

# Using hybrid automata for set-membership state estimation with uncertain nonlinear continuous-time systems

Nacim Meslem<sup>a</sup>, Nacim Ramdani<sup>b</sup>, Yves Candau<sup>b</sup>

<sup>a</sup>*MIG INRA Jouy-en-Josas, Domaine de Vilvert 78352 Jouy-en-Josas, France.*

<sup>b</sup>*CERTES, Université Paris-Est Créteil Val de Marne, 94000 Créteil, France.*

---

## Abstract

This paper deals with set membership state estimation for continuous-time systems from discrete-time measurements, in the unknown but bounded error framework. The classical predictor-corrector approach to state estimation uses interval Taylor methods for solving the prediction phase, which are known to have poor performance in presence of large model or input uncertainty. In this paper, we show how to derive more efficient predictors by using a nonlinear hybridization method which builds hybrid automata to characterize the boundaries of reachable sets. The derived continuous-discrete set membership predictor-corrector estimator is then tested with simulated data from a bioreactor. Our method is compared to classical continuous-time interval observers and is shown to have promising performance.

### *Key words:*

bounded error, continuous-time systems, differential inequalities, hybrid systems, interval analysis, nonlinear systems, state estimation

---

## 1 Introduction

State estimation is an important issue when addressing control or diagnosis issues with dynamical systems. For many systems, such as bioreactors for instance, it is more natural to assume that all uncertain quantities – measurement noise, model uncertainty and modelling errors – belong to a known set.

---

*Email addresses:* [nmeslem@jouy.inra.fr](mailto:nmeslem@jouy.inra.fr) (Nacim Meslem), [nacim.ramdani@u-pec.fr](mailto:nacim.ramdani@u-pec.fr) (Nacim Ramdani), [yves.candau@u-pec.fr](mailto:yves.candau@u-pec.fr) (Yves Candau).

This assumption is rather natural and requires much less data than statistical assumptions. In such an *unknown-but-bounded-error* (UBBE) framework, the estimation problem no longer has a unique solution, but there exists a set of state vectors that are consistent with measured data, the model structure and the prior error bounds. *Set-membership estimation* (SME) techniques allow the characterization of this solution set (Schweppe, 1968).

SME techniques have reached a mature stage when dealing with linear or nonlinear discrete-time models (see (Milanese et al., 1996) and the references therein). For the linear case, the solution set is a convex polyhedron which can be characterized either exactly or superscribed in simple-shaped forms, such as ellipsoids (Chernousko and Rokityanskii, 2000; Durieu et al., 2001; Maksarov and Norton, 2002; Chernousko, 2005), parallelotopes (Chisci et al., 1996) or zonotopes (Alamo et al., 2005). When the discrete-time model is nonlinear, Alamo et al. (2008) used DC programming with zonotopes (a DC function is a function that can be expressed as the difference of two convex functions), while Jaulin et al. (2001) and Kieffer et al. (2002a) developed new tools using interval analysis, consistency techniques and constraint propagation. Chen et al. (1997) introduced an extension of the Kalman filter to intervals.

To the contrary, SME with continuous-time nonlinear systems needs further investigations, and even more when the models, in the form of differential equations, encompass uncertainty. In the SME literature, there are mainly two types of SME techniques:

(i) The first one addresses the case of continuous-time state estimation from continuous-time measurements, and was proposed for some classes of continuous-time systems (such as bioreactors for instance), for the first time by Gouzé et al. (2000), then further investigated by Hadj-Sadok and Gouzé (2001), Rapaport and Gouzé (2003), Dochain (2003), Bernard and Gouzé (2004), Rapaport and Dochain (2005) and Moisan et al. (2009). Such observers will be denoted *interval observers* in the sequel.

(ii) The second one addresses the case of continuous-time state estimation from discrete-time measurements and was introduced by Jaulin et al. (2001), then improved by Raïssi et al. (2004), Raïssi et al. (2005), Kieffer et al. (2006) and Goffaux et al. (2009). In the sequel, we will investigate this type of approach, which we denote as the *prediction-correction approach*.

The prediction-correction approach relies on a two-stage methodology. The first stage is a *prediction* stage, which relies on *guaranteed set integration*, i.e. *validated* numerical integration methods for solving the initial value problem (IVP) for ordinary differential equations (ODE). Prediction stage yields a guaranteed numerical evaluation of the state vector as the solution of the ODE at measurement time steps – (Nedialkov et al., 1999) is a very good review on *validated* numerical integration methods based on interval Taylor series. The second stage is a *correction* stage, which consists in studying the consistency

between the feasible domain for actual output and the feasible one for model output, in order to prune inconsistent parts from the two outputs. In fact, the over-approximations induced by interval computations in the validated numerical integration methods (Moore, 1966; Anguelov and Markov, 1998) used in the prediction stage, reduce the performance of the prediction/correction approach, and more importantly when system model encompasses uncertainty in either parameters or inputs. Even though, several validated numerical integration methods (Lohner, 1987; Rihm, 1994; Berz and Makino, 1998; Nedialkov and Jackson, 1999; Deville et al., 2002; Lin and Stadtherr, 2006; Kletting et al., 2007) are available, they usually fail (computing algorithm stops) as long as the magnitude of model uncertainties is large. To date, the method most often used to control over-approximations proceeds by bisecting state or parameter vectors uncertainty sets. Unfortunately, bisection strategies lead to combinatorial complexity, hence large computation time and memory usage, which is a severe drawback if an on-line implementation is required for state estimation.

In, this paper we address the improvement of the prediction stage by using validated numerical methods for set integration which do not revert to bisection when used with large uncertainty sets. The idea is to use comparison theorems for differential inequalities, and in particular the classical Müller’s theorem (Müller, 1927; Marcelli and Rubbioni, 1997; Walter, 1997; Singer and Barton, 2006) which makes it possible to derive two *bracketing* dynamical systems which enclose the original uncertain dynamical system and thus bound the solution set between a minimal solution, i.e. a flow that is always lower than the solution flow pipe, and a maximal solution, i.e., a flow that is always larger. Since the two bounding systems involve no more uncertainty, classical interval Taylor methods can be used for the guaranteed computation of the minimal and maximal solutions, hence the solution set. As a matter of fact, such bracketing methods are the very ones used within the classical interval observers recalled above. Moreover, Goffaux et al. (2009) improved recently the prediction-correction approach by using the Müller’s theorem to build a set of framers, i.e. bracketing systems, for computing solution sets for the prediction stage, while assuming that system’s Jacobian off-diagonal matrix elements are of fixed sign. It remains that for the general case where the latter Jacobian matrix elements may change sign over a given time horizon, the previous methods do not work, hence obtaining tight bracketing systems, or framers, remains an issue.

In our previous works (Ramdani et al., 2008, 2009), we addressed this issue and introduced a hybrid bounding approach. Given a time grid,  $t_0 < t_1 < t_2 < \dots < t_{n_T}$ , our method analyzes the signs of the off-diagonal Jacobian matrix elements. Over each time interval  $[t_j, t_{i+1}]$ , where these signs remain fixed, our method uses the Müller’s theorem to build the two bracketing dynamical systems. Over each time interval  $[t_j, t_{i+1}]$ , where the sign of at least one partial

derivative changes, our method reverts to classical interval Taylor method and solves the prediction stage by using whole domains. Finally, when the obtained bounding systems are analyzed over the whole time interval  $[t_0, t_{n_T}]$ , they behave as the subsystems of a hybrid system, which switches from one subsystem to another each time a partial derivative changes sign. In fact, the ability of our hybrid bounding approach to yield effective results in general, is driven by its ability to ascertain the signs of the partial derivatives. Therefore, when the size of the domains taken for initial state vector or parameter vector are large, one expects the method to have difficulties to ascertain the signs, and hence to seldom use the bounding systems approach. A simple idea to circumvent such a drawback may consist again in allowing the domains to be partitioned in a way that renders easier the determination of the signs of the partial derivatives. But then, method complexity would grow exponentially.

In the sequel, we will show how to modify our nonlinear hybridization approach in order to improve the performance of the prediction stage, hence the set-membership prediction-correction state estimator without using neither parameter nor state bisection. The main improvement resides in the development of a new rule, a generic rule for deriving the bracketing systems over time intervals when our original rule does not work, i.e. when the sign of off-diagonal elements of system's Jacobian matrix changes with time. A nice consequence of the improvement is that it may no longer be necessary to revert to interval Taylor methods for solving the prediction stage.

The organization of the paper is as follows. In section 2, we overview set-membership state estimation with continuous-time systems from discrete-time data. We recall our hybrid bounding method and give the improved rule for building the bracketing systems in section 3. Finally, we show the potential of our improved hybrid bounding method for set-membership state estimation for bioreactor, in section 4. We compare also the outcome of our method with the one of a classical interval observer, by using simulated data.

## 2 Set-membership state estimation in presence of uncertainty

Consider the uncertain continuous dynamical system (1) where uncertainties are represented by bounded sets with *a priori* known bounds,

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}, \mathbf{p}, t) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}, \mathbf{p}, t) \\ \mathbf{x}(t_0) \in \mathbb{X}_0 \subset \mathbb{D}, \mathbf{p} \in \mathbb{P}, t \in I = [t_0, t_{n_T}] \end{cases} \quad (1)$$

Functions  $\mathbf{f} : \mathbb{D} \times \mathbb{P} \times \mathbb{R}^+ \rightarrow \mathbb{R}^n$  and  $\mathbf{g} : \mathbb{D} \times \mathbb{P} \times \mathbb{R}^+ \rightarrow \mathbb{R}^m$  are possibly nonlinear.  $\mathbf{f} \in C^{k-1}(\mathbb{D} \times \mathbb{P} \times \mathbb{R}^+)$ ,  $\mathbb{D} \times \mathbb{P} \times \mathbb{R}^+ \subseteq \mathbb{R}^{n+n_p+1}$  is an open set;  $n$ ,  $m$  and  $n_p$  are the dimensions of respectively the state vector  $\mathbf{x}$ , the output vector  $\mathbf{y}$  and the parameter vector  $\mathbf{p}$ . The initial state  $\mathbf{x}(t_0)$  is assumed to belong to an a priori known set  $\mathbb{X}_0$  and we assume that measurements  $\mathbf{y}_j$  of the output vector are available at sampling times  $t_j \in \{t_1, t_2, \dots, t_{n_T}\}$  in  $[t_0, t_{n_T} = T] = I$ . Note that sampling interval needs not be constant. Measurement noise is a discrete time signal assumed additive and bounded with known bounds. Denote by  $\mathbb{E}_j$  a feasible domain for output error at time  $t_j$ , the feasible domain for output model at time  $t_j$  is then given by

$$\mathbb{Y}_j = \mathbf{y}_j + \mathbb{E}_j . \quad (2)$$

SME aims at characterizing, in a guaranteed way, the set  $\mathbb{X}$  of all state trajectories consistent with model (1), its uncertainties, the actual data, and their feasible domains. Denote  $\mathbb{X}(t; t_0, \mathbb{X}_0, \mathbb{P})$ , the solution set at time  $t$ ,  $t \geq t_0$ , of (1), originating from each initial condition in  $\mathbb{X}_0$  at time  $t_0$  and each parameter vector in  $\mathbb{P}$ .

By using a method for characterizing the reachable set for uncertain nonlinear continuous-time systems, the *prediction* stage computes a conservative over-approximation  $\mathbb{X}_{j+1}^p$  of the set  $\mathbb{X}(t_{j+1}; t_j, \mathbb{X}_j, \mathbb{P})$  of all possible solutions of (1) at time  $t_{j+1}$  originating from each initial condition in  $\mathbb{X}_j$  at time  $t_j$  and each parameter vector in  $\mathbb{P}$ .

The *correction* stage uses contractors and consistency techniques to characterize  $\mathbb{X}_{j+1}^{inv}$ , which is an over-approximation of the reciprocal image of the output feasible domain  $\mathbb{Y}_{j+1}$  by function  $\mathbf{g}$ , at time  $t_{j+1}$ , then reduces the predicted state set by computing the intersection between the reciprocal image  $\mathbb{X}_{j+1}^{inv}$  and the predicted set  $\mathbb{X}_{j+1}^p$ , as follows

$$\mathbb{X}_{j+1}^c = \mathbb{X}_{j+1}^{inv} \cap \mathbb{X}_{j+1}^p \quad (3)$$

Finally, in the next step, the prediction phase is initialized with  $\mathbb{X}_{j+1} = \mathbb{X}_{j+1}^c$ .

The algorithm below summarizes the procedure of this set-membership state estimation approach.

### Algorithm 1 *Prediction-Correction Set-Membership Estimator*

- (1) **Input:**  $(\mathbb{X}_0, \mathbb{P}, \mathbf{f}, \mathbf{g}, \mathbb{Y}_1, \dots, \mathbb{Y}_{n_T})$
- (2)  $t_j = t_0; \mathbb{X}_j = \mathbb{X}_0;$
- (3) **while**  $(t_j < t_{n_T})$  **do**  
     \*\*\* Prediction phase \*\*
- (4)      $\{t_{j+1}, \mathbb{X}_{j+1}^p\} = \mathbf{Guaranteed-Set-Integration}(\mathbf{f}, \mathbb{X}_j, \mathbb{P}, t_j);$   
     \*\*\* Correction phase \*\*

- (5)  $\mathbb{X}_{j+1}^{inv} = \mathbf{Guaranteed-Set-Inversion}(\mathbf{g}, \mathbb{Y}_{j+1}, \mathbb{P}, t_{j+1});$   
(6)  $\mathbb{X}_{j+1}^c = \mathbb{X}_{j+1}^{inv} \cap \mathbb{X}_{j+1}^p;$   
\*\*\* *Re-initialization* \*\*  
(7)  $\mathbb{X}_{j+1} = \mathbb{X}_{j+1}^c$   
(8)  $j = j + 1;$   
(9) **end**  
(10) **Output:**  $(\mathbb{X}_1^c, \mathbb{X}_2^c, \dots, \mathbb{X}_{n_T}^c).$

The ability of this algorithm to solve practical problems depends essentially on the performances (size of enclosures, computation time, ...) of the computing reachable set methods used. Hence, the main contribution of this paper, that is the use of our nonlinear hybridization approach to reachable set computation (Ramdani et al., 2009) allied with an enhanced rule for deriving the bracketing systems.

**Remark 1** *The set inversion step needed for correction stage may be achieved by using interval analysis, branching and reduction functions based on constraint satisfaction and consistency techniques (see (Jaulin et al., 2001) and the references therein). When on-line implementation is desired, set inversion can be achieved efficiently without bisection nor branching by using a smart combination of differential algebraic and Taylor model methods (Berz and Makino, 1998; Lin and Stadtherr, 2007).*

### 3 Guaranteed set integration using a nonlinear hybridization

In a previous work (Ramdani et al., 2008, 2009), we have introduced a *hybrid bounding* approach to nonlinear reachability computation for uncertain nonlinear systems. This approach relies on (i) an adapted version of the classical Müller's existence theorem (Müller, 1927; Walter, 1997) given in (Kieffer et al., 2006); (ii) and a partition of the reachable state set for the nonlinear uncertain systems into regions over which it is possible to implement the Müller's theorem. In other words, our method analyzes the signs of the partial derivatives  $(\partial f_i / \partial x_j)(\mathbf{x}, \mathbf{p}, t)$  and  $(\partial f_i / \partial p_k)(\mathbf{x}, \mathbf{p}, t)$ , evaluated over the current reached state space  $\mathbb{X}(t; t_0, \mathbb{X}_0, \mathbb{P})$  and for any  $\mathbf{p} \in [\mathbf{p}]$ . Then, over each time interval  $[t_i, t_{i+1}]$  where these signs remain constant it designs a bracketing coupled dynamical system thanks to Müller's theorem, which we write as follows

$$\mathcal{M}_i = \begin{cases} \dot{\bar{\mathbf{x}}} = \bar{\mathbf{f}}(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{p}}, \underline{\mathbf{p}}, t) \\ \dot{\underline{\mathbf{x}}} = \underline{\mathbf{f}}(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{p}}, \underline{\mathbf{p}}, t) \\ \bar{\mathbf{x}}(t_0) = \bar{\mathbf{x}}_0, \\ \underline{\mathbf{x}}(t_0) = \underline{\mathbf{x}}_0 \end{cases} \quad (4)$$

where there is no uncertainty in either state or parameter vectors and which yields a guaranteed enclosure of all the possible solutions for the original uncertain system (1). That is, the flow pipe of the original uncertain system is enclosed between a minimal and a maximal solution, obtained as the solution of the bracketing coupled system (4). In appendix A, we recall the adapted version of the Müller theorem (Walter, 1997; Kieffer et al., 2006) and the practical rule (Kieffer et al., 2006; Ramdani et al., 2009) based on a monotonicity test which makes it possible to design, in the general case, the bracketing coupled system (4) over time intervals where the partial derivatives have constant sign.

Now, for each time interval where the sign of at least one partial derivative is not constant for all  $t$  in  $[t_j, t_{j+1}]$ , that is,

$$\begin{aligned} & \exists t_1, t_2 \in [t_j, t_{j+1}], \exists \mathbf{x}(t_1) \in \mathbb{X}(t_1) \exists \mathbf{x}(t_2) \in \mathbb{X}(t_2), \exists \mathbf{p} \in [\mathbf{p}], \exists i | \\ & \exists l \text{ (sign } ((\partial f_i / \partial x_l)(\mathbf{x}(t_1), \mathbf{p}, t_1)) \cdot \text{sign } ((\partial f_i / \partial x_l)(\mathbf{x}(t_2), \mathbf{p}, t_2)) < 0) \vee \\ & \exists k \text{ (sign } ((\partial f_i / \partial p_k)(\mathbf{x}(t_1), \mathbf{p}, t_1)) \cdot \text{sign } ((\partial f_i / \partial p_k)(\mathbf{x}(t_2), \mathbf{p}, t_2)) < 0) \end{aligned} \quad (5)$$

then rule 2 (see appendix A) cannot be used. The first version of our hybridization uses interval Taylor methods and computes the solution set using *whole* domains. To address this severe shortcoming, we introduce a new rule.

### 3.1 New bracketing methods

In this section, we consider the time intervals  $[t_j, t_{j+1}]$  over which condition (5) is satisfied. Thus, let us consider real variable  $z \in [\underline{z}, \bar{z}]$  and let us assume that the range of  $\frac{\partial f_i}{\partial z}(\cdot, z, \cdot)$  over  $[\underline{z}, \bar{z}]$  contains zero. Here,  $z$  denotes either  $x_l$  or  $p_k$ . Our aim consists in building two functions  $\underline{f}_i$  and  $\bar{f}_i$  from the algebraic expression of  $f_i$ , which satisfy

$$\forall z \in [\underline{z}, \bar{z}] \quad \underline{f}_i(\cdot, \underline{z}, \bar{z}, \cdot) \leq f_i(\cdot, z, \cdot) \leq \bar{f}_i(\cdot, \bar{z}, \underline{z}, \cdot). \quad (6)$$

To derive the two bracketing functions  $\underline{f}_i$  and  $\bar{f}_i$ , we can use the following property

**Property 1 (P1)** (Moisan and Bernard, 2005). *Any Lipschitz function  $f_i$  can be written as the sum of one decreasing function  $d(\cdot, z, \cdot)$  and one increasing function  $c(\cdot, z, \cdot)$ , i.e.:*

$$\forall z \in [\underline{z}, \bar{z}] = [z], \quad f_i(\cdot, z, \cdot) = d(\cdot, z, \cdot) + c(\cdot, z, \cdot) \quad (7)$$

**PROOF.** Denoting  $\lambda$  the Lipschitz constant of  $f_i$  with respect to  $z$ , one can write  $f_i(., z, .) = d(., z, .) + c(., z, .)$ , where

$$c(., z, .) = \lambda z, \tag{8}$$

$$d(., z, .) = -(\lambda z - f_i(., z, .)). \tag{9}$$

See the detailed proof in (Moisan and Bernard, 2005).

In fact, there are several ways of decomposing function  $f_i$ . Some decompositions may introduce large overapproximation when writing inequalities (6), hence very loose bounds for the computed reachable set. Therefore, we must look for decompositions that do not introduce the latter overapproximations, decompositions which are obtained using what we define as a *pragmatic* way. The proof of property (P1) gives one way to decompose function  $f_i$  as the sum of one increasing function and one decreasing one (eqs. (8)-(9)). When Lipschitz constant  $\lambda$  is large, the computed bounds may be loose (though not necessarily). Therefore, it is also interesting to investigate the possibility to decompose function  $f_i$  as the product of one increasing function and one decreasing one. In both cases, we will assume that the decompositions are obtained using *pragmatic* ways, which we summarize in the following assumption (H1).

**Hypothesis 1 (H1)** *The function  $f_i$  can be written in a pragmatic way, as the sum or the product of one decreasing function  $d(., z, .)$  and one increasing function  $c(., z, .)$ , i.e.:*

$$\forall z \in [\underline{z}, \bar{z}] = [z], \quad f_i(., z, .) = d(., z, .) \star c(., z, .) \tag{10}$$

where  $\star \in \{+, \times\}$ .

**Proposition 1**

$$\forall z \in [\underline{z}, \bar{z}], \quad d(., \bar{z}, .) \star c(., \underline{z}, .) \leq f_i(., z, .) \leq d(., \underline{z}, .) \star c(., \bar{z}, .). \tag{11}$$

where  $\star \in \{+, \times\}$  if (H1) is true, or  $\star \in \{+\}$  if not. □

**PROOF.** If (H1) is true, it is easy to check that

$$\begin{aligned} \max_{z \in [z]}(f_i(., z, .)) &= \max_{z \in [z]}(d(., z, .) \star c(., z, .)) \\ &\leq \max_{z \in [z]}(d(., z, .)) \star \max_{z \in [z]}(c(., z, .)) \\ &\leq d(., \underline{z}, .) \star c(., \bar{z}, .). \end{aligned}$$

$$\begin{aligned}
\min_{z \in [z]}(f_i(\cdot, z, \cdot)) &= \min_{z \in [z]}(d(z) \star c(z)) \\
&\geq \min_{z \in [z]}(d(\cdot, z, \cdot)) \star \min_{z \in [z]}(c(\cdot, z, \cdot)) \\
&\geq d(\cdot, \bar{z}, \cdot) \star c(\cdot, \underline{z}, \cdot).
\end{aligned}$$

If (H1) is not true, then one can use property P1 and the above still holds with  $\star \equiv +$ . This completes the proof.  $\square$

**Remark 2** *One may also improve this method by using DC functions theory (Carrizosa, 2001), (Tuy, 1995).*

We can now state our enhanced rule for designing the bracketing dynamical systems which encompass the original uncertain system (1) in all possible situations. Hence, over each time interval  $[t_j, t_{j+1}]$ , we use the following enhanced rule.

**Rule 1 (The enhanced rule)**

For each component  $f_i(\cdot, \cdot, \cdot)$  of the field vectors  $\mathbf{f}(\cdot, \cdot, \cdot)$  of the uncertain system (1), one must apply one of the two rules below to obtain the framing functions  $\underline{f}_i(\cdot, \cdot, \cdot)$  and  $\bar{f}_i(\cdot, \cdot, \cdot)$ ,

- (1) If (eq.(5) is not satisfied) then use rule 2 presented in appendix A,
- (2) else if (H1) is true, then use (11) (proposition 1) with (10),
- (3) else use (11) (proposition 1) with (8)-(9) (property P1).

Consequently, when the bracketing systems obtained by the enhanced rule are analyzed over the whole time interval  $[t_0, t_{n_T}]$ , they behave as subsystems of a hybrid system

$$\mathcal{H} = \{\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_l\} \quad (12)$$

constituted by a collection of bracketing systems and the discrete transitions of the hybrid system  $\mathcal{H}$  is governed by sign change of the partial derivatives,

$$\begin{aligned}
Gc_{i,k}(\mathbf{x}, \mathbf{p}, t) &= \text{sign}(\partial f_i(\mathbf{x}, \mathbf{p}, t) / \partial p_k) \quad \text{and} \\
Gc_{i,j}(\mathbf{x}, \mathbf{p}, t) &= \text{sign}(\partial f_i(\mathbf{x}, \mathbf{p}, t) / \partial x_j)
\end{aligned} \quad (13)$$

which act as the guard conditions that drive the discrete transitions from one bracketing system to another one. So, given initial conditions, the execution of this hybrid deterministic automaton (12) characterizes a conservative over-approximation of the reachable set of the analyzed uncertain nonlinear continuous system (1).

### 3.2 Illustrative example

Let us consider the following uncertain system

$$\begin{cases} \dot{x}_1 = f_1(x_2) = \frac{x_2}{4+x_2} \\ \dot{x}_2 = f_2(x_1, x_2) = \frac{x_2 x_1}{(1+x_1)(2+x_1)} \end{cases} \quad (14)$$

where  $x_1(t_0)$  and  $x_2(t_0)$  belong in positive intervals  $[x_1(t_0)]$  and  $[x_2(t_0)]$  respectively. Note that, for these initial conditions all the state trajectories  $x_1(t)$  and  $x_2(t)$  stay positive for any  $t \geq t_0$  because the face  $x_2(t_0) = 0$  is repulsive.

Now, to design a deterministic hybrid system which encompasses the uncertain system (14), we must analyze the signs of the two following partial derivatives  $\frac{\partial f_1}{\partial x_2}(x_2)$  and  $\frac{\partial f_2}{\partial x_1}(x_1, x_2)$ . Thus, for the first partial derivative it is clear that it is always positive for any  $x_2(t) \in [x_2(t)]$ . However, for the second partial derivative there exist three situations with respect to the following guard condition

$$Gc_{2,1}(x_1, x_2) = \text{sign}\left(\frac{\partial f_2}{\partial x_1}(x_1, x_2)\right) = \text{sign}(2 - x_1^2). \quad (15)$$

There are two situations where the sign of the guard condition can be ascertained:

- $\forall x_1(t) \in [x_1(t)]$ ,  $2 - x_1^2 \geq 0$  then by using the rule 2 we obtain as bracketing system  $\mathcal{M}_1$

$$\mathcal{M}_1 = \begin{cases} \dot{\bar{x}}_1 = \frac{\bar{x}_2}{4+\bar{x}_2} \\ \dot{\bar{x}}_2 = \frac{\bar{x}_2 \bar{x}_1}{(1+\bar{x}_1)(2+\bar{x}_1)} \\ \dot{\underline{x}}_1 = \frac{\underline{x}_2}{4+\underline{x}_2} \\ \dot{\underline{x}}_2 = \frac{\underline{x}_2 \underline{x}_1}{(1+\underline{x}_1)(2+\underline{x}_1)} \end{cases} \quad (16)$$

- $\forall x_1(t) \in [x_1(t)]$ ,  $2 - x_1^2 < 0$  then by using the rule 2 we obtain as bracketing system  $\mathcal{M}_2$

$$\mathcal{M}_2 = \begin{cases} \dot{\bar{x}}_1 = \frac{\bar{x}_2}{4+\bar{x}_2} \\ \dot{\bar{x}}_2 = \frac{\bar{x}_2 \underline{x}_1}{(1+\underline{x}_1)(2+\underline{x}_1)} \\ \dot{\underline{x}}_1 = \frac{\underline{x}_2}{4+\underline{x}_2} \\ \dot{\underline{x}}_2 = \frac{\underline{x}_2 \bar{x}_1}{(1+\bar{x}_1)(2+\bar{x}_1)} \end{cases} \quad (17)$$

The third situation corresponds to the case where there exist  $x'_1(t)$  and  $x''_1(t)$  in  $[x_1(t)]$  such that

$$2 - x_1'^2(t) > 0 \quad \text{and} \quad 2 - x_1''^2(t) < 0.$$

Hence, to obtain the third bracketing system  $\mathcal{M}_3$ , we use rule 2 to frame  $f_1$  and  $f_2$  with respect to  $x_2$  and we use the new bracketing method (e.g. we use proposition 1 under (H1) with  $\star = +$ ) to frame  $f_2$  with respect to  $x_1$ , in a pragmatic way. For this, we rewrite  $f_2$  as

$$f_2(x_1, x_2) = x_2 \left( \frac{2}{x_1 + 2} - \frac{1}{x_1 + 1} \right)$$

and so we obtain the following double inequalities  $\forall x_1(t) \in [x_1(t)]$

$$x_2 \left( \frac{2}{\underline{x}_1 + 2} - \frac{1}{\underline{x}_1 + 1} \right) \leq f_2(x_1, x_2) \leq x_2 \left( \frac{2}{\bar{x}_1 + 2} - \frac{1}{\bar{x}_1 + 1} \right),$$

which allow to obtain the below third bracketing system  $\mathcal{M}_3$

- if  $2 \in [x_1^2(t)]$ , then by using the enhanced rule we obtain as bracketing system  $\mathcal{M}_3$

$$\mathcal{M}_3 = \begin{cases} \dot{\bar{x}}_1 = \frac{\bar{x}_2}{4 + \bar{x}_2} \\ \dot{\bar{x}}_2 = \bar{x}_2 \left( \frac{2}{\bar{x}_1 + 2} - \frac{1}{\bar{x}_1 + 1} \right) \\ \dot{\underline{x}}_1 = \frac{\underline{x}_2}{4 + \underline{x}_2} \\ \dot{\underline{x}}_2 = \underline{x}_2 \left( \frac{2}{\underline{x}_1 + 2} - \frac{1}{\underline{x}_1 + 1} \right) \end{cases} \quad (18)$$

Thus, according to the sign of the guard condition (15), the hybrid system  $\mathcal{H} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}$  which encompasses the uncertain system (14) is governed by the automaton depicted in the Figure 1.

### 3.3 Comment on convergence analysis

It is well established today that contrarily to statistical state estimators for which the convergence issues are handled through the asymptotic properties of the point estimations when the number of data measurements tends to infinity, the convergence of set membership state estimation or state interval observers is connected to the properties of guaranteedness and arbitrary precision. The guaranteedness property is the fact that the estimated set for the state vector is guaranteed to enclose the actual state vector of the system. The property of arbitrary precision means that the estimated set converges from the outside to the actual state vector. These issues are now well-established for bounded-error observers for discrete-time systems, see (Kieffer et al., 2002b) for details. For the type of systems under study in the present paper, the arbitrary precision convergence property is subject to the one of the prediction step, i.e. the verified integration of the ODE. To address this issue, one can analyze the boundedness of the reachable set as over-approximated by our nonlinear

hybridization method, which in turn can be addressed by analyzing the convergence of the size of the derived over-approximation. In (Ramdani et al., 2009) we have shown how to tackle this problem via the  $\epsilon$ -practical stability notion for hybrid and switching systems developed in (Xu and Zhai, 2005), i.e. conditions which keep system trajectories within given bounds. Indeed, we have given in Proposition 7 of (Ramdani et al., 2009) a condition on the eigenvalues of the Jacobian matrix of the nonlinear uncertain system which ensures practical stability for the size of the estimated reachable set over a bounded time horizon. In other words, this result makes it possible to analyze whether the size of the reachable set computed between two measurement time instants remains within given bounds.

## 4 Application

In order to emphasize the performance of our nonlinear hybridization approach allied with the improved bracketing rule, when used for set-membership state estimation, we will consider state estimation with bioreactors. Furthermore, we will compare the performance of our prediction-correction method with the interval observer proposed in (Moisan and Bernard, 2005). The latter assumes continuous time measurements, or at least very fast sampling of them. To the contrary, our method considers discrete time or rare data. We will show that our set-membership state estimation approach yields better results than interval observers when sampling periods for actual data are either small or fairly large.

A bioreactor is based on fermentation principles which consist in exploiting metabolic reactions that take place in cells of microorganisms (bacteria, yeast, phytoplankton, etc.). The derivation of a model for bioreactors is in general difficult. This is mainly due to the presence of living organisms whose behavior is only poorly represented by uncertain parametric functions. It is therefore interesting to model the uncertain variables and uncertain parameters of these systems by interval vectors.

Here, we address a simple model where only one population of microorganisms is taken into account. Two state variables are necessary to describe the state of the bioreactor. The first one represents the microorganism concentration called biomass and denoted by  $x_1$ , the second one represents the substrate concentration, denoted by  $x_2$ . The growth rate is given by the Haldane model, Thus one gets the following standard equations for the dynamics of the two state bioreactor

$$\begin{cases} \dot{x}_1(t) = \mu_0\mu(x_2)x_1 - \alpha D(t)x_1 \\ \dot{x}_2(t) = -k_1\mu_0\mu(x_2)x_1 + D(t)(s_{in}(t) - x_2) \end{cases} \quad (19)$$

where  $\mu_0\mu(x_2)$  is the growth rate of biomass, modeled by non-monotonic Haldane law:

$$\mu_0\mu(x_2) = \mu_0 \frac{x_2}{x_2 + k_s + x_2^2/k_i} \quad (20)$$

with an uncertain bounded parameter  $\mu_0$

$$\underline{\mu}_0 \leq \mu_0 \leq \bar{\mu}_0 .$$

This type of bioreactor is fed by a solution containing substrate in concentration  $s_{in}(t)$  which is not known exactly

$$\underline{s}_{in}(t) \leq s_{in}(t) \leq \bar{s}_{in}(t)$$

and we assume also that biomass measurement is available at discrete time instants  $t_j$  ( $j \in \{1, \dots, n_f\}$ ), and than its feasible domain is given by

$$[y](t_j) = x_1(t_j) + [-e, +e](t_j), \quad j \in \{1, \dots, n_T\} \quad (21)$$

where  $e$  is the maximal absolute measurement error at instant  $t_j$ .

#### 4.1 Building the hybrid bracketing system

In order to establish the local bracketing systems for bioreactor (19), the analysis of the variations of its partial derivatives signs is necessary. It is easy to show that the signs of the partial derivatives are always constant, except for  $(\partial\dot{x}_1/\partial x_2)$ . Indeed, the sign of the latter depends on substrate  $x_2(t)$ :

$$\text{sign}((\partial\dot{x}_1)/(\partial x_2)(t)) = \text{sign}(\sqrt{k_s k_i} - x_2(t)) \quad (22)$$

We have to use our nonlinear hybridization approach for guaranteed set integration with (19). According to the sign of  $(\partial\dot{x}_1)/(\partial x_2)$ , system (19) admits three local bracketing systems which form the hybrid bracketing system. Indeed, the first system  $\mathcal{M}_1$  is valid over time intervals  $[t_j, t_{j+1}]$  when the derivative is negative

$$(\mathcal{M}_1) \begin{cases} \dot{\bar{x}}_1(t) = \bar{\mu}_0\mu(\underline{x}_2)\bar{x}_1 - \alpha D(t)\bar{x}_1 \\ \dot{\bar{x}}_2(t) = -k_1\underline{\mu}_0\mu(\bar{x}_2)\underline{x}_1 + D(t)(\bar{s}_{in}(t) - \bar{x}_2) \\ \dot{\underline{x}}_1(t) = \underline{\mu}_0\mu(\bar{x}_2)\underline{x}_1 - \alpha D(t)\underline{x}_1 \\ \dot{\underline{x}}_2(t) = -k_1\bar{\mu}_0\mu(\underline{x}_2)\bar{x}_1 + D(t)(\underline{s}_{in}(t) - \underline{x}_2) \end{cases} \quad (23)$$

and the second system  $\mathcal{M}_2$  is valid over time intervals when the derivative is positive

$$(\mathcal{M}_2) \begin{cases} \dot{\bar{x}}_1(t) = \bar{\mu}_0 \mu(\bar{x}_2) \bar{x}_1 - \alpha D(t) \bar{x}_1 \\ \dot{\bar{x}}_2(t) = -k_1 \underline{\mu}_0 \mu(\bar{x}_2) \bar{x}_1 + D(t) (\bar{s}_{in}(t) - \bar{x}_2) \\ \dot{\underline{x}}_1(t) = \underline{\mu}_0 \mu(\underline{x}_2) \underline{x}_1 - \alpha D(t) \underline{x}_1 \\ \dot{\underline{x}}_2(t) = -k_1 \bar{\mu}_0 \mu(\underline{x}_2) \bar{x}_1 + D(t) (\underline{s}_{in}(t) - \underline{x}_2) \end{cases} \quad (24)$$

Recall that systems (23) and (24) are built locally by means of rule 2. Now, over the time intervals where the sign of  $(\partial \dot{x}_1)/(\partial x_2)$  is not constant, we can use the new bracketing method to frame  $\mu(x_2)$  and so obtain a third bracketing system  $\mathcal{M}_3$ . Indeed, for  $k_s = 9.28$  and  $k_i = 256$ , we can rewrite  $\mu(x_2)$  as

$$\mu(x_2) = \frac{266.43}{x_2 + 246.35} - \frac{10.43}{x_2 + 9.64}, \quad (25)$$

then we have the following double inequalities to frame  $\mu(x_2)$ :

$$\frac{266.43}{\bar{x}_2 + 246.35} - \frac{10.43}{\underline{x}_2 + 9.64} \leq \mu(x_2) \leq \frac{266.43}{\underline{x}_2 + 246.35} - \frac{10.43}{\bar{x}_2 + 9.64} \quad (26)$$

Consequently, we consider as third bracketing system  $\mathcal{M}_3$ , the following differential equations:

$$(\mathcal{M}_3) \begin{cases} \dot{\bar{x}}_1(t) = \bar{\mu}_0 \left( \frac{266.43}{\bar{x}_2 + 246.35} - \frac{10.43}{\bar{x}_2 + 9.64} \right) \bar{x}_1 - \alpha D(t) \bar{x}_1 \\ \dot{\bar{x}}_2(t) = -k_1 \underline{\mu}_0 \mu(\bar{x}_2) \bar{x}_1 + D(t) (\bar{s}_{in}(t) - \bar{s}_2) \\ \dot{\underline{x}}_1(t) = \underline{\mu}_0 \left( \frac{266.43}{\bar{x}_2 + 246.35} - \frac{10.43}{\underline{x}_2 + 9.64} \right) \underline{x}_1 - \alpha D(t) \underline{x}_1 \\ \dot{\underline{x}}_2(t) = -k_1 \bar{\mu}_0 \mu(\underline{x}_2) \bar{x}_1 + D(t) (\underline{s}_{in}(t) - \underline{s}_2) . \end{cases} \quad (27)$$

We have succeeded in building the bracketing systems needed to solve the prediction phase of the **Prediction-Correction Set-Membership Estimator**. In the **Guaranteed-Set-Integration** function, the hybrid system  $\mathcal{H} = \{\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3\}$  with guard condition  $G_{c_{1,2}}(x_2) = \text{sign}(\sqrt{k_s k_i} - x_2(t))$  is implemented.

#### 4.2 Test and results

The data considered in this example are as follows:  $\alpha = 0.5$ ,  $k = 42.14$ ,  $k_s = 9.28 \text{ mmol/l}$ ,  $k_i = 256 \text{ mmol/l}$ ,  $\mu_0 \in [0.703, 0.777]$ ,  $x_1(t_0 = 0) \in [0, 10]$ ,

$x_2(t_0 = 0) \in [0, 100]$ ,  $s_{in}(t) \in [0.95, 1.05](50 + 15 \cos(1/5t))$ ,

$$D(t) = \begin{cases} 2 & \text{if } 0 \leq t < 5 \\ 0.5 & \text{if } 5 \leq t < 10 \\ 1.067 & \text{if } 10 \leq t \leq 20, \end{cases}$$

and the feasible measurement domain is given by  $[y](t_j) = [0.95y(t_j), 1.05y(t_j)]$ , with a constant measurement time step  $h = t_{j+1} - t_j = 0.3$  days. Almost all these data are the same as those given in (Moisan and Bernard, 2005).

First, we use the interval Hermite-Obreschkoff series with variable step control as implemented in the VNODE software (Nedialkov, 1999; Nedialkov et al., 2001), for guaranteed set integration within the **Prediction-Correction Set-Membership Estimator**. The interval Taylor method fails to solve guaranteed set integration.

To the contrary, our nonlinear hybridization approach successfully solves the prediction stage as shown below. The red bold continuous curves on Figure 2 and Figure 3 show a guaranteed enclosure of all the possible state trajectories of (19) consistent with all the uncertainties and the feasible domains for measurements. The blue dashed curves represent the pseudo-actual data for state vector of (19) as obtained by running the model with the following values for the uncertain parameters and the initial state:  $\mu_0 = 0.74$ ,  $x_1(t_0 = 0) = 5$ ,  $x_2(t_0 = 0) = 40$  and  $s_{in}(t) = 50 + 15 \cos(1/5t)$ . Figure 4 shows the autonomous switching signal which controls the switching between the three bracketing systems. This signal depends only on the initial conditions.

As first conclusion, our nonlinear hybridization method successfully addresses set membership state estimation for system (19) whereas interval Taylor methods fail to do so.

**Remark 3** *If one uses the old nonlinear hybridization method to deal with this example, it is necessary to combine the interval Taylor method with a bisection strategy to obtain satisfactory results even when the magnitude of the uncertainties on the initial state vector and the parameter vector is small.*

#### 4.3 Comparison with interval observer

In this subsection, we compare, by using simulated data, the results of the set-membership state estimation obtained with a prediction-correction estimator using our enhanced nonlinear hybridization, and those obtained with the bundle interval observer as developed in (Moisan and Bernard, 2005). Recall that

the main advantage of the prediction-correction approach w.r.t interval observers is the fact that it considers discrete-time measurements which is more realistic in practice.

To make this comparison we have, made the following assumptions

- For the continuous-time measurement case: we use a bundle of estimations obtained by considering in parallel 70 interval observers working with various gains taken in  $(k_1, k_2)^T \in [-14, 0] \times [0, 140]$  and with a reinitialization period  $r_p = 1$  days. The structure of these interval observers is given in (Moisan and Bernard, 2005). Note that, we have used consistency techniques to eliminate the negative values of the lower bound of the substrate generated by the interval observers.
- For the discrete-time measurement case: we use the prediction-correction estimator allied with the nonlinear hybridization and we set the measurement time step  $h = 0.1$  days.

The other parameters, uncertainties and initial state vector are the same as in the previous subsection.

As shown in Figure 5 and Figure 6, the prediction-correction estimator gives better results than the bundle interval observers. The red curves correspond to the estimated interval state vector obtained by the prediction-correction estimator and the green dotted curves correspond to the estimated interval state vector as obtained by the bundle of interval observers.

Now, let us reduce the reinitialization period of the bundle interval observers to  $r_p = 0.5$  days and increase the time measurement step to  $h = 0.5$  days. Here also the prediction-correction estimator works better than the bundle interval observers. See Figure 7 and Figure 8.

In a last test, we keep reinitialization period  $r_p = 0.5$  days and we take a larger measurement time step  $h = 5$  days. Figure 9 and Figure 10 show that if the prediction-correction estimator works with very large measurement time step, then it may give results which are not better than the bundle interval observers.

## 5 Conclusion

In this paper, we have addressed set membership state estimation for continuous-time systems from discrete-time measurement data. We used a nonlinear hybridization approach to set integration with an improved rule for deriving bracketing systems. The prediction-correction set-membership estimator can

now be used with a broader class of nonlinear systems. We successfully illustrated the obtained estimator with examples involving bioreactors and we compared the results of our method with the ones given by a classical method which uses a bundle of interval observers.

Future work should focus on the use of consistency techniques to improve further the accuracy of estimated state boxes, and to filter the feasible domain for the uncertain parameters.

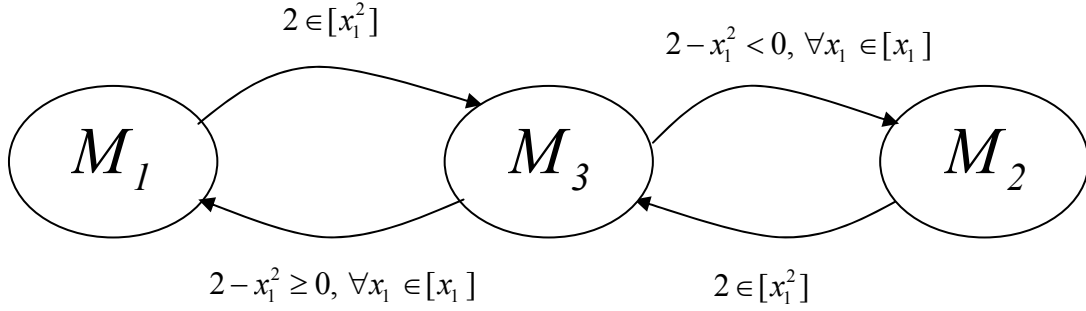


Fig. 1. The hybrid automaton.

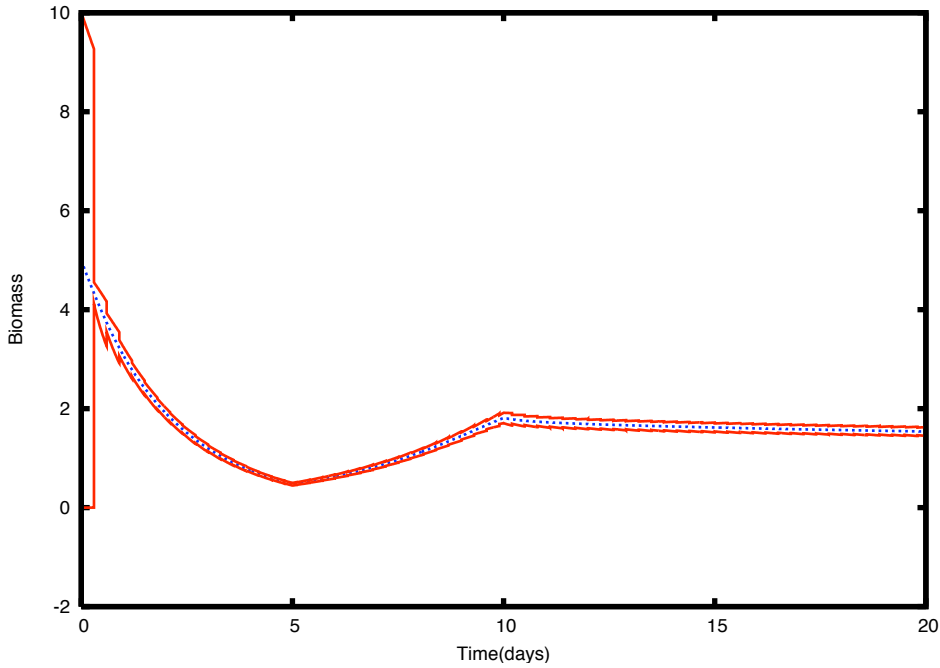


Fig. 2. The estimated interval biomass. Red continuous curves represent the lower and the upper bounds of this interval, blue dashed curves represent the real value of biomass.  $h = 0.3$  days.

## A Appendix. An adapted version of Müller theorem

Here, we recall Müller's theorem (Müller, 1927; Marcelli and Rubbioni, 1997; Walter, 1997; Singer and Barton, 2006) as reported in (Kieffer and Walter, 2006).

**Theorem 1** ((Walter, 1997; Kieffer et al., 2006)) *If  $\bar{x}_i(t)$  and  $\underline{x}_i(t)$  satisfy the following inequalities for all  $i \in \{1, \dots, n\}$*

- *the left DINI derivatives  $D^- \underline{x}_i(t)$  and  $D^- \bar{x}_i(t)$  and the right DINI derivatives*

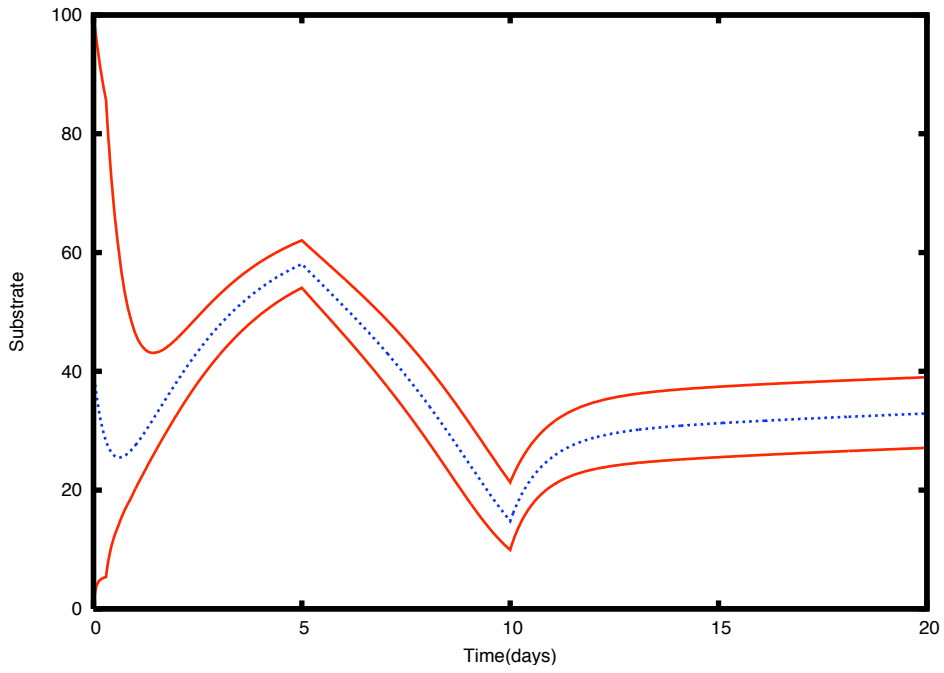


Fig. 3. The estimated interval substrate. Red continuous curves represent the lower and the upper bounds of this interval, blue dashed curves represent the real value of substrate.  $h = 0.3$  days.

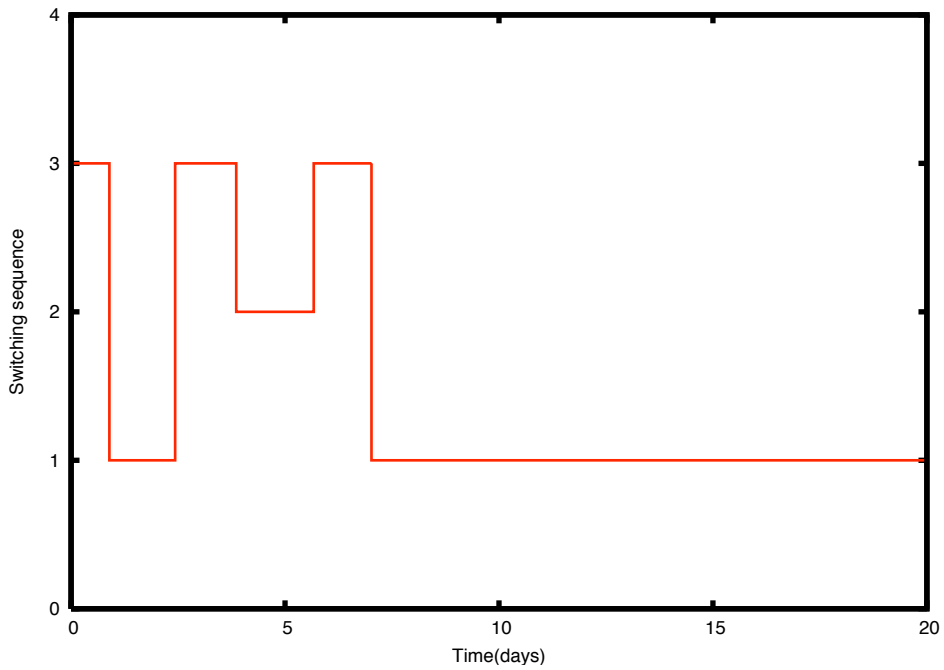


Fig. 4. Autonomous switching signal

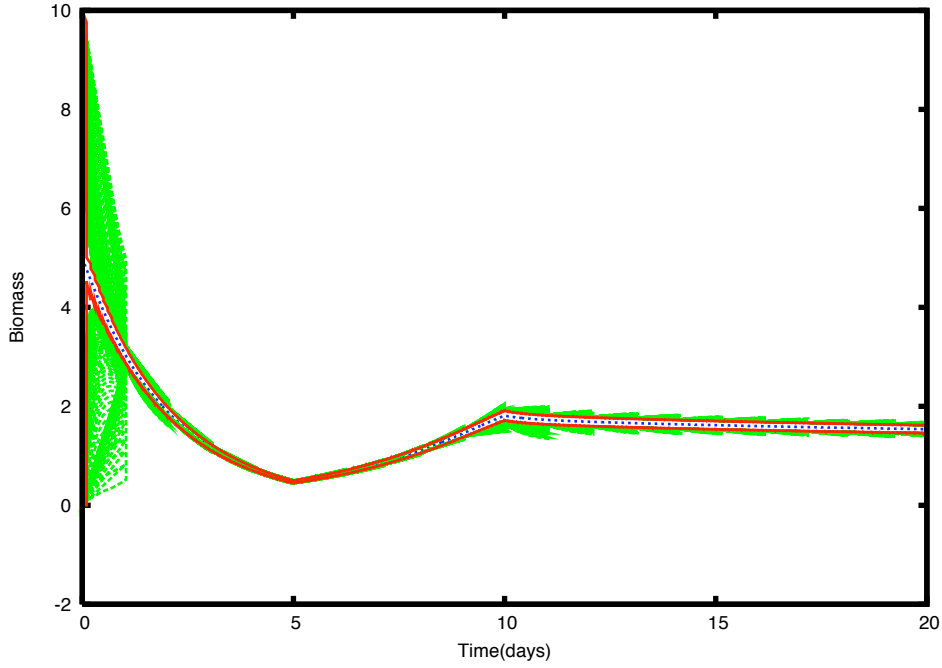


Fig. 5. The estimated interval biomass. Red continuous curves represent the lower and the upper bounds of this interval obtained by the prediction-correction estimator. Green dots curves represent lower and the upper bounds of this interval obtained by the bundle observers.  $h = 0.1$  days and  $r_p = 1$  days.

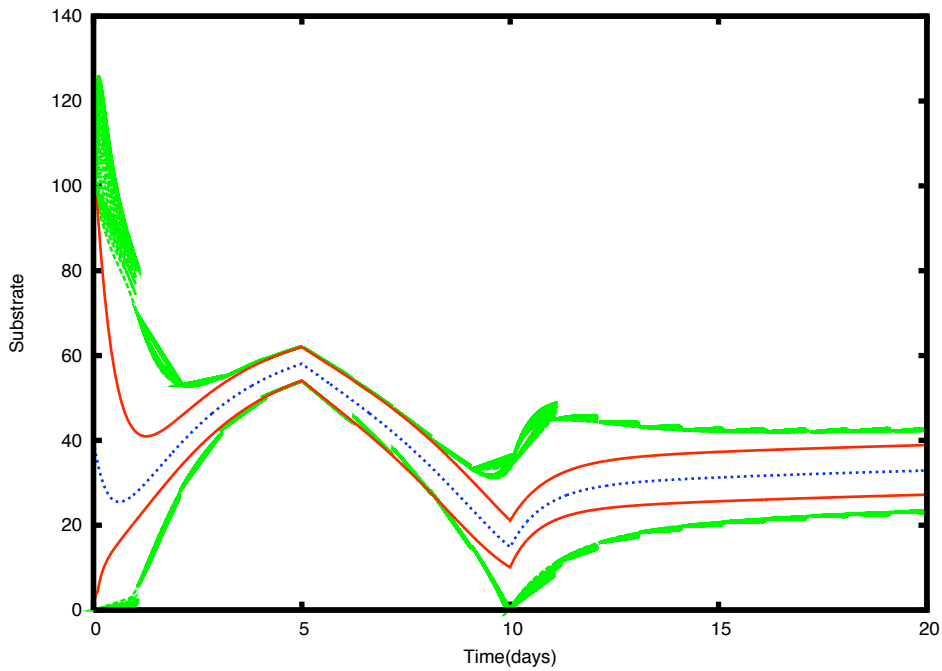


Fig. 6. The estimated interval substrate. Red continuous curves represent the lower and the upper bounds of this interval obtained by the prediction-correction estimator. Green dots curves represent lower and the upper bounds of this interval obtained by the bundle observers.  $h = 0.1$  days and  $r_p = 1$  days

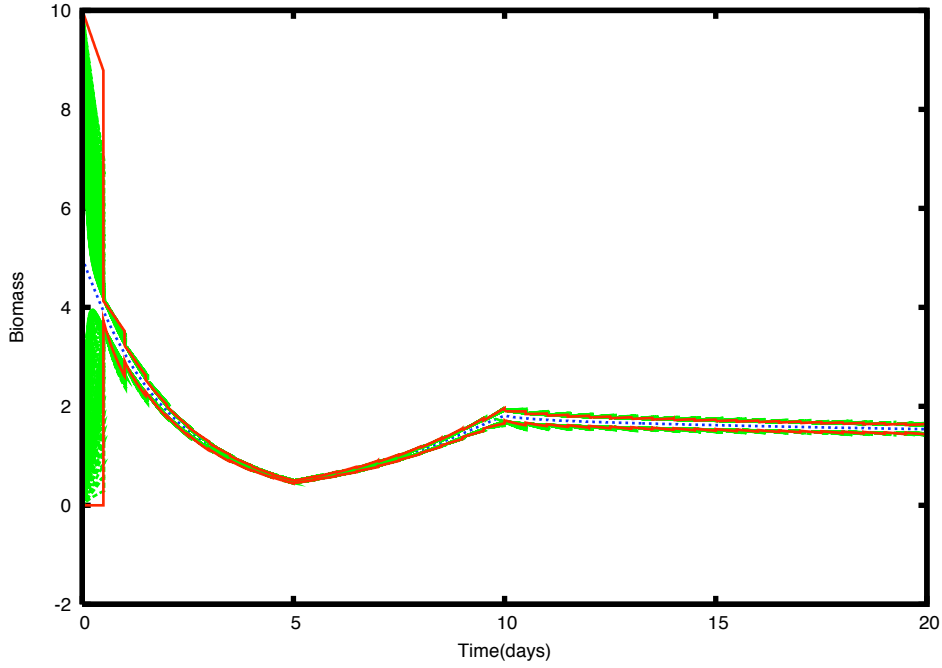


Fig. 7. The estimated interval biomass. Red continuous curves represent the lower and the upper bounds of this interval obtained by the prediction-correction estimator. Green dots curves represent lower and the upper bounds of this interval obtained by the bundle observers.  $h = r_p = 0.5$  days.

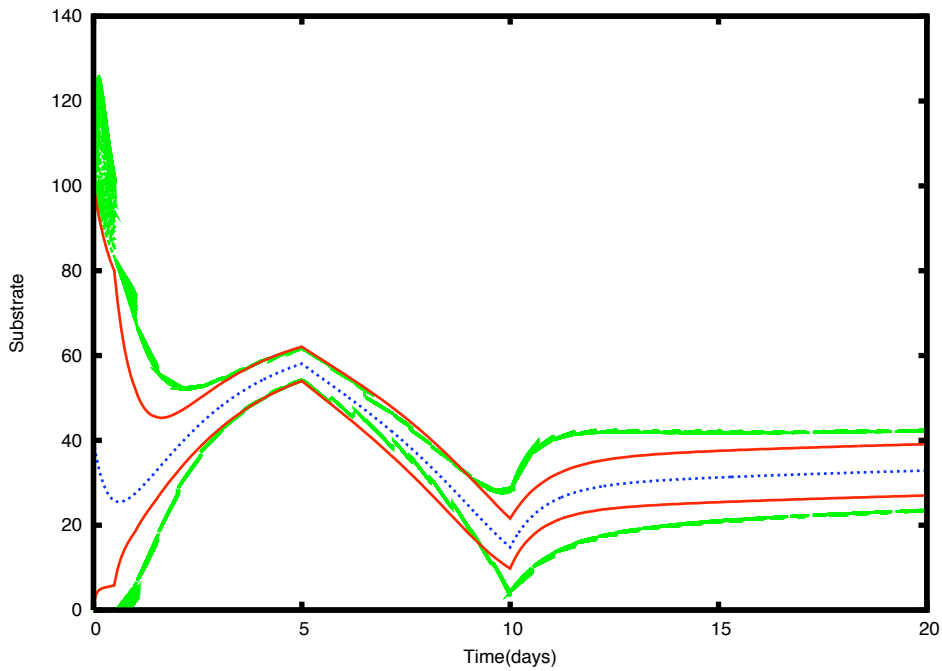


Fig. 8. The estimated interval substrate. Red continuous curves represent the lower and the upper bounds of this interval obtained by the prediction-correction estimator. Green dots curves represent lower and the upper bounds of this interval obtained by the bundle observers.  $h = r_p = 0.5$  days

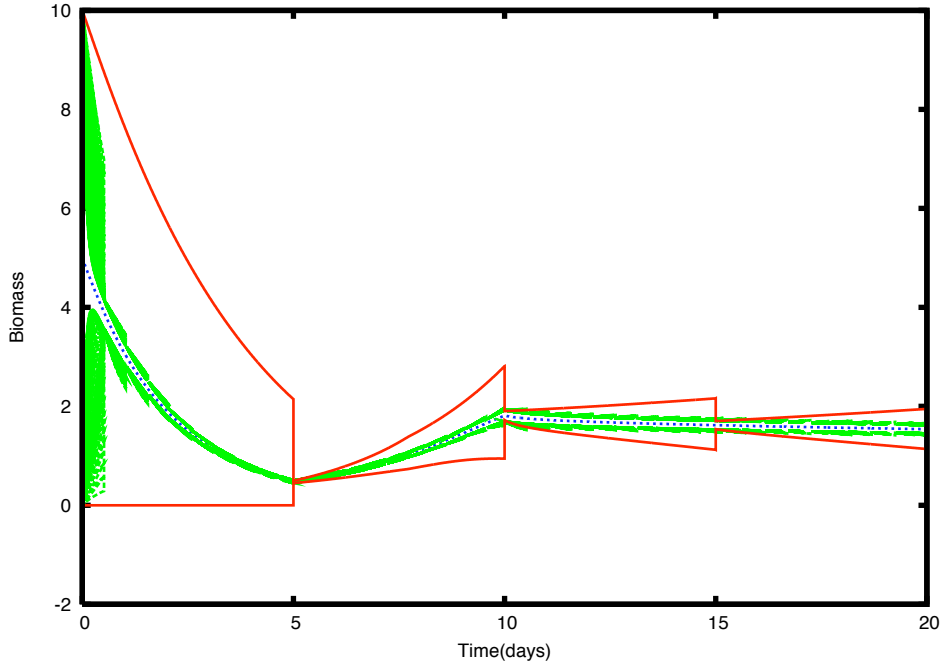


Fig. 9. The estimated interval biomass. Red continuous curves represent the lower and the upper bounds of this interval obtained by the prediction-correction estimator. Green dots curves represent lower and the upper bounds of this interval obtained by the bundle observers.  $h = 5$  days and  $r_p = 0.5$  days.

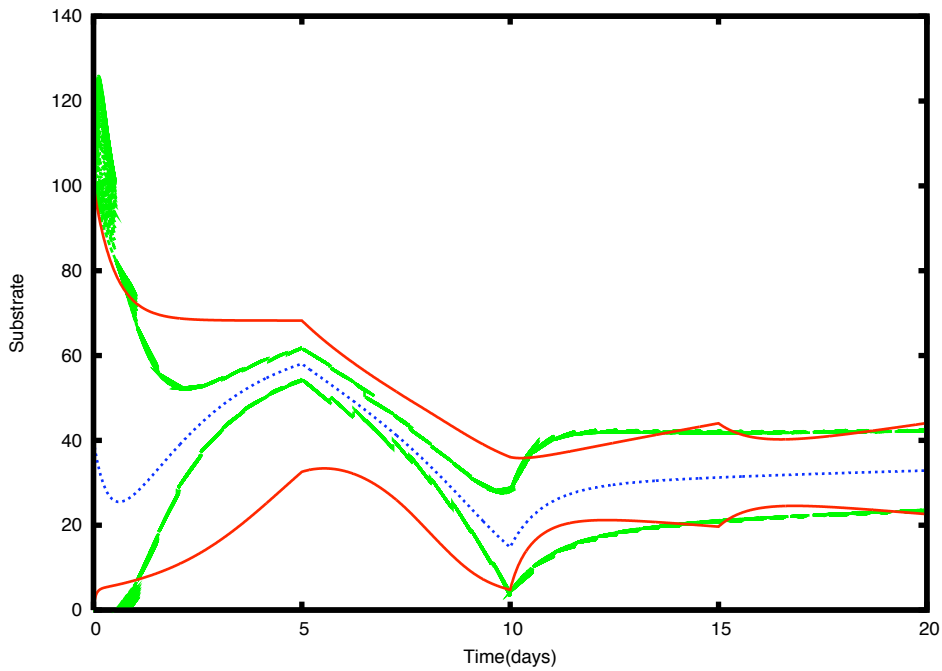


Fig. 10. The estimated interval substrate. Red continuous curves represent the lower and the upper bounds of this interval obtained by the prediction-correction estimator. Green dots curves represent lower and the upper bounds of this interval obtained by the bundle observers.  $h = 5$  days and  $r_p = 0.5$  days

$D^+ \underline{x}_i(t)$  and  $D^+ \bar{x}_i(t)$  of  $\underline{x}_i(t)$  and  $\bar{x}_i(t)$  are such that

$$D^+ \underline{x}_i(t) \leq \min_{\mathbb{D}(t)} f_i(\mathbf{x}, \mathbf{p}, t) \quad (\text{A.1})$$

$$D^+ \bar{x}_i(t) \geq \max_{\mathbb{D}(t)} f_i(\mathbf{x}, \mathbf{p}, t) \quad (\text{A.2})$$

where  $\mathbb{D}(t)$  is the subset of  $\mathbb{D}(t)$  defined by

$$\mathbb{D}_i : \begin{cases} x_i = \underline{x}_i(t) \\ \underline{x}_j(t) \leq x_j \leq \bar{x}_j(t), j \neq i \\ \underline{\mathbf{p}} \leq \mathbf{p} \leq \bar{\mathbf{p}} \end{cases} \quad (\text{A.3})$$

and where  $\bar{\mathbb{D}}(t)$  is the subset of  $\mathbb{D}(t)$  defined by

$$\bar{\mathbb{D}}_i : \begin{cases} x_i = \bar{x}_i(t) \\ \underline{x}_j(t) \leq x_j \leq \bar{x}_j(t), j \neq i \\ \underline{\mathbf{p}} \leq \mathbf{p} \leq \bar{\mathbf{p}} \end{cases} \quad (\text{A.4})$$

Then for all  $\mathbf{x}_0 \in [\underline{\mathbf{x}}_0, \bar{\mathbf{x}}_0]$ ,  $\mathbf{p} \in [\underline{\mathbf{p}}, \bar{\mathbf{p}}]$ , the following double inequalities hold

$$\bar{\mathbf{f}}(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{p}}, \underline{\mathbf{p}}, t) \leq \mathbf{f}(\mathbf{x}, \mathbf{p}, t) \leq \underline{\mathbf{f}}(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \bar{\mathbf{p}}, \underline{\mathbf{p}}, t), \quad \forall t_0 \geq t \quad (\text{A.5})$$

and the solution  $\mathbf{x}(t)$  of the system (1) exists and remains bounded by

$$\underline{\mathbf{x}}(t) \leq \mathbf{x}(t) \leq \bar{\mathbf{x}}(t), \quad \forall t_0 \geq t \quad (\text{A.6})$$

where  $\underline{\mathbf{x}}(t)$  and  $\bar{\mathbf{x}}(t)$  are the solution of the system (4). In addition, if for all  $\mathbf{p} \in [\underline{\mathbf{p}}_0, \bar{\mathbf{p}}_0]$ , the function  $\mathbf{f}(\mathbf{x}, \mathbf{p}, t)$  is Lipschitz with respect to  $\mathbf{x}$  over  $\mathbb{D}$  then this solution is unique for any given  $\mathbf{p}$ .

**Remark 4** The comparison operator  $\leq$  applied between vectors should be understood as a collection of inequalities applied component by component.

**PROOF.** See (Walter, 1997; Singer and Barton, 2006).

One manner to implement the above theorem when all the partial derivatives have constant sign, is given in (Ramdani et al., 2008, 2009). Indeed, we use the following rule.

**Rule 2** [Analysis of the partial derivatives signs]: The inequalities below are meant  $\forall t \in [t_j, t_{j+1}]$ ,  $\forall \mathbf{x}(t) \in \mathbb{X}(t)$  and  $\forall \mathbf{p} \in \mathbb{P}$ . Define  $\bar{\delta}^i(p_k)$  as follows

$$\bar{\delta}^i(p_k) = \begin{cases} \bar{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} \geq 0 \\ \underline{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} < 0 \end{cases} \quad (\text{A.7})$$

and  $\bar{\delta}^i(\mathbf{p}) = [\bar{\delta}^i(p_1), \dots, \bar{\delta}^i(p_k), \dots]^T$ . In a similar way, define  $\underline{\delta}^i(p_k)$  as follows

$$\underline{\delta}^i(p_k) = \begin{cases} \underline{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} \geq 0 \\ \bar{p}_k & \text{if } \frac{\partial f_i}{\partial p_k} < 0 \end{cases} \quad (\text{A.8})$$

and  $\underline{\delta}^i(\mathbf{p}) = [\underline{\delta}^i(p_1), \dots, \underline{\delta}^i(p_k), \dots]^T$ . Now define  $\bar{\gamma}^i(x_j)$  as follows

$$\bar{\gamma}^i(x_j) = \begin{cases} \bar{x}_i & \text{if } i = j \\ \bar{x}_j & \text{if } (i \neq j) \wedge \frac{\partial f_i}{\partial x_j} \geq 0 \\ \underline{x}_j & \text{if } (i \neq j) \wedge \frac{\partial f_i}{\partial x_j} < 0 \end{cases} \quad (\text{A.9})$$

and  $\bar{\gamma}^i(\mathbf{x}) = [\bar{\gamma}^i(x_1), \dots, \bar{\gamma}^i(x_j), \dots]^T$ . In a similar way, define  $\underline{\gamma}^i(x_j)$  as follows

$$\underline{\gamma}^i(x_j) = \begin{cases} \underline{x}_i & \text{if } i = j \\ \underline{x}_j & \text{if } (i \neq j) \wedge \frac{\partial f_i}{\partial x_j} \geq 0 \\ \bar{x}_j & \text{if } (i \neq j) \wedge \frac{\partial f_i}{\partial x_j} < 0 \end{cases} \quad (\text{A.10})$$

and  $\underline{\gamma}^i(\mathbf{x}) = [\underline{\gamma}^i(x_1), \dots, \underline{\gamma}^i(x_k), \dots]^T$ . Now the components of the differential equations which make it possible to compute the upper and lower solutions are obtained as follows

$$i = 1, \dots, n, \quad \begin{cases} \dot{\underline{x}}_i(t) = f_i(\underline{\gamma}^i(\mathbf{x}), \underline{\delta}^i(\mathbf{p}), t) \\ \dot{\bar{x}}_i(t) = f_i(\bar{\gamma}^i(\mathbf{x}), \bar{\delta}^i(\mathbf{p}), t) \end{cases} \quad (\text{A.11})$$

Denote

$$\underline{f}_i(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \underline{\mathbf{p}}, \bar{\mathbf{p}}, t) = f_i(\underline{\gamma}^i(\mathbf{x}), \underline{\delta}^i(\mathbf{p}), t) \quad (\text{A.12})$$

$$\bar{f}_i(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \underline{\mathbf{p}}, \bar{\mathbf{p}}, t) = f_i(\bar{\gamma}^i(\mathbf{x}), \bar{\delta}^i(\mathbf{p}), t) \quad (\text{A.13})$$

then obviously  $\underline{\mathbf{x}}(t)$  and  $\bar{\mathbf{x}}(t)$  are in general, solutions of a system of coupled differential equations, that is

$$\begin{cases} \dot{\underline{\mathbf{x}}}(t) = \underline{f}(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \underline{\mathbf{p}}, \bar{\mathbf{p}}, t), & \underline{\mathbf{x}}(t_j) = \underline{\mathbf{x}}_j, & t \in [t_j, t_{j+1}] \subset I \\ \dot{\bar{\mathbf{x}}}(t) = \bar{f}(\underline{\mathbf{x}}, \bar{\mathbf{x}}, \underline{\mathbf{p}}, \bar{\mathbf{p}}, t), & \bar{\mathbf{x}}(t_j) = \bar{\mathbf{x}}_j \end{cases} \quad (\text{A.14})$$

which involves no uncertain quantity.  $\square$

## References

- Alamo, T., Bravo, J., Camacho, E., 2005. Guaranteed state estimation by zonotopes. *Automatica* 41, 1035–1043.
- Alamo, T., Bravo, J., Redondo, M., Camacho, E., 2008. A set-membership state estimation algorithm based on dc programming. *Automatica* 44, 216–224.
- Anguelov, R., Markov, S., 1998. Wrapping effect and wrapping function. *Reliable Computing* 4, 311–330.
- Bernard, O., Gouzé, J., 2004. Closed loop observers bundle for uncertain biotechnological models. *Journal of process control* 14, 765–774.
- Berz, M., Makino, K., 1998. Verified integration of odes and flows with differential algebraic methods on taylor models. *Reliable Computing* 4 (4), 361–369.
- Carrizosa, E., 2001. An optimal bound for d.c. programs with convex constraints. *Mathematical Methods of Operations Research* 54 (1), 47–51.
- Chen, G., Wang, J., Shieh, L., 1997. Interval kalman filtering. *IEEE Transaction on Aerospace and Electronic Systems* 33(1), 250–258.
- Chernousko, F., 2005. Ellipsoidal state estimation for dynamical systems. *Non-linear analysis*.
- Chernousko, F., Rokityanskii, D. Y., 2000. Ellipsoidal bounds on reachable sets of dynamical systems with matrices subjected to uncertain perturbations. *Journal of optimization theory and applications* 104, 1–19.
- Chisci, L., Garulli, A., Zappa, G., 1996. Recursive state bounding by parallelotopes. *Automatica* 32, 1049–1055.
- Deville, Y., Janssen, M., Hentenryck, P. V., 2002. Consistency techniques in ordinary differential equations. *Reliable Computing* 7, 289–315.
- Dochain, D., 2003. State and parameter estimation in chemical and biochemical processes: a tutorial. *Journal of process control* 13, 801–818.
- Durieu, C., Walter, E., Polyak, B., 2001. Multi-input multi-output ellipsoidal state bounding. *Journal of Optimization Theory and Applications* 111 (2), 273–303.
- Goffaux, G., Wouwer, A. V., Bernard, O., 2009. Improving continuous-discrete interval observers with application to microalgae-based bioprocesses. *Journal of Process Control* 19, 1182–1190.
- Gouzé, J., Rapaport, A., Hadj-Sadok, Z. M., 2000. Interval observers for uncertain biological systems. *Journal of Ecological Modelling* 133, 45–56.
- Hadj-Sadok, M., Gouzé, J., 2001. Estimation of uncertain models of activated sludge processes with interval observers. *Journal of Process Control* 11, 299–310.
- Jaulin, L., Kieffer, M., Didrit, O., Walter, E., 2001. *Applied interval analysis: with examples in parameter and state estimation, robust control and robotics*. Springer-Verlag, London.
- Kieffer, M., Jaulin, L., Walter, E., 2002a. Guaranteed recursive non-linear state bounding using interval analysis. *International Journal of Adaptive*

- Control and Signal Processing 16, 193–218.
- Kieffer, M., Jaulin, L., Walter, E., 2002b. Guaranteed recursive nonlinear state estimation using interval analysis. *International journal of adaptive control and signal processing* 16, 193–218.
- Kieffer, M., Walter, E., 2006. Guaranteed nonlinear state estimation for continuous-time dynamical models from discrete-time measurements. In: *Proceedings 6th IFAC Symposium on Robust Control*. Toulouse.
- Kieffer, M., Walter, E., Simeonov, I., 2006. Guaranteed nonlinear parameter estimation for continuous-time dynamical models. In: *Proceedings 14th IFAC Symposium on System Identification*. Newcastle, Aus, pp. 843–848.
- Kletting, M., Rauh, A., Aschemann, H., Hofer, E. P., 2007. Consistency tests in guaranteed simulation of nonlinear uncertain systems with application to an activated sludge process. *Journal of computational and applied mathematics* 199, 213–219.
- Lin, Y., Stadtherr, M., 2006. Validated solutions of initial value problems for parametric odes. *Applied Numerical Mathematics*.
- Lin, Y., Stadtherr, M. A., 2007. Guaranteed state and parameter estimation for nonlinear continuous-time systems with bounded-error measurements. *Industrial and Engineering Chemistry Research* 46, 7198–7207.
- Lohner, R. J., 1987. Enclosing the solutions of ordinary initial and boundary value problems. In: Wiley (Ed.), *Computer Arithmetic: Scientific Computation and Programming Languages*. Stuttgart, pp. 255–286.
- Maksarov, D., Norton, J., 2002. Computationally efficient algorithms for state estimation with ellipsoidal approximation. *Int. J. Adapt. Control Signal Process* 16 (6), 411–434.
- Marcelli, C., Rubbioni, P., 1997. A new extension of classical Müller theorem. *Nonlinear Analysis, Theory, Methods & Applications* 28(11), 1759–1767.
- Milanese, M., Norton, J., Piet-Lahanier, H., (Eds.), E. W., 1996. *Bounding Approaches to System Identification*. Plenum Press, New York.
- Moisan, M., Bernard, O., 2005. Interval observers for non monotone systems. application to bioprocess models. In: *In proceedings of the 16th IFAC world conference*. Prague, pp. 43–48.
- Moisan, M., Bernard, O., Gouzé, J., 2009. Near optimal interval observers bundle for uncertain bioreactors. *Automatica* 45, 291–295.
- Moore, R., 1966. *Interval analysis*. Prentice-Hall, Englewood Cliffs.
- Müller, M., 1927. Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen. *Mathematische Zeitschrift* 26, 619–645.
- Nedialkov, N., 1999. Computing rigorous bounds on the solution of an initial value problem for an ordinary differential equation. PhD University of Toronto.
- Nedialkov, N., Jackson, K., Corliss, G., 1999. Validated solutions of initial value problems for ordinary differential equations. *Applied Mathematics and Computation* 105, 21–68.
- Nedialkov, N., Jackson, K., Pryce, J., 2001. An effective high-order interval

- method for validating existence and uniqueness of the solution of an ivp for an ode. *Reliable Computing* 7, 449–465.
- Nedialkov, N. S., Jackson, K. R., 1999. An interval hermite-obreschkoff method for computing rigorous bounds on the solution of an initial value problem for an ordinary differential equation. *Reliable Computing* 5, 289–310.
- Raïssi, T., Ramdani, N., Candau, Y., 2004. Set membership state and parameter estimation for systems described by nonlinear differential equations. *Automatica* 40(10), 1771–1777.
- Raïssi, T., Ramdani, N., Candau, Y., 2005. Bounded-error moving horizon state estimator for non-linear continuous-time systems : application to a bioprocess system. *Journal of Process Control* 15 (5), 537–545.
- Ramdani, N., Meslem, N., Candau, Y., 2008. Reachability of uncertain nonlinear systems using a nonlinear hybridization. *Hybrid Systems: Computation and Control, HSCC’08, St. Louis, MO, USA*, In : M. Egerstedt and B. Mishra (Eds.) : HSCC 2008, LNCS 4981, pp. 415–428. Springer-Verlag Berlin Heidelberg.
- Ramdani, N., Meslem, N., Candau, Y., 2009. A hybrid bounding method for computing an over-approximation for the reachable set of uncertain nonlinear systems. *IEEE Transactions on automatic control* 54 (10), 2352–2364.
- Rapaport, A., Dochain, D., 2005. Interval observers for biochemical processes with uncertain kinetics and inputs. *Mathematical Biosciences* 193, 235–253.
- Rapaport, A., Gouzé, J., 2003. Parallelotopic and practical observers for nonlinear uncertain systems. *International Journal of Control* 76, 237–251.
- Rihm, R., 1994. Interval methods for initial value problems in odes. In: in *Computational Mathematics, E. S. (Ed.), Validated computations: proceedings of the IMACS-GAMM International Workshop on Validated Computations*. J. Herzberger, Amsterdam, pp. 173–208.
- Schweppe, F., 1968. Recursive state estimation: unknown but bounded errors and system inputs. *IEEE Trans. On automatic Control* 13(1), 22–28.
- Singer, A., Barton, P., 2006. Bounding the solutions of parameter dependent nonlinear ordinary differential equations. *SIAM J. Sci. Comput* 27(6), 2167–2182.
- Tuy, H., 1995. DC optimization: Theory, methods and algorithms. *Handbook of global optimization*. Dordrecht: Kluwer Academic Publishers, pp. 149–216.
- Walter, W., 1997. Differential inequalities and maximum principles: Theory, new methods and applications. *Nonlinear Analysis, Theory, Methods & Applications* 30 (8), 4695–4711.
- Xu, X., Zhai, G., 2005. Practical stability and stabilisation of hybrid and switching systems. *IEEE Transaction on automatic control* 50 (11), 1897–1903.