the set of neighbors of $v$. A vertex of degree $k$ (resp. at least $k$, at most $k$) is called a $k$-vertex (resp. $\geq k$-vertex, $\leq k$-vertex). If a vertex $u$ is adjacent to a $k$-vertex (resp. $\geq k$-vertex, $\leq k$-vertex) $v$, then $v$ is a $k$-neighbor (resp. $\geq k$-neighbor, $\leq k$-neighbor) of $u$. A path of length $k$ (i.e. formed by $k$ edges) is called a $k$-path. If two graphs $G$ and $H$ are isomorphic, we denote it by $G \cong H$.

The paper is organized as follows. The next section is devoted to the target graph $T_{16}$ and some of its properties. We prove Theorem 2 in Section 3. We finally give some concluding remarks in the last section.
2. The Tromp graph $T_{16}$

In this section, we describe the construction of the target graph $T_{16}$ used to prove Theorem 2 and give some useful properties.

*Tromp's construction* was proposed by Tromp [17]. Let $G$ be an oriented graph and $G'$ be an isomorphic copy of $G$. The Tromp graph $Tr(G)$ has $2|V(G)| + 2$ vertices and is defined as follows:

- $V(Tr(G)) = V(G) \cup V(G') \cup \{\infty, \infty'\}$
- $\forall u \in V(G): \overrightarrow{uv}, \overrightarrow{vu}, \overrightarrow{vu'}$, $\overrightarrow{vu'} \in A(Tr(G))$
- $\forall u, v \in V(G), \overrightarrow{uv} \in A(G): \overrightarrow{uv}, \overrightarrow{vu}, \overrightarrow{vu'}, \overrightarrow{vu'} \in A(Tr(G))$.

Fig. 1 illustrates the construction of $Tr(G)$. We can observe that, for every $u \in V(G) \cup \{\infty\}$, there is no arc between $u$ and $u'$. Such pairs of vertices will be called twin vertices, and we denote by $t(u)$ the twin vertex of $u$. Remark that $t(t(u)) = u$. This notion can be extended to sets in a standard way: for a given $W \subseteq V(G)$, $W = \{v_1, v_2, \ldots, v_k\}$, then $t(W) = \{t(v_1), t(v_2), \ldots, t(v_k)\}$.

By construction, the graph $Tr(G)$ satisfies the following property:

\[
\forall u \in Tr(G): N^+(u) = N^-(t(u)) \quad \text{and} \quad N^-(u) = N^+(t(u)).
\]

In the remainder, we focus on the specific graph family obtained via the Tromp’s construction applied to Paley tournaments. For a prime power $p \equiv 3 \pmod{4}$, the *Paley tournament* $QR_p$ is defined as the oriented graph whose vertices are the integers modulo $p$ and such that $\overrightarrow{uv}$ is an arc if and only if $v - u$ is a non-zero quadratic residue of $p$. For instance, the Paley tournament $QR_7$ has vertex set $V(QR_7) = \{0, 1, \ldots, 6\}$ and $\overrightarrow{uv} \in A(QR_7)$ whenever $v - u \equiv r \pmod{7}$ for $r \in \{1, 2, 4\}$; see Fig. 2. Note that the bounds of Theorems 1(2), 1(3), and 1(4) have been obtained by proving that all the graphs of the considered classes admit a homomorphism to the Paley tournaments $QR_7$, $QR_{11}$, and $QR_{19}$, respectively.

Let $T_{16} = Tr(QR_7)$ be the Tromp graph on sixteen vertices obtained from $QR_7$. In the remainder of this paper, the vertex set of $T_{16}$ is $\{0, 1, \ldots, 6, \infty, 0', 1', \ldots, 6', \infty'\}$ where $\{0, 1, \ldots, 6\}$ is the vertex set of the first copy of $QR_7$ and $\{0', 1', \ldots, 6'\}$ is the vertex set of the second copy of $QR_7$; thus, for every $u \in \{0, 1, \ldots, 6, \infty\}$, we have $t(u) = u'$. In addition, for every $u \in V(T_{16})$, we have by construction $|N^+_T(u)| = |N^-_{T_{16}}(u)| = 7$. The graph $T_{16}$ has remarkable symmetry and some useful properties given below.

**Proposition 5** ([8]). For any $QR_p$, the graph $Tr(QR_p)$ is such that:

\[
\forall u \in V(Tr(QR_p)): N^+(u) \cong QR_p \quad \text{and} \quad N^-(u) \cong QR_p.
\]
Proposition 6 ([8]). For any QRₚ, if \{a₁, a₂, a₃\} and \{b₁, b₂, b₃\} span triangles \(t₁ \) and \( t₂ \) respectively in \( Tr(QRₚ) \) and the map \( \psi \) taking \( a_i \) to \( b_i \) (1 ≤ i ≤ 3) is an isomorphism \( t₁ \rightarrow t₂ \), then \( \psi \) can be extended to an automorphism of \( Tr(QRₚ) \).

It is then clear that \( Tr(QRₚ) \) is vertex-transitive and arc-transitive.

Proposition 7. Let \( G \) be an oriented graph such that \( G \rightarrow T₁₆ \). Then, for any vertex \( v \) of \( G \), the graph \( G' \) obtained from \( G \) by reversing the orientation of every arc incident to \( v \) admits a homomorphism to \( T₁₆ \).

Proof. Let \( \psi \) be a \( T₁₆ \)-coloring of \( G \). For every \( w \in V(T₁₆) \), we have \( N_{T₁₆}^+(w) = N_{T₁₆}^-(t(w)) \) and \( N_{T₁₆}^-(w) = N_{T₁₆}^+(t(w)) \). Therefore, the mapping \( \psi' : V(G') \rightarrow V(T₁₆) \) defined by \( \psi'(u) = \psi(u) \) for all \( u \in V(G') \setminus \{v\} \) and \( \psi'(v) = t(\psi(v)) \) is clearly a \( T₁₆ \)-coloring of \( G' \). □

An orientation \( n \)-vector is a sequence \( \alpha = (\alpha₁, \alpha₂, \ldots, \alphaₙ) \in [0, 1]ⁿ \) of \( n \) elements. Let \( S = (v₁, v₂, \ldots, vₙ) \) be a sequence of \( n \) (not necessarily distinct) vertices of \( T₁₆ \), a vertex \( u \) is said to be an \( \alpha \)-successor of \( S \) if for any \( i \), \( 1 \leq i \leq n \), we have \( u_i \in A(T₁₆) \) whenever \( \alpha_i = 1 \) and \( u_i \in A(T₁₆) \) otherwise. For instance, the vertex \( 3' \) of \( T₁₆ \) is a \((1, 1, 0, 1, 0, 0)\)-successor of \((1, 2, 6, 1, \infty, 2')\) since the arcs \( 3'1, 3'2, 2'3, \infty3', \) and \( 2'3 \) belong to \( A(T₁₆) \).

In the remainder, we say that \( T₁₆ \) has Property \( Pₐₖ \) if, for every sequence \( S \) of \( n \) distinct vertices of \( T₁₆ \) and any orientation \( n \)-vector \( \alpha \) which is compatible with \( S \), there exist \( k \) \( \alpha \)-successors of \( S \). This set of \( k \) \( \alpha \)-successors is denoted by \( Succₐ(S) \).

Proposition 8. The graph \( T₁₆ \) has Properties \( P₁₇, P₂₃, \) and \( P₃₁ \).

Proof. (1) Property \( P₁₇ \) is trivial since every vertex of \( T₁₆ \) has seven successors and seven predecessors.

(2) To prove that \( T₁₆ \) has Property \( P₂₃, \) we have to show that, for every sequence \( S = (u, v) \) and any compatible orientation \( 2 \)-vector \( \alpha \), there exist at least three \( \alpha \)-successors of \( S \). We have two cases to consider: the case \( uv \in A(T₁₆) \) and the case \( u = t(v) \). Since \( T₁₆ \) is arc-transitive, we will consider w.l.o.g. \( S = (0, 1) \) and \( S = (\infty, \infty) \).

A case study shows that the three \( \alpha \)-successors of \( S = (0, 1) \) are \( 2, 6', \) and \( \infty \) (resp. \( 2', 6, \) and \( \infty' \); \( 3', 4, \) and \( 5' \)) for \( \alpha = (0, 0) \) (resp. \( (1, 1); \) \( (0, 1); \) \( (1, 0) \)).

Consider now the case \( S = (\infty, \infty') \). By definition, only the two compatible orientation \( 2 \)-vectors with \( S = (0, 1) \) and \( (1, 0) \). It is then clear by construction of \( T₁₆ \) that we have seven \( \alpha \)-successors of \( S \) in each case.

(3) Property \( P₃₁ \) was proved by Marshall [8]. □

Proposition 9. Let \( u, v₁, \) and \( v₂ \) be three distinct vertices of \( T₁₆ \), and \( S₁ = (u, v₁) \) for every \( 1 ≤ i ≤ 2 \). Let \( \alpha \) be an orientation \( 2 \)-vector compatible with \( S₁ \) and \( S₂ \). Then \( Succₐ(S₁) \neq Succₐ(S₂) \).

Proof. Suppose to the contrary that there exist such \( S₁ \) and \( S₂ \) with \( Succₐ(S₁) = Succₐ(S₂) \).

By Proposition 7, we may assume w.l.o.g. that \( \alpha₂ = 0 \). If \( v₁ = t(v₂) \), we clearly have \( Succₐ(S₁) \neq Succₐ(S₂) \) since \( N_{T₁₆}^+(v₁) \cap N_{T₁₆}^-(v₂) = \emptyset \). Thus, we may assume w.l.o.g. that \( v₁v₂ \in A(T₁₆) \), and since \( T₁₆ \) is arc-transitive, we assume w.l.o.g. that \( v₁ = 0 \) and \( v₂ = 1 \).

Therefore, the vertices of \( Succₐ(S₁) = Succₐ(S₂) \) must be the common successors of \( 0 \) and \( 1 \). We have \( N_{T₁₆}^+(0) \cap N_{T₁₆}^+(1) = \{2, 6', \infty\}. \) If \( α₁ = 0 \), then a case study allows us to check that \( T₁₆ \) has no vertex \( u \) distinct from \( 0 \) and \( 1 \) having \( 2, 6', \) and \( \infty \) as successors. Therefore, we should have \( α₁ = 1 \) and then we can check that \( u \) should necessarily be either \( 0' \) or \( 1' \). However, in each case, we will have \( |Succₐ(S₁)| = 7 \) and \( |Succₐ(S₁−₁)| = 3 \) for some \( i \in [1, 2] \). □

3. Proof of Theorem 2

In this section, we prove Theorem 2, that is that every graph \( G \) with \( mad(G) < \frac{10}{7} \) and girth at least 5 admits a homomorphism to \( T₁₆ \).

Let us define the partial order \( \leq \). Let \( n₃(G) \) be the number of \( ≥ 3 \)-vertices in \( G \). For any two graphs \( G₁ \) and \( G₂ \), we have \( G₁ \leq G₂ \) if and only if at least one of the following conditions hold:

- \( G₁ \) is a proper subgraph of \( G₂ \);
- \( n₃(G₁) < n₃(G₂) \).

Note that this partial order is well-defined, since if \( G₁ \) is a proper subgraph of \( G₂ \), then \( n₃(G₁) ≤ n₃(G₂) \). So \( ≤ \) is a partial linear extension of the subgraph poset.

Let \( H \) be a hypothetical minimal counterexample to Theorem 2 according to \( <. \) We first prove that \( H \) does not contain a set of thirteen configurations. Then, using a discharging procedure, we show that every graph with girth 5 which contains none of these thirteen configurations has an average degree at least \( \frac{10}{7} \), this implies that \( H \) has \( mad(H) ≥ \frac{10}{7} \), a contradiction.
3.1. Structural properties of $H$

A weak 5-vertex is a 5-vertex adjacent to three 2-vertices. A weak 4-vertex is a 4-vertex adjacent to one 2-vertex.

**Lemma 10.** The graph $H$ does not contain the following configurations:

- (C1) a $\leq 1$-vertex;
- (C2) a $k$-vertex adjacent to $(k - 2)$ 2-vertices for $3 \leq k \leq 4$;
- (C3) a $k$-vertex adjacent to $(k - 1)$ 2-vertices for $2 \leq k \leq 7$;
- (C4) a $k$-vertex adjacent to $k$ 2-vertices for $1 \leq k \leq 15$;
- (C5) a 3-vertex;
- (C6) a $k$-vertex adjacent to $(k - 2)$ 2-vertices and one weak 5-vertex for $5 \leq k \leq 6$;
- (C7) a 4-vertex adjacent to three weak 5-vertices;
- (C8) a weak 5-vertex adjacent to two weak 4-vertices;
- (C9) a 5-vertex adjacent to two 2-vertices and two weak 5-vertices;
- (C10) a 5-vertex adjacent to one 2-vertex and four weak 5-vertices;
- (C11) a 6-vertex adjacent to three 2-vertices and three weak 5-vertices;
- (C12) a 7-vertex adjacent to five 2-vertices and two weak 5-vertices;
- (C13) an 8-vertex adjacent to seven 2-vertices and one weak 5-vertex.

The drawing conventions for a configuration $C$ contained in a graph $G$ are the following. If $u$ and $v$ are two vertices of $C$, then they are adjacent in $G$ if and only if they are adjacent in $C$. Moreover, the neighbors of a white vertex in $G$ are exactly its neighbors in $C$, whereas a black vertex may have neighbors outside of $C$. Two or more black vertices in $C$ may coincide in a single vertex in $G$, provided they do not share a common white neighbor. Finally, an edge will represent an arc with any of its two possible orientations.

Let $G$ be an oriented graph, $v$ be a $k$-vertex with $N(v) = \{v_1, v_2, \ldots, v_k\}$ and $\alpha$ be an orientation $k$-vector such that $\alpha_i = 0$ whenever $\bar{v}_i \bar{v}_j \in A(G)$ and $\alpha_i = 1$ otherwise. Let $\varphi$ be a $T_{16}$-coloring of $G \setminus \{v\}$ and $S = (\varphi(v_1), \varphi(v_2), \ldots, \varphi(v_k))$. Recall that a necessary condition to have $\alpha$-successors of $S$ is that $\alpha$ must be compatible with $S$, that is, for any pair of vertices $v_i$ and $v_j$, $\varphi(v_i) \neq \varphi(v_j)$ whenever $\alpha_i \neq \alpha_j$ and $\varphi(v_i) \neq t(\varphi(v_j))$ whenever $\alpha_i = \alpha_j$. Hence, every vertex $v_i$ forbids one color for each vertex $v_j$, $i \in [1, k]$, $i \neq j$. We define $f_i^S(v_j)$ to be the forbidden color for $v_i$ by $\varphi(v_j)$ (i.e. $f_i^S(v_j) = \varphi(v_j)$ whenever $\alpha_i \neq \alpha_j$ and $f_i^S(v_j) = t(\varphi(v_j))$ whenever $\alpha_i = \alpha_j$). Therefore, $\alpha$ is compatible with $S$ if and only if we have $\varphi(v_i) \neq f_i^S(v_j)$ for every pair $i, j$, $1 \leq i < j \leq k$. Note that if $\varphi(v_i) \neq f_i^S(v_j)$, then we necessarily have $\varphi(v_i) \neq f_i^S(v_i)$.

For each configuration, we suppose that $H$ contains it and we consider a reduction $H'$ with a girth at least 5 such that $H' \prec H$ and $\text{mad}(H') < \frac{10}{3}$; therefore, by minimality of $H$, $H'$ admits a $T_{16}$-coloring $\varphi$. We will then show that we can choose $\varphi$ so that it can be extended to $H$ by Proposition 8, contradicting the fact that $H$ is counterexample.

In what follows, if $H$ contains a configuration, then $H^*$ will denote the graph obtained from $H$ by removing all the white vertices of this configuration.

**Proof of Configurations (C1)–(C4).** Trivial. □

For Configurations (C1)–(C4), the reductions $H'$ have been obtained from $H$ by removing some vertices and/or arcs; therefore, we clearly had $\text{mad}(H') \leq \text{mad}(H)$. To prove that Configuration (C5) is forbidden in $H$, we considered a reduction $H'$ obtained from $H$ by removing one 3-vertex and by adding new vertices and arcs. The following lemma shows that this reduction $H'$ has nevertheless a maximum average degree less than $\frac{10}{3}$.

Let $G$ be a graph containing a 3-vertex $v$ adjacent to three vertices $u_1$, $u_2$, and $u_3$; see Fig. 4(a). We denote by $R(G)$ the graph obtained from $G \setminus \{v\}$ by adding 2-paths joining respectively $u_1$ and $u_2$, $u_2$ and $u_3$, and $u_3$ and $u_1$; see Fig. 4(b).

**Lemma 11.** If $\text{mad}(G) < \frac{10}{3}$, then $\text{mad}(R(G)) < \frac{10}{3}$.

The proof of this lemma is left to the reader.

**Proof of Configuration (C5).** Suppose that $H$ contains the configuration depicted in Fig. 3(d). Since Configurations (C1) and (C2) are forbidden, $u_1$, $u_2$, and $u_3$ are $\leq 3$-vertices. Let $H'$ be the graph obtained from $H^*$ by adding, for every $1 \leq i < j \leq 3$, a 2-path joining $u_i$ to $u_j$ in such a way that its orientation is the same orientation of the path $[u_i, v, u_j]$ in $H$. We have $H' \prec H$ since $n_3(H') = n_3(H) - 1$, $\text{mad}(H') < \frac{10}{3}$ by Lemma 11, and $H'$ has clearly girth at least 5. Any $T_{16}$-coloring $\varphi$ of $H'$ induces a coloring of $H^*$ such that $\varphi(u_i) \neq f_i^S(u_j)$ for any $i, j$, $1 \leq i < j \leq 3$. □

Configurations (C6)–(C13) all contain a weak 5-vertex. To shorten the proofs, we will often use the following lemma, later called Main Lemma.

**Lemma 12 (Main Lemma).** Let $G$ be an oriented graph containing a weak 5-vertex $u$ (see Fig. 5) and let $\varphi$ be a $T_{16}$-coloring of $G^*$. Then, for a fixed coloring of $u_1', u_2', u_3'$, and $v_1$, at most two colors are forbidden for $v_2$. 


(a) \( C_2 \). (b) \( C_3 \). (c) \( C_4 \). (d) \( C_5 \).

Fig. 3. Configurations \( C_2 \)–\( C_5 \).

(a) The graph \( G \). (b) The graph \( R(G) \).

Fig. 4. Configurations of Lemma 11.

Fig. 5. Configuration of Lemma 12.

Fig. 6. Configuration \( (C_6) \): a \( k \)-vertex adjacent to \((k - 2)\) 2-vertices and one weak 5-vertex for \( 5 \leq k \leq 6 \).

Fig. 7. Configuration \( (C_7) \): a 4-vertex adjacent to three weak 5-vertices.

**Proof.** The color \( \varphi(v_1) \) together with each of the fifteen colors for \( v_2 \) distinct from \( f_{v_2}^\varphi(v_1) \) give three possible colors for \( u \) by Property \( P_{2,3} \). Proposition 9 ensures that at most one of these fifteen colors, say \( f_{v_2} \), gives the three colors \( f_{u_1}^\varphi(u_1') \), \( f_{u_2}^\varphi(u_2') \), and \( f_{u_3}^\varphi(u_3') \) for \( u \). Thus, for any \( \varphi(v_2) \not\in \{ f_1, f_2, \ldots, f_{15} \} \), we have three available colors for \( u \) whose one of them is distinct from \( f_{u_1}^\varphi(u_1'), f_{u_2}^\varphi(u_2'), f_{u_3}^\varphi(u_3') \). □

**Proof of Configuration (C6).** Suppose that \( H \) contains the configuration depicted in Fig. 6 and let \( \varphi \) be a \( T_{16} \)-coloring of \( H^* \). By Main Lemma, the weak 5-vertex \( u \) forbids two colors for \( v \), say \( f_1 \) and \( f_2 \). By Property \( P_{1,7} \), we can choose \( \varphi \) such that \( \varphi(v) \not\in \{ f_1, f_2, f_{u_1}^\varphi(u_1'), \ldots, f_{u_3}^\varphi(u_3') \} \). □

**Proof of Configuration (C7).** Suppose that \( H \) contains the configuration depicted in Fig. 7 and let \( \varphi \) be a \( T_{16} \)-coloring of \( H^* \). By Main Lemma, each of the weak 5-vertices \( u, v, \) and \( w \) forbids two colors for \( x \), say \( f_1, f_2, \ldots, f_6 \). By Property \( P_{1,7} \), we can choose \( \varphi \) such that \( \varphi(x) \not\in \{ f_1, f_2, \ldots, f_6 \} \). □
Proof of Configuration (C8). Suppose that $H$ contains the configuration depicted in Fig. 8 and let $H' = H \setminus \{u, u_1, u_2, u_3, v_1, v_1\}$. Let $\varphi$ be a $T_{16}$-coloring of $H'$. We clearly have $\varphi(v') \neq f^\alpha_1(v'')$ (resp. $\varphi(w') \neq f^\alpha_2(w'')$) since $v$ (resp. $w$) is colored in $H'$. Property $P_{2,3}$ ensures that we have two available colors for $v$ (resp. $w$), say $c_1$ and $c_2$ (resp. $d_1$ and $d_2$), distinct from $f^\alpha_1(v')$ (resp. $f^\alpha_2(w')$).

Therefore, we have to show that there exists $i, j \in [1, 2]$ such that assigning $c_i$ to $v$ and $d_j$ to $w$ allows us to extend $\varphi$ to a $T_{16}$-coloring of $H$. Let $W = \{f^\alpha_1(u'_i), f^\alpha_1(u'_j), f^\alpha_2(u'_i), f^\alpha_2(u'_j)\}$, and $V_i = \{c_1, c_2\}, V_2 = \{d_1, d_2\}$. Let $\alpha = (\alpha_1, \alpha_2)$ be an orientation 2-vector such that $\alpha_1 = 0$ (resp. $\alpha_2 = 0$) whenever $\varphi(u') \in A(G)$ (resp. $\varphi(u') \in A(G)$), and $\alpha_1 = 1$ (resp. $\alpha_2 = 1$) otherwise.

Note that we must have $\varphi(u) \notin W$. Suppose first that $V_1 = V_2$ (more precisely, $c_i = d_i$ for every $i \in [1, 2]$). If $\alpha_1 = \alpha_2$, then we set $\varphi(v') = \varphi(w')$, and we get $|\text{Succ}_\varphi(\varphi(v), \varphi(w))| = 7$. Thus, $\varphi$ can be extended to $H$. If $\alpha_1 \neq \alpha_2$, then let $S_1 = (c_1, d_2)$ and $S_2 = (c_2, d_1)$. The sequences $S_1$ and $S_2$ are compatible with $\alpha$, and by Property $P_{2,3}$ we have $|\text{Succ}_\varphi(S_i)| \geq 3$ for every $i \in [1, 2]$. Moreover, a case study shows that $\text{Succ}_\varphi(S_1) \neq \text{Succ}_\varphi(S_2)$.

Therefore, there exists $i \in [1, 2]$ such that $\text{Succ}_\varphi(S_i) \neq W$, and so $\varphi$ can be extended to $H$.

Proof of Configuration (C9). Suppose that $H$ contains the configuration depicted in Fig. 9 and let $\varphi$ be a $T_{16}$-coloring of $H'$. The weak 5-vertices $u$ and $v$ forbid four colors for $x$, say $f_1, f_2, f_3, f_4$, by Lemma 12. By Property $P_{1,7}$, we can choose $\varphi$ such that $\varphi(x) \notin \{f_1, f_2, f_3, f_4, f^\alpha_1(x'_1), f^\alpha_2(x'_2)\}$. □

Proof of Configuration (C10). Suppose that $H$ contains the configuration depicted in Fig. 10 and let $\varphi$ be a $T_{16}$-coloring of $H'$. By Main Lemma, the weak 5-vertices $u, v, w$, and $x$ forbid eight colors for $y$, say $f_1, \ldots, f_8$. We clearly can choose $\varphi$ such that $\varphi(y) \notin \{f_1, \ldots, f_8, f^\alpha_1(y'_1), f^\alpha_2(y'_2)\}$. □

Proof of Configuration (C11). Suppose that $H$ contains the configuration depicted in Fig. 11 and let $\varphi$ be a $T_{16}$-coloring of $H'$. The weak 5-vertices $v, w$, and $x$ forbid six colors for $u$, say $f_1, \ldots, f_6$, by Lemma 12. We clearly can choose $\varphi$ such that $\varphi(u) \notin \{f_1, \ldots, f_6, f^\alpha_1(u'_1), f^\alpha_2(u'_2), f^\alpha_3(u'_3)\}$. □
Proof of Configuration (C12). Suppose that $H$ contains the configuration depicted in Fig. 12 and let $\varphi$ be a $T_{16}$-coloring of $H^*$. The weak 5-vertices $v$ and $w$ forbid four colors for $u$, say $f_1, \ldots, f_4$, by Lemma 12. We clearly can choose $\varphi$ such that $\varphi(u) \notin \{f_1, \ldots, f_4, \varphi(v)(u'_1), \varphi(v)(u'_2), \ldots, \varphi(w)(u'_5)\}$. □

Proof of Configuration (C13). Suppose that $H$ contains the configuration depicted in Fig. 13 and let $\varphi$ be a $T_{16}$-coloring of $H^*$. The weak 5-vertex $u$ forbids two colors for $v$, say $f_1, f_2$, by Lemma 12. We clearly can choose $\varphi$ such that $\varphi(v) \notin \{f_1, f_2, \varphi(v)(v'_1), \varphi(v)(v'_2), \ldots, \varphi(w)(v'_7)\}$. □
Therefore, no such counterexample $H$ exists.

The discharging rules are defined as follows:

(R1) Each weak 4-vertex gives 2 to its 2-neighbor.

(R2) Each non weak 4-vertex gives 1 to their weak 5-neighbors.

(R3) Each $\geq 5$-vertex gives 2 to their 2-neighbors and 1 to their weak 5-neighbors.

Let $v$ be a $k$-vertex of $H$. Note that $k > 1$ by (C1) and $k \neq 3$ by (C5).

- If $k = 2$, then $\omega(v) = -4$. Since every 2-vertex of $H$ has two $\geq 4$-neighbors by (C2) and (C3), $v$ receives 2 from each neighbor by (R1) and (R3). Hence $\omega^*(v) = 0$.
- If $k = 4$, then $\omega(v) = 2$. By (C2), a 4-vertex has at most one 2-neighbor. If $v$ has one 2-neighbor (i.e. $v$ is weak), then it gives 2 by (R1). If $v$ has no 2-neighbor, then it has at most two weak 5-neighbors by (C7). Therefore, $v$ gives at most $1 \times 2$ by (R2). Hence $\omega^*(v) \geq 2 - \max\{2; 1 \times 2\} = 0$.
- If $k = 5$, then $\omega(v) = 5$. By (C3), a 5-vertex has at most three 2-neighbors. If $v$ has three 2-neighbors (i.e. $v$ is weak), then it has no weak 5-neighbors by (C6); it thus gives $2 \times 3$ by (R3). Moreover, by (C8), $v$ has at most one weak 4-neighbor; therefore, $v$ has at least either one non weak 4-neighbor or one $\geq 5$-neighbor; thus, $v$ receives at least 1 by (R2) or (R3). If $v$ has two 2-neighbors, then it has at most one weak 5-neighbor by (C9), and then gives at most $2 \times 2 + 1$ by (R3). If $v$ has one 2-neighbor, then it has at most three weak 5-neighbors by (C10), and then gives at most $2 + 1 \times 3$ by (R3). Finally, if $v$ has no 2-neighbor, it gives at most $1 \times 5$ by (R3). Hence, $\omega^*(v) \geq 5 - \max\{2 \times 3 - 1; 2 \times 2 + 1; 2 + 1 \times 3; 1 \times 5\} = 0$.
- If $k = 6$, then $\omega(v) = 8$. By (C3), a 6-vertex has at most four 2-neighbors. If $v$ has four 2-neighbors, then it has no weak 5-neighbor by (C6), and then gives $2 \times 4$ by (R3). If $v$ has three 2-neighbors, then it has at most two weak 5-neighbors by (C11), and then gives at most $2 \times 3 + 1 \times 2$ by (R3). Finally, if $v$ has 2-neighbors, $0 \leq l \leq 2$, then $v$ has at most $(6 - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (6 - l)$ by (R3). Hence, $\omega^*(v) \geq 8 - \max\{2 \times 4; 2 \times 3 + 1 \times 2; 2 \times l + 1 \times (6 - l)\} = 0$ for any $0 \leq l \leq 2$.
- If $k = 7$, then $\omega(v) = 11$. By (C3), a 7-vertex has at most five 2-neighbors. If $v$ has five 2-neighbors, then it has at most one weak 5-neighbor by (C12) and then gives at most $2 \times 5 + 1$ by (R3). Finally, if $v$ has 2-neighbors, $0 \leq l \leq 4$, then it has at most $(7 - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (7 - l)$ by (R3). Hence, $\omega^*(v) \geq 11 - \max\{2 \times 5 + 1; 2 \times l + 1 \times (7 - l)\} = 0$ for any $0 \leq l \leq 4$.
- If $k = 8$, then $\omega(v) = 14$. By (C4), an 8-vertex has at most seven 2-neighbors. If $v$ has seven 2-neighbors, then it has no weak 5-neighbor by (C13) and then gives $2 \times 7$ by (R3). Finally, if $v$ has 2-neighbors, $0 \leq l \leq 6$, then it has at most $(8 - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (8 - l)$ by (R3). Hence, $\omega^*(v) \geq 14 - \max\{2 \times 7; 2 \times l + 1 \times (8 - l)\} = 0$ for any $0 \leq l \leq 6$.
- If $k = 9$, then $\omega(v) = 17$. By (C4), a 9-vertex has at most eight 2-neighbors. If $v$ has 2-neighbors, $0 \leq l \leq 8$, then it has at most $(9 - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (9 - l)$ by (R3). Hence, $\omega^*(v) \geq 17 - 2 \times l + 1 \times (9 - l) \geq 0$ for any $0 \leq l \leq 8$.
- If $k \geq 10$, then $\omega(v) = 3k - 10$. If $v$ has 2-neighbors, $0 \leq l \leq k$, then $v$ has at most $(k - l)$ weak 5-neighbors and then gives at most $2 \times l + 1 \times (k - l)$ by (R3). Hence, $\omega^*(v) \geq 3k - 10 - 2 \times l + 1 \times (k - l) \geq 0$ for any $0 \leq l \leq k$.

Thus, for every $v \in V(H)$, we have $\omega^*(v) \geq 0$ once the discharging is finished, that completes the proof.
4. Concluding remarks

4.1. Graphs with maximum average degree of less that \( \frac{10}{3} \)

In this paper, we proved that every oriented graph with maximum average degree less than \( \frac{10}{3} \) and girth at least 5 has oriented chromatic number at most 16. We recently proved in a companion paper that the restriction of girth can be dropped:

**Theorem 13** ([12]). Let \( G \) be an oriented graph with maximum average degree less than \( \frac{10}{3} \). Then, \( \chi_o(G) \leq 16 \).

The proof of this theorem is based on the same techniques as the one used in this paper. The discharging procedure is the same. The difference lies in the forbidden configurations. Recall that the case where two black vertices coincide in a configuration (provided they do not share a white neighbor) is taken into account in the proofs. However, the cases where black vertices coincide with white vertices (creating cycles of lengths 3 and 4) was not taken into account here since we considered graphs with girth at least 5. So, to drop the restriction of girth, and thus get Theorem 13, we considered configurations with cycles of lengths 3 and 4.

4.2. Strong oriented coloring

In 1999, Nešetřil and Raspaud [9] introduced the notion of strong oriented coloring, which is a stronger version of the notion of oriented coloring studied in this paper.

Let \( M \) be an additive abelian group. An \( M \)-strong-oriented coloring of an oriented graph \( G \) is a mapping \( \varphi \) from \( V(G) \) to \( M \) such that \( \varphi(u) \neq \varphi(v) \) whenever \( uv \) is an arc in \( G \) and \( \varphi(v) - \varphi(u) \neq -(\varphi(t) - \varphi(z)) \) whenever \( uv \) and \( tz \) are two arcs in \( G \). The strong oriented chromatic number of an oriented graph is the minimal order of a group \( M \) such that \( G \) has an \( M \)-strong-oriented coloring. It is clear that any strong oriented coloring of an oriented graph \( G \) is an oriented coloring of \( G \) and therefore the oriented chromatic number of \( G \) is less than its strong oriented chromatic number.

Nešetřil and Raspaud showed that a strong oriented coloring of an oriented graph \( G \) can be equivalently defined as a homomorphism \( \varphi \) from \( G \) to \( H \), where \( H \) is an oriented graph with \( k \) vertices labeled by the \( k \) elements of an abelian additive group \( M \), such that for any pair of arcs \( uv \) and \( tz \) of \( A(H) \), \( v - u \neq -(t - z) \). For every prime power \( p \equiv 3 \pmod{4} \), the Paley graph \( QR_p \) (defined in Section 2.) is clearly an oriented graph with \( p \) vertices labeled by the \( p \) elements of the field \( \mathbb{Z}/p\mathbb{Z} \) and such that for any pair of arcs \( uv \) and \( tw \) of \( A(QR_p) \), \( v - u \neq -(t - w) \).

Borodin et al. [4] proved that the oriented chromatic number of the graphs with maximum average degree less than \( \frac{10}{3} \) is at most 19 by showing that these graphs admit a homomorphism to the Paley graph \( QR_{19} \). Therefore, their result applies for the strong oriented chromatic number: the graphs with maximum average degree of less that \( \frac{10}{3} \) have a strong oriented chromatic number at most 19. So, a natural question to ask is:

**Question 14.** Does there exist an abelian additive group \( M \) on 16 elements such that we can label the vertices of \( T_{16} \) with the elements of \( M \) in such a way that \( v - u \neq -(v - z) \) whenever \( uv \) and \( tz \) are two arcs of \( T_{16} \)?

It is true, that would imply that 16 colors are enough for a strong oriented coloring of an oriented graph with maximum average degree less than \( \frac{10}{3} \), and therefore, for a strong oriented coloring of a planar graph with girth at least 5.

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References


