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Perfect matchings and series-parallel graphs: multiplicatives proof nets as R&B-graphs

[Extended Abstract]

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Abstract

A graph-theoretical look at multiplicative proof nets lead us to two new descriptions of a proof net, both as a graph endowed with a perfect matching.

The first one is a rather conventional encoding of the connectives which nevertheless allows us to unify various sequentialisation techniques as the corollaries of a single graph theoretical result.

The second one is more exciting: a proof net simply consists in the set of its axioms — the perfect matching — plus one single series-parallel graph which encodes the whole syntactical forest of the sequent. We thus identify proof nets which only differ because of the commutativity or associativity of the connectives, or because final par have been performed or not. We thus push further the program of proof net theory which is to get closer to the proof itself, ignoring as much as possible the syntactical "bureaucracy".

1 Presentation

This paper introduces two new ways of looking at proof structures and nets, and their correctness criteria. Our basic tool for describing proof nets is edge-bicoloured graph, that we call R&B-graphs: one of the colours, B, defines a perfect matching or 1-factor of the graph, — a standard topic in graph theory: a matching B is a set of pairwise non-adjacent edges, and it is said to be perfect whenever each vertex is incident to an edge of B. An edge not in B is in R. We then consider æ-cycles — alternate elementary cycles — i.e. the even cycles with edges alternatively in B and in R, which does not use twice the same edge. We prove a theorem related to one by Kotzig [7] which characterises the R&B-graph without æ-cycles as an inductively defined class of R&B-graphs which recursively contain a B-isthmus.

In the first of our two approaches, the connectives are directly encoded in the R&B-graph. The criterion is the absence of æ-cycle. Using our theorem, we obtain a B-isthmus, and this is actually enough for establishing sequentialisation. We then consider two mappings of a proof net into a R&B-graph without æ-cycle. This enables us to obtain from the same graph theoretical theorem the existence of a splitting `tensor` link (sequentialisation à la Girard, [5]) and the existence of a section or splitting `par` link (sequentialisation à la Danos-Regnier, [2,3]).

The second approach, inspired by the first, is a more abstract representation of proof nets. A proof net is still a R&B-graph which simply consists in a perfect matching B which encodes the axiom links, and a single series-parallel graph R (this inductive class of graphs is rather famous [6,10]) which encodes the whole of the syntactical forest of the sequent, while the criterion is that any æ-cycle should contain a given configuration.

This presentation identifies proof nets which only differ because of commutativity and associativity, or because final `par`s have been or not performed. So we push further the research program associated with proof net which is to get as close as possible from the proof itself, ignoring as much as we can the syntactical "bureaucracy".

One can wonder whether we admit or not the `mix` rule. Actually, these results apply to both systems (with or without the `mix` rule). Nevertheless, in the body of the paper we concentrate on proof nets with `mix`, because their theory is a bit more general and their sequentialisation is a bit more difficult. We then explain in a short section how the connectivity condition which excludes the `mix` rule may be added to our presentation: it is simple and harmless.

As usual in this kind of study, we ignore the cut-rules and links, viewing them as `tensor` rules or links.

The combinatorial proofs are more developed than the logical ones, for which we assume some familiarity.

2 Multiplicative proof structures and nets.

Let $\mathcal{P}$ be a set of propositional variables, and let $\mathcal{N} = \mathcal{P} \cup \mathcal{P}^\perp$ be the set of atoms. The multiplicative formulae $\mathcal{F}$ are defined by $\mathcal{F} ::= \mathcal{N} \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} ⅋ \mathcal{F}$.

A proof structure simply consists in a multiset of formulae where atoms — elements of $\mathcal{N}$ — have been indexed in such a way that, for any index $_x$, either $_x$ does not appear or $_x$ appears exactly twice: one on a_x and one on $a_x^\perp$ for some $a \in \mathcal{P}$.

From this we easily get a graph as follows. Turn each formula into its sub-formula tree. Add an edge between propositional variables who share the same index.

The proof structure is said to be a proof net whenever each cycle of this graph contains the two edges of some `par` branching of one of the sub-formula tree. They exactly correspond to the proofs of the multiplicative sequent calculus enriched with the `mix` rule [2,4]. We refer to this description of proof structures and nets as DR proof structures and nets.

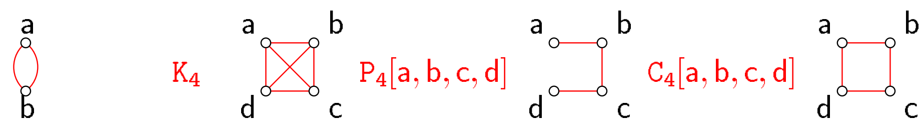
Here is an example of a proof structure with three conclusions, which is a proof net: $\mathbf{EX} = \alpha_x \ , \ (((\alpha_x^\perp \otimes \alpha_y) \parr \alpha_y^\perp) \otimes (\alpha_z \parr (\alpha_z^\perp \otimes \alpha_u))) \ , \ \alpha_u^\perp$

3 Basic terminology: graphs, matchings, series-parallel graphs

I recall the basic terminology that I use, because there are a lot of little variations that can be puzzling sometimes — I mostly follow [9].

3.1 *Graphs and matchings*

A **graph** consists of a finite non-empty set of elements called **vertices** written $\mathsf{a}, \mathsf{b}, \ldots, \mathsf{u}, \mathsf{v}, \ldots$ and of a multi-set of unordered pairs of vertices called **edges**. An edge is written xy, possibly with an integral index when there are multiple xy edges. We do not allow edges of the form xx (except in the appendix). A graph is said to be **simple** whenever there are no multiple edges — i.e. in case the multi-set of edges is a set. Here are a few graphs and nicknames for some particular simple graphs:



A bijective function f mapping the vertices of G onto the vertices of H such that both f and f^{-1} preserve the number of edges joining each pair of vertices is called an **isomorphism**, and when there exists an isomorphism from G to H, G and H are said to be isomorphic.

If there is an edge xy in a graph G, xy is said to join vertices x and y, to be be **incident** with vertices x and y, and vertices x and y are said to be **adjacent**. Two edges which share a vertex are also said two be **adjacent**. A set of edges is said to be **independent** if no two edges are adjacent.

The **degree** of a vertex x is the number of edges incident to x. In case the degree of x is one, the vertex and its unique incident edge are said to be be **pendant**.

A **path** is an alternating sequence of vertices and edges, beginning and ending with vertices, two consecutive items being incident. A path is said to **join** its first and last vertices. If all vertices are distinct the path is said to be **elementary**, and if all edges are distinct the path is said to be **simple**. The **length** of a path is the number of occurences of edges in it. A **cycle** is a path of length at least two whose end vertices are equal. A cycle is said to be **elementary** if all its vertices are distinct but the first and last. A **chord** of a cycle is an edge joining two vertices of the cycle, but not in the cycle.

If G is a graph and H is also a graph the vertices and edges of which are vertices and edges of G, H is said to be a **subgraph** of G. If H is a subgraph of G and if every edge joining two vertices of H which lies in G also lies in H, we call H an **induced** subgraph of G. Given a graph H, a graph G having no induced subgraph isomorphic to H is said to be H-**free**.

A graph is **connected** if every two vertices are joined by a path. The maximal connected induced subgraphs of G are called its **components.** An edge xy is called an **isthmus** whenever $G - \mathsf{xy}$ has more components than G.

A set of edges in a graph G is called a **matching** if no two edges are adjacent. A matching is said to be **perfect** if every vertex is incident to an edge of the matching.

Given a graph G and a matching B, a path p is said to be **alternating** if the edges of p are alternately in B and not in B.

Given a graph and a matching, an alternating elementary path will be written an **æ-path.** An alternating elementary cycle of odd length is called an **æ-loop.** An alternating elementary cycle of even length is called an **æ-cycle**, An æ-cycle is said to be **minimal** when none of its chords induces a shorter æ-cycle.

3.2 Series-parallel graphs, the `SymPa` *class*

Two vertices x and y of a graph $G = (V; R)$ are said to be **equivalent** if the bijective function mapping x onto y, y onto x and any other vertex onto itself is an isomorphism from G onto itself. If further more xy is an edge in G, they are said to be **\$-equivalent**, and otherwise to be **||-equivalent.**

Write $G_1 \| G_2$ (instead of $G_1 \oplus G_2$) for the disjoint union of the simple graphs G_1 and G_2 and call disjoint union `Pa`**rallel composition.** [1]

Define the `Sym`metrical **series composition** of two disjoint simple graphs $G_1 = (V; R_1)$ and $G_2 = (V_2; R_2)$ as the simple graph obtained from $G_1 \| G_2$ by adding an edge $\mathsf{x}_1\mathsf{x}_2$ for all x_1 in V_1 and x_2 in V_2.

The class `SymPa` of **series-parallel** graphs is the smallest class of simple graphs which contains the one-vertex graphs and which is closed under parallel and symmetrical series composition. [2]

Proposition 3.1

(i) *If a graph is* $\mathtt{P_4}$*-free, then any of its full subgraph is* $\mathtt{P_4}$*-free as well.*

(ii) *A graph is* $\mathtt{P_4}$ *free iff its complement is.*

(iii) *A graph is in* `SymPa` *if and only if it is* $\mathtt{P_4}$ *free.*

(iv) *In a* `SymPa` *graph there always exists two equivalent vertices, either \$-equivalent or ||-equivalent. If we identify them, we also obtain a* `SymPa` *graph, and the resulting vertex is respectively denoted by* $X \otimes Y$ *or* $X \otimes Y$.

(v) *Given a* `SymPa` *graph, its decomposition by means of \$ and || is unique up to the associativity and commutativity of \$ and ||.*
Examples: $\mathtt{K_4} = \mathsf{a\$b\$c\$d}$ $\quad \mathtt{C_4}[\mathsf{a,b,c,d}] = (\mathsf{a}\|\mathsf{c})\$(\mathsf{b}\|\mathsf{d})$.

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[1] These series-parallel graphs have been (re)discovered and studied many times, firstly in the forties [13] for electronic circuits, hence the name series-parallel, but also, for scheduling, concurrency and graph decomposition.

[2] This `SymPa` class of graphs exactly are the "contractile" coherent spaces, that Girard studied and characterised, i.e. the one defined from **1** by & (\$) and ⊕ (||). But I think that such an additive notation would be misleading.

Proof. Points (i) and (ii) are obvious. Using (i) and (ii) we easily prove (iii), by showing that the P_4 freeness implies that either the graph or its complement is not connected, by induction on the number of vertices. Points (iv) and (v) are also not difficult. All this is more or less known, see among others [6,10].□

4 R&B-graphs and SymPa-R&Bgraphs

4.1 R&B-*graph*

Definition 4.1 An R&B-**graph** $G = (V; B, R)$ consists of two simple graphs $G_R = (V, R)$ and $G_B = (V, B)$ with the same vertices, such that B is a perfect matching of the underlying graph $\underline{G} = (V; B \oplus R)$. The R&B-graph G is said to be *simple* or *connected* whenever $\underline{G}$ is.

Thus G is not simple if and only if there exists an edge xy common to B and R.

An R&B-graph can clearly be pictured as an edge-bicoloured graph: the B-edges, the ones of G_B will be Blue or Bold, while the R-edges, the ones G_R will be Red or Regular.

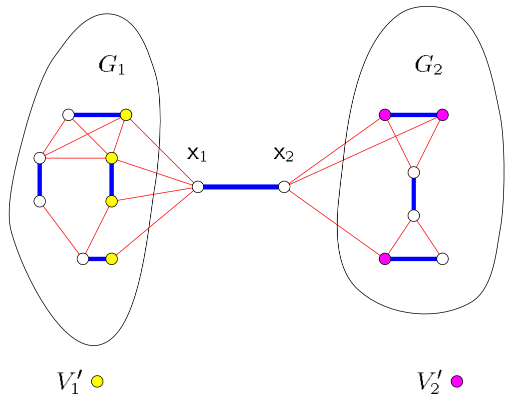
Definition 4.2 R&B$^+$is the smallest class of R&B-graphs which contains all the one edge R&B-graphs $G = (\{\mathsf{x}, \mathsf{y}\}; \mathsf{xy}, \emptyset)$ (notice this unique B-edge must be a B-edge) and which is closed under disjoint union [3] and the following operation:

$$\textit{Let} \begin{cases} G_1 = (V_1; B_1, R_1) \textit{ and } G_2 = (V_2; B_2, R_2) \textit{ be two disjoint } \mathsf{R\&B}\textit{-graphs} \\ \mathsf{x}_1 \textit{ and } \mathsf{x}_2 \textit{ be two new vertices} \\ V_1' \textit{ be a non-empty subset of } V_1 \\ V_2' \textit{ be a non-empty subset of } V_2 \end{cases}$$

define a new R&B-*graph from the* R&B-*graph* $G_1 \oplus \mathsf{x}_1 \oplus \mathsf{x}_2 \oplus G_2$ *by adding:*

- *the* B-*edge* $\mathsf{x}_1\mathsf{x}_2$
- *all* R-*edges* $\mathsf{v}_1\mathsf{x}_1$ *for vertices* v_1 *in* V_1'
- *all* R-*edges* $\mathsf{x}_2\mathsf{v}_2$ *for vertices* v_2 *in* V_2'

[3] if we skip the closure under disjoint union, we exactly obtain the connected R&B-graphs of R&B$^+$



Notice that all R&B-graph of R&B$^+$ are simple.

Theorem 4.3 *Given an* R&B*-graph* $G = (V; B, R)$ *the following properties are equivalent:*

(i) *G contains no æ-cycle (alternating elementary cycle)*

(ii) *B is the unique perfect matching of the underlying graph $\underline{G}$*

(iii) *G belongs to* R&B$^+$

Proof. If G is not simple, there is an R-edge xy_R and a B-edge xy_B and

(i) is false: x, xy_B, y, yx_R, x is an æ-cycle.

(ii) is false too: exchange the colours B and R of the two xy edges.

(iii) is false as well: as noticed above, a R&B-graph of R&B$^+$is simple.

Thus we can assume that G is simple.

$\neg$(**i**) $\Rightarrow \neg$(**ii**) Assume G contains an æ-cycle c. Every edge incident to a vertex of c but not in c is an R-edge. Exchanging the colours of the edges of c, we obtain an other perfect matching of G — notice that the fact that an æ-cycle is *elementary* is necessary: otherwise we could obtain adjacent B-edges.

$\neg$(**ii**) $\Rightarrow \neg$(**i**) Assume $\underline{G}$ is also the underlying graph of $G' = (V; B', R')$ with x_0x_1 in B but not in B' — or the converse, the question being symmetrical.

We extend a path of $\underline{G}$ starting with x_0x_1 which will be an æ-path both in G and G': the $(2p+1)^{\text{th}}$ edge is in B but not in B', hence in R' and the $2p^{\text{th}}$ edge is in B' but not in B, hence in R.

Assume the path already built is of odd length: its last edge e is in R'; since B' is a *perfect* matching of G', there must be a (unique) edge e' in B' adjacent to e. Because e is in B, e and e' are incident, and B is a (perfect) matching e' is in R. When the path already built is of even length, the argument is symmetrical.

Since $\underline{G}$ is finite, we meet a vertex x again, and thus we found an æ-path from a vertex to itself. The first and last edge may not be of the same colour in either of the R&B-graph G and G': they would be both in B or both in B' while B and B' are (perfect) matching.

Therefore this æ-path is an æ-cycle (both in G and G').

(**iii**) $\Rightarrow$ (**i**) Straightforward induction.

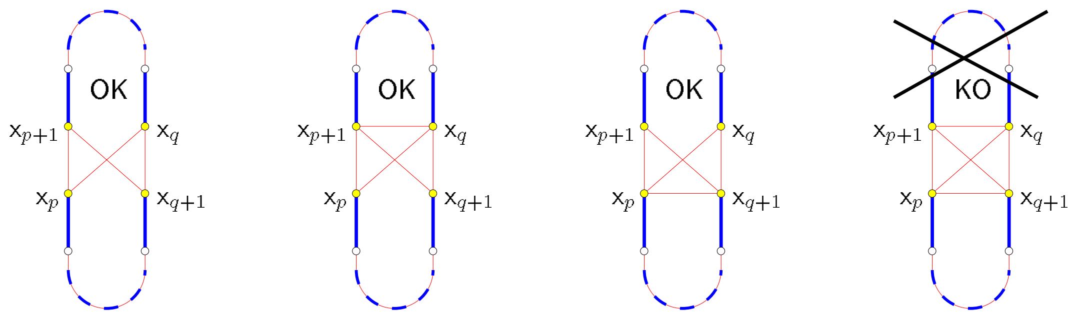
(**i**) $\Rightarrow$ (**iii**) May be deduced from $(i) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$, which is known [7]. In the literature it is deduced from difficult results: “cathedral structure theorem" [9] or Tütte's theorem [1]. That is the reason why we give in appendix our own simple proof of $(i) \to (iii)$ — which yields $(ii) \Rightarrow (iii)$ together with $(ii) \Rightarrow (i)$; furthermore it is a simple and algorithmic proof. □

Proposition 4.4 *The equivalent clauses of the previous theorem are checked in less than $O(|B|^3) = O(|V|^3)$, using the characterisation (i) and a standard breadth search algorithm.*

4.2 Series-parallel R&B*-graphs:* SymPa-R&B*-graphs*

Definition 4.5 A symmetrical series-parallel R&B-graph or SymPa-R&B-**graph** $G = (V; B, R)$ is an R&B-graph such that $G_R = (V; R)$ is a series parallel graph.

An æ-path is said to **contain a bow tie** if it contains two R-edges $x_p x_{p+1}$ and $x_q x_{q+1}$ such that the R-induced subgraph on $x_p, x_{p+1}, x_q, x_{q+1}$, contains $C_4[x_p, x_{p+1}, x_{q+1}, x_q]$ and is not K_4.



An æcycle is said to *strictly* contain a bow tie whenever the R-induced subgraph on $x_p, x_{p+1}, x_q, x_{q+1}$, *is* $C_4[x_p, x_{p+1}, x_{q+1}, x_q]$ (left most picture).

Proposition 4.6 *Given a* SymPa-R&B-*graph $G = (V; B, R)$ the two following properties are equivalent:*

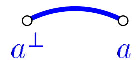
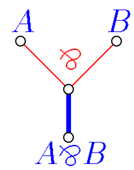
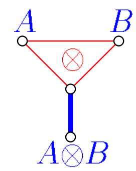
(i) *each* æ-*cycle contains a bow tie*

(ii) *each minimal* æ-*cycle strictly contains a bow tie*

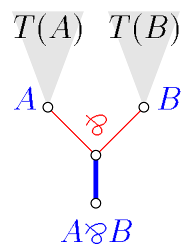
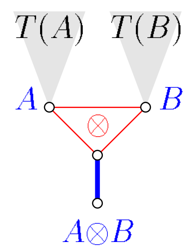
Proof. $(i) \Rightarrow (ii)$ is obvious. $(i) \Rightarrow (ii)$ is proved by induction on the number of symbols in R plus the size of the æ-cycle plus the number of B-edges: it is too lengthy to be given here. □

5 Proof structures and nets as R&B-graphs, and a first sequentialisation

Let us defines the **links** as R&B-graphs in the following way:

Links			
Name	`axiom`	`par`	`tensor`
Premises	none	A and B	A and B
`R&B`-graph			
Conclusions	$a^\perp$ and a	$A⅋B$	$A\otimes B$

The `R&B`-**tree** $T(C)$ of a formula C is defined inductively as follows:

Formula C	$a \in \mathcal{N}$	$A⅋B$	$A\otimes B$
`R&B`-tree $T(C)$			

Definition 5.1 A `R&B` **proof structure** or `R&B-PS` is a simple `R&B`-graph such that there exists a partition of its edges such that each class together with its incident vertices is isomorphic to a `R&B`-link — in such a way that the labels of the labelled vertices unify. Pendant vertices are called the **conclusions** of the `R&B-PS`.

An alternative definition is to say that a `R&B-PS` consists of the `R&B`-trees of some formulae $C_1, \ldots, C_n$, together with a matching of `B`-edges joining each atom to a dual atom.

A `R&B` proof structure is said to be a `R&B` proof net whenever it does not contain any æ-cycle.

Theorem 5.2 (sequentialisation) *Any proof of the sequent calculus is mapped onto a proof net. Conversely, any proof net corresponds to at least one proof of the sequent calculus.*

Proof. Firstly, a straightforward induction shows that a `PS` inductively constructed according to a sequent calculus proof can not contain any æ-cycle.

The converse is a consequence of the theorem 4.3, which shows that there exists a non-pendant `B`-edge which is a `B`-isthmus.

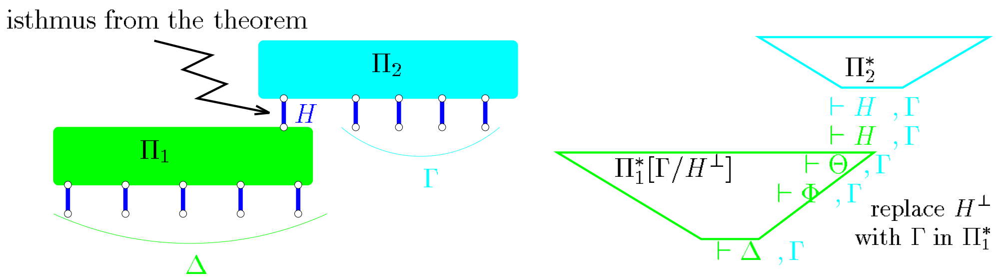
As usual, we may assume that every conclusion is the conclusion of a `tensor` or `axiom` link, and that Π is connected — otherwise apply `mix`-rules to the sequentialisations of its components.

If we suppress the pendant vertices C_i (i.e. the conclusions), their unique B-adjacent vertex $\mathtt{c}_i$, and the R-edges incident to $\mathtt{c}_i$ we obtain a simple R&B-graph Π° with no æ-cycle.

Because of the link structure, there are at most two R-edge incident to $\mathtt{c}_i$, and each time we suppress two incident R-edges $\mathtt{c}_i\mathtt{x}$ and $\mathtt{c}_i\mathtt{y}$ there is an R-edge xy. Thus a B-isthmus of Π° is a B-isthmus of Π. For the same reason if Π° is connected since Π is.

If Π° is empty, the PN Π consists of an axiom-link, or of a tensor-link between two axiom-links, and the sequentialisation of Π is trivial.

If Π° is not empty we apply the theorem 4.3 to Π°: it contains a B-isthmus, which is an inner B-isthmus of Π as well. By induction we obtain a sequent calculus proof for each part, and by "plugging" them we obtain a sequent calculus proof corresponding to Π.



As we only defined axiom links for atoms, because of η-expansion, one should first substitute a variable for H, and then replace it again with H. □

6 Two other mappings from proof structures to R&B-graphs and sequentialisation techniques

In this (sketchy) section, Π denotes a proof net à la Danos-Regnier [2,3].

6.1 Sequentialisation à la Girard: finding a splitting tensor *link*

We map Π onto a R&B-graph $\Pi_\otimes$ as follows. The vertices of $\Pi_\otimes$ correspond to the premises of tensor links. The pair of premises are linked via a B-edge: thus we have one B-edge per tensor link. We put a R-edge between a premise A of a tensor link $A \otimes B$ and a premise A' of another tensor link $A' \otimes B'$ whenever an atom of A is linked via an axiom link to an atom of A'. There is an obvious bijection between splitting tensor and B-isthmuses — notice that $\Pi_\otimes$ ignores the final par links. It is easily seen that this R&B-graph $\Pi_\otimes$ contains no æ-cycle if and only of if Π is a proof net. If it is so, $\Pi_\otimes$ is in R&B$^+$and thus, *still using Theorem 4.3*, Π contains a B-isthmus, i.e. a splitting tensor link. Thus we can perform sequentialisation as in Girard original paper [5], possibly using the mix rule when $\Pi_\otimes$ is not connected.

6.2 *Sequentialisation à la Danos-Regnier: finding a section, a.k.a. splitting* `par` *link*

We can also map Π onto a `R&B`-graph $\Pi_{⅋}$.

Remember that a **block** [2,4] in a DR proof structure is a component of the graph minus the `axiom` and `tensor` edges, and write $b(A)$ for the block of the vertex (occurrence of formula) A. Thus $b(A) \cap b(B) \neq \emptyset$ is equivalent to $b(A) = b(B)$.

$\Pi_{⅋}$ is defined as follows.

Vertices are pairs of premises of `par` links and conclusions of `par` links. We put a `B`-edge between the vertex corresponding to the pair of premises of a given `par` link and the vertex corresponding to the conclusion of the same `par` link. We put an `R`-edge between

(A, B) and (A', B') iff $b(A) \cup b(B)$ intersects $b(A') \cup b(B')$

(A, B) and $A'⅋B'$ iff $b(A) \cup b(B))$ intersects $b(A'⅋B')$

$A⅋B$ and $A'⅋B'$ iff $b(A⅋B)$ intersects $b(A'⅋B')$

There is an obvious bijection between the `B`-isthmuses of $\Pi_{⅋}$ and the "sections" or splitting `par` links of [2]. Whenever we start with a proof net Π, the resulting `R&B`-graph $\Pi_{⅋}$ contains no æ-cycle, i.e. is in `R&B⁺`. Thus *still using Theorem 4.3*, we find a section, and we can perform sequentialisation as in Danos' thesis. The correctness of $\Pi_{⅋}$ does not, in this case, imply the correctness of Π, unless Π contains no æ-cycle in its blocks.

7 `SymPa-R&B` proof structures and nets: a perfect matching plus a series-parallel graph

7.1 *Definition*

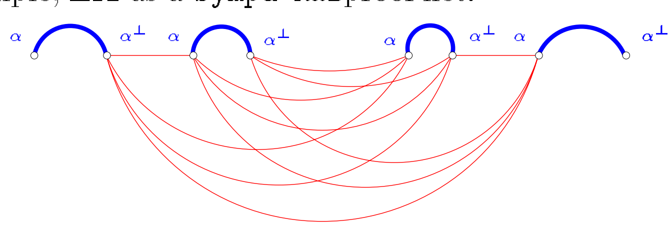
Definition 7.1 A `SymPa-R&B` **proof structure** with conclusions $C_1, \ldots, C_n$ is a `SymPa-R&B`-graph $G = (V; B, R)$ whose vertices are the occurences of atoms in $C_1, \ldots, C_n$ and which satisfies (i) and (ii):

(i) an edge of the perfect matching `B` joins two dual vertices

(ii) there is an `R`-edge between two vertices whenever they meet on a $\otimes$ in the syntactical forest

A `SymPa-R&B` proof structure is said to be a `SymPa-R&B` proof net whenever every æ-cycle contains a bow tie.

Our example, **EX** as a `Sympa-R&B`proof net:



$R = \alpha \,\|((\alpha^{\perp}\$\alpha)\|\alpha^{\perp})\ \$\ (\alpha\|(\alpha^{\perp}\$\alpha))\ \|\ \alpha^{\perp}$

The conclusions are defined as follows: any partition of the components of R defines the conclusions up to associativity and commutativity of the connectives. To find an expression for a conclusion, write R from the atoms by means of $\|$ and \$, and turn $\|$ into $⅋$, and \$ into $\otimes$.

As usual there is no particular difficulty in proving that translating rule by rule a proof of the sequent calculus into a `Sympa-R&B`proof structure, we obtain a proof *net*, i.e. a `Sympa-R&B`-graph such that every æ-cycle contains a bow tie. The `par`-rule is translated into identity, the `mix`-rule into disjoint union. The `tensor`-rule $A \otimes B$ is translated into the symmetrical series composition applied to the conclusion A (a union of some `R`-components) of one premise, and the conclusion B of the other premise. However this is a straightforward consequence of Proposition 7.3 of next subsection.

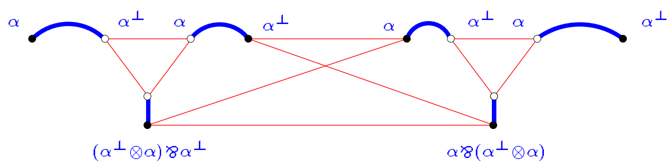
7.2 Sequentialisation: from `R&B-PS` *to* `R&B-PS` *and vice-versa*

In order to prove sequentialisation we introduce a generalisation of both `R&B-PS` and `R&B-PS`, which may have its own interest. Let us call them `Sympa+R&B` proof structures.

They consists in a `R&B`-proof structure plus some `SymPa` graph/relation R between its conclusions. We write $\Pi\frac{\Gamma}{R}$ for a `Sympa+R&B` proof structure whose conclusions are Γ and whose `SymPa` relation on Γ is R.

A `Sympa+R&B` proof structure $\Pi\frac{\Gamma}{R}$ is said to be a `SymPa-R&B` proof net or to be correct whenever every æ-cycle of $\Pi\frac{\Gamma}{R}$ contains a bow tie.

An example:



$$R = \alpha \parallel [\ ((\alpha^\perp \otimes \alpha) \parallel \alpha^\perp)\ \$\ (\alpha \parallel (\alpha^\perp \otimes \alpha))\ ] \parallel \alpha^\perp$$

If R is empty then it is a `R&B` proof structure, which is a `R&B` proof net if and only if it is a `Sympa+R&B` proof net: since there are no C_4 in a `R&B`- proof structure, to say that each æ-cycle contains a bow tie means that there is no æ-cycle.

If there is no link but `axiom`-link, it is a `SymPa-R&B` proof structure, and it is a `SymPa-R&B` proof net if and only if it is a `Sympa+R&B` proof net.

Now, we consider the following invertible transformation between `Sympa+R&B` proof structure:

Definition 7.2 Let $*$ be either \$ or $\|$ and correspondingly let $\star$ be either $\otimes$ (when $* = \$$) or $⅋$ (when $* = \|$).

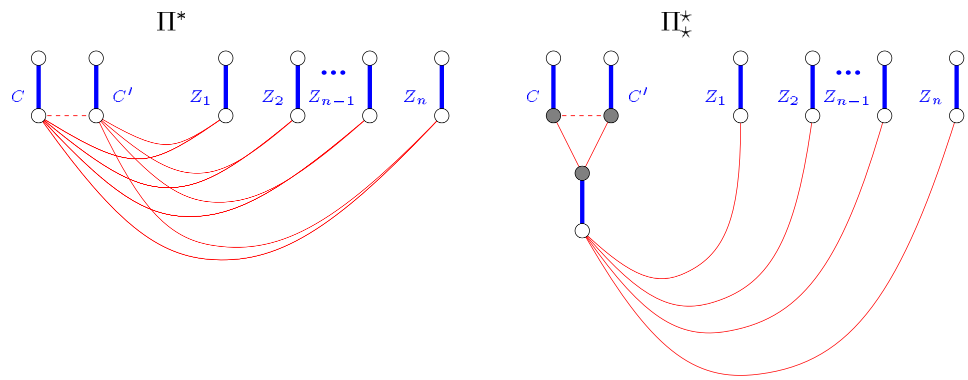
Let $\Pi\frac{\Gamma, C, C'}{S[C * C']}$ be a `Sympa+R&B` proof structure, with `SymPa` relation $R = S[C * C']$ on its conclusions — this entails that C and C' are $*$-equivalent with respect to the `SymPa` relation $R = S[C * C']$ on conclusions. The `Sympa+R&B` proof structure $\Pi\frac{\Gamma, C \star C'}{S[C \star C']}$ is obtained from the `Sympa-R&B` proof structure $\Pi\frac{\Gamma, C, C'}{S[C * C']}$ by performing the following operations :

(i) add to the underlying `R&B` proof structure $\Pi\frac{\Gamma, C, C'}{S[C * C']}$ a $\star$-link with premises

C and C': this yields a R&B proof structure $\Pi\frac{\Gamma,C\star C'}{\emptyset}$

(ii) add to this R&B proof structure the SymPa relation $S[C\star C']$ obtained by identifying the two $*$-equivalent vertices C and C' and calling it $C\star C'$

Performing these operations in the reverse order, — this time with no condition on $S[C\star C']$ — we see that this transformation obviously has an inverse, leading from $\Pi\frac{\Gamma,C\star C'}{S[C\star C']}$ to $\Pi\frac{\Gamma,C,C'}{S[C*C']}$.



Proposition 7.3

$$\Pi_* = \Pi\frac{\Gamma, C, C'}{S[C * C']} \textit{ is correct } \Longleftrightarrow \Pi_\star^\star = \Pi\frac{\Gamma, C\star C'}{S[C\star C']} \textit{ is correct}$$

Proof. We must check that the transformation and its converse preserve the two following properties:

○ each time the transformation introduces an æ-cycle, it contains a bow tie

⋈ each time an R-edge appearing in the bow tie of an æ-cycle is (re)moved, either the æ-cycle vanishes or it still contains a bow tie.

Let $Z_1, \ldots, Z_k$ be the conclusions Z such that $ZC, ZC' \in R$ — since C and C' are equivalent, ZC is in R if and only if ZC' is in R.

Remember that an æ-cycle contains a bow tie whenever it contains two R-edges $\mathtt{x}_p\mathtt{x}_{p+1}$ and $\mathtt{x}_q\mathtt{x}_{q+1}$ such that the R-induced subgraph on $\mathtt{x}_p, \mathtt{x}_{p+1}, \mathtt{x}_q, \mathtt{x}_{q+1}$ contains $\mathtt{C_4}[\mathtt{x}_p, \mathtt{x}_{p+1}, \mathtt{x}_q, \mathtt{x}_{q+1}]$ and is not $\mathtt{K_4}$. The bow tie is said to **depend on** the R-edges $\mathtt{x}_p\mathtt{x}_{p+1}$ and $\mathtt{x}_q\mathtt{x}_{q+1}$, which are said to be **essential** to the bow tie.

Π_* correct $\quad\Longrightarrow\quad \Pi_\star^\star$ correct

○ There are fewer æ-cycles in $\Pi_\star^\star$, because some are no more elementary in Π: they use twice the B-edge incident to $(C\star C')$.

⋈ Assume there is in Π_* an æ-cycle whose bow tie depends on two R-edges one of which is either CC' (if $* = \$$) or a CZ_i or a $C'Z_j$. Firstly, in case $* = \$$, a bow tie of Π_* may not depend on the R-edge CC': indeed, because C and C' are \$-equivalent the bow tie would necessary be $\mathtt{K_4}$, which is not possible. Assume the bow tie depends on the R-edges CZ_i and XY, containing $\mathtt{C_4}[C, Y, X, Z_i]$ then Y is some Z_j. Thus, if $X = C'$ the æ-cycle vanishes. Otherwise because C and C' are equivalent, we also have the R-edges $C'Z_i$ and $C'Z_j$. Thus, in $\Pi_\star^\star$ the edges $C\star C'Z_i$ and XZ_j define a bow tie in the image of the æ-cycle. Indeed we may not have both the R-edge Z_iZ_j

and $X(C{\star}C'$ in $\Pi^\star_\star$, since we would have both CX and Z_iZ_j, which conflicts with $CZ_i\ Z_jX$ being a bow tie. Hence we also have a bow tie in $\Pi(C * C')$.

$\Pi^\star_\star$ correct $\Longrightarrow$ Π_* correct

$\bowtie$ Firstly any R-edge of the suppressed $\star$-link may not belong to a bow tie. So we can assume that the bow tie depend on an R-edge $(C{\star}C')Z_k$ of an æ-cycle of $\Pi^\star_\star$, and because æ-cycle are elementary the other R-edge it depends may not be a $(C{\star}C')Z_l$ R-edge. Thus this bow tie contains some $\mathtt{C_4}[(C{\star}C'), Z_k, X, Z_n]$, with X being none of the Z_l.

The æ-cycle c of $\Pi^\star_\star$ either pass by C or by C', say C, and is mapped is mapped onto an æ-cycle of Π_* passing through C, while the bow tie of $\Pi^\star_\star$ is mapped onto a bow tie of Π_* depending on CZ_k and Z_lX.

$\circ$ If an æ-cycle appears in Π_*, while there was none in $\Pi^\star_\star$ then it contains two R-edges CZ_i and $C'Z_j$ with $i \neq j$. If $* = \|$ or $Z_iZ_j \notin R$ there is a bow tie containing $\mathtt{C_4}[C, Z_i, Z_j, C']$ — the condition ensures it is not a $\mathtt{K_4}$. If $* = \$$ we have in $\Pi^\star_\star$ an æ-cycle c, containing CC' as well as its two adjacent B-edges. This æ-cycle c contains a bow tie in $\Pi^\star_\star$, which is mapped on a bow tie of $\Pi^\star_\star$, because of the previous alinea (ii).

□

Out of this proposition we easily obtain sequentialisation for `SymPa-R&B` proof nets. Let Π be `Sympa-R&B` proof net. Perform the previous transformation in order to obtain a `R&B` proof net Π_{BR} — not in a unique way, due to commutativity and associativity of ⅋/|| and ⊗/\$, and to the possibility of stopping as soon as R is empty, or of going on until we have a single conclusion.

For instance, the example that we gave of a `Sympa+R&B` proof structure is an intermediate state between the `R&B` proof structure form and the `SymPa-R&B` form of our example **EX.**

The sequentialisation theorem for a `Sympa-R&B` proof net follows from the easy observation that any sequent calculus proof corresponding to any `R&B` proof net Π_{BR} associated with Π translates into the `SymPa-R&B` proof net Π.

8 What about the `mix` rule?

Actually, if one wants to exclude the **mix** rule and to have the standard multiplicative sequent calculus, it is quite simple. To the various criteria we introduced, one must always add:

There exists a bow tie free æ-path between any two vertices.

For `R&B` proof structures which contain no $\mathtt{C_4}$ and therefore no bow tie, it simply means that there is an æ-path between any two vertices.

9 Conclusions

This work was actually developed for pomset logic [12,8] in order to obtain a sequentialisation theorem. In this case we also have to take into account *directed* series composition which corresponds to the non-commutative and self-dual connectivebefore. A first step would be to have a close look at a direct proof of sequentialisation for the Sympa-R&B proof nets.

A direct look at cut-eliminations is rather amusing: it remind us of Girard's turbo cut-elimination, but it even seems that this time there are multiple inlet valves! Notice that two conclusions, one the negation of the other, correspond to complementary series-parallel relations.

Finally we have the feeling that this presentation of proof net as SymPa-R&B-graphs is really a meeting point between syntax and semantics.

Appendix: an algorithmic proof of a theorem by Kotzig

Coming back to theorem 4.3, here we prove $(i) \Rightarrow (iii)$. Together with $\neg(i) \Rightarrow \neg(ii)$ that we already gave, it gives a simple proof of $(ii) \Rightarrow (iii)$, known as a theorem by Kotzig [7].

I actually obtained this proof on the dual structures, for which it was a bit more difficult to formalise [12,11].

Definition 9.1 Let $G = (V; B, R)$ a R&B-graph, and ℓ be an æ-loop on v: ℓ is an æ-path of odd length whose end vertices are the same vertex v — thus the two edges of ℓ incident to v are R-edges.

Contracting the loop ℓ consists in identifying all its vertices with v (quotient graph).

Lemma 9.2 *Assume a* B *and* R *coloured graph G' is obtained from a* R&B-*graph $G = (V; B, R)$ by contracting a loop on* v *. Then:*

(i) *G' is a graph (possibly with an* R-*edge* vv*) with a perfect matching — the* B-*edges of G not in ℓ.*

(ii) *If G' is not a* R&B-*graph, then it is because it is not anymore a simple graph, and, if so, G contains an æ-cycle.*

(iii) *If there exists an æ-path between two vertices of G' then there exists one in G too, with endings of the same colour.*

(iv) *If G contains no æ-cycle so does G'.*

(v) *Whenever a* B-*edge is a* B-*isthmus in G', it is a* B-*isthmus in G too.*

Proof.

(i) The fact that the B-edges of G' still define a perfect matching is clear. Two cases may occur: either G' contains an R-edge between v and itself, or a R-edge between v and u, the other ending of the unique B-edge incident to v.

(ii) If the æ-path in G' does not use any of the new edges vw, than it is itself an æ-path in G. Otherwise, notice the R-edge vw of the æ-path may be replaced by an æ-path of G starting and ending with a R-edge: p : vR...BRw . When replacing the R-edge vw of the æ-path in G' with the æ-path p in G, there is no risk of getting a non-elementary path in G, since it only uses B-edges which do not belong to G'.

(iii) Because of the previous remark, if there was an æ-path from v to itself there would be one in G.

(iv) Notice the loop and v belong to the same 2-edge-connected block of G.

□

Theorem 9.3 *Given a* R&B*-graph* $G = (V; B, R)$*, and a vertex* $\mathtt{x} \in V$*, there exists an* æ*-path from* x *to an* æ*-cycle, or to a* B*-isthmus.*

In particular if G *has no* æ*-cycle, then there exists a* B*-isthmus.*

Proof. We extend an æ-path starting with the unique B-edge incident to x, using the following algorithm, which stops when it finds one of the two wanted configurations. An easy induction on the number of B-edges proves its termination.

1.When ending on an R-edge, we can only extend the path with *the* B-edge incident to the end vertex, which may not be already met, since each vertex of the path is incident to a B-edge, while B-edges are a matching.

2.When ending on a B-edge,

2.1 if there is no R-edge incident to it, we are done: this B-edge is a B-isthmus.

2.2. Otherwise we randomly choose an R-edge extending the path.

2.2.1. If it is still elementary, we extend the æ-path.

2.2.2. If this path is no more elementary,

2.2.2.1. either we have an æ-cycle, and an æ-path from x to this elementary cycle,

2.2.2.2. or an æ-loop ℓ on the end vertex v of the æ-path. In this latter case we contract this æ-loop on v.

2.2.2.2.1. If the graph is not a R&B-graph, Lemma 9.2 (iii) shows that G contains an æ-path from x to v and an æ-cycle containing v.

2.2.2.2.2. Otherwise, we proceed with G' which has at least one B-edge less, remembering that a wanted configuration in G' defines a similar configuration in G, by lemma 9.2. Hence, by induction on the number of B-edges we are done. □

Proposition 9.4 *This algorithm works in polynomial time. When used to sequentialise a proof net as in section 4, we first check if the proof structure is correct in* $O(|V|^3)$ *and then we use this algorithm to find a isthmus: as the cases 2.2.1. and 2.2.2.2.1. may not appear, the isthmus is found in* $O(|V|^2)$*. Thus the sequentialisation is performed in* $O(|V|^3)$*.*

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