POSITIVE AND NEGATIVE

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ABSTRACT. An introduction to the distinction between positive rules and negative rules, positive operators and negative operators, positive formulas and negative formulas: so, a first introduction to ludics.

1. SEQUENTS. REPRESENTATIONS OF A SEQUENT. FORMULAS

The first exercise to do in order to introduce in our topic - the positive and negative in logic - is to become familiar with a good definition of sequents, to understand that the representations of sequents include the focusing on some formulas, to see the negation not as a connective but as a metalinguistic operation. Indeed, our consideration of positive and negative in logic does not depend on the linguistic connective of negation.

It is convenient - in classical logic as well as in linear logic - to consider a sequent as a finite multiset of formulas (i.e. a set of occurrences of formulas), and to see the usual representation of a sequent

$$\vdash \Gamma$$

(where Γ is a finite sequence of formulas) as a presentation of the finite multiset of formulas containing all and only all the formulas occurring in Γ. (The consideration of sequents as ordered finite multisets of formulas leads to non-commutative logic.

Given a sequent, we may change its representation, i.e. to write another representation of the same sequent, a representation which differs from the previous one only in the order the occurrences of formulas are listed, i.e. a representation which is obtained from the previous one by a permutation of the occurrences of formulas. The change of the representation of a sequent is called *exchange rule*: this rule plays an important role when the sequents are considered as ordered finite multisets of formulas.

Usually, each representation of a sequent is induced or motivated by a particular attention we want to put on particular formulas, (in general on one or two formulas) in order to get a proof or to search a proof; when we want to put particular attention on some formulas in order to get a proof or to search a proof, we say that we are focusing on these formulas, or that some formulas are focused. Focused formulas in the representation of a sequent are placed at the end or at the beginning of the representation; e.g.:

$$\vdash \Gamma, A \text{ (focus on } A) \text{ or } \vdash \Gamma, A, B \text{ (focus on } A \text{ and } B).$$

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When we consider the sequents with the right side only (i.e. one-sided sequents), we do not consider the negation as a linguistic connective and we require that, for each formula A, there must be one and only one formula B which is called *negation of* A and is denoted in classical logic $\neg A$ and in linear logic $A^{\perp}$ or $\sim A$, in such a way that

$$\neg\neg A = A \text{ in classical logic, } A^{\perp\perp} = A \text{ in linear logic.}$$

A formula may be:

- an atomic non logical formula (its negation is also an atomic non logical formula);
- a logical constant, i.e.
  - in linear logic, 1 (one, the multiplicative true), $\top$ (top, the additive true), $\perp$ (bottom, the multiplicative false), 0 (zero, the additive false), with the negations defined as follows:
    $$1^{\perp} = \perp,\ \perp^{\perp} = 1,\ \top^{\perp} = 0,\ 0^{\perp} = \top$$
  - in classical logic, V (true) and F (false), i.e. $V = 1 = \top$ and $F = \perp = 0$, with the negations defined as follows:
    $$\neg V = F,\ \neg F = V.$$
- a conjunction or a disjunction of two formulas A, B, i.e.
  - in linear logic, $A \otimes B$(times, multiplicative conjunction) , $A \invamp$ (par, multiplicative disjunction), $A \& B$(with, additive conjunction) , $A \oplus$ (plus, additive disjunction) with the negations defined as follows
    $$(A\&B)^{\perp} = B^{\perp} \oplus A^{\perp}\ ,\ (A \oplus B)^{\perp} = B^{\perp}\&A^{\perp}$$
  - in classical logic, $A \wedge B$(and, conjunction) , $A \vee$ (or, disjunction), i.e. $\wedge = \otimes = \&$ and $\vee = \invamp = \oplus$, with the negation defined as follows
    $$\neg(A \wedge B) = \neg B \vee \neg A\ ,\ \neg(A \vee B) = \neg B \wedge \neg A\ (A\&B)^{\perp} = B^{\perp} \oplus A^{\perp}\ ,$$
    $$(A \oplus B)^{\perp} = B^{\perp}\&A^{\perp}$$
- an universal or existential quantification of a formula A, i.e. (both in classical logic and in linear logic) $\forall x A$ (all, $(A \otimes B)^{\perp} = B^{\perp} \invamp A^{\perp}$, $(A \invamp B)^{\perp} = B^{\perp} \otimes A^{\perp}$ universal quantifier) and $\exists x A$, with the negation defined as follows in linear logical
  $$(\forall x A)^{\perp} = \exists x A^{\perp}\ ,\ (\exists x A)^{\perp} = \exists x A^{\perp}$$
  and in classical logic
  $$\neg(\forall x A) = \exists x \neg A\ ,\ \neg(\exists x A) = \exists x \neg A$$

2. Rules. Focused formulas

The second exercise to do in order to introduce to our topic (positive and negative in logic) is to classify usual rules in logic and to understand that in each rule some formulas are focused in premisses and in the conclusion of the rule.

The usual rules of sequent calculus for first-order classical logic - written by taking into account what has been understood by means of linear logic - are presented in figure 1; these rules have one of these forms.

- 0-ary rules: no premise (i.e. no provability condition in order to prove the conclusion) and one conclusion where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent. For example,
  - identity rule (the conclusion contains only two formulas, a formula A and the negation of A, and both the formulas are focused),
  - multiplicative rule for "true" (the conclusion contains only the formula "true" which is focused),
  - additive rule for "truth" (the conclusion must contain the formula "truth" which is focused, and may contain a finite number of other formulas).
- Strictly unary rules: one premise where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent, one conclusion where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent, if the premise is provable then the conclusion is provable as well. For example,
  - multiplicative rule for "falsehood" (where no formula is focused in the premise, and the formula "falsehood" is focused in the conclusion),
  - the multiplicative disjunction rule or "par-rule" (two formulas are focused in the premise, and the disjunction of these formulas is focused in the conclusion),
  - the existential quantifier rule (where one formula with a term - say $A[t]$ - is focused in the premise, and the formula $\exists x A[x/t]$ is focused in the conclusion),
  - the universal quantifier rule (where one formula A together with a variable x not occurring in the context is focused in the premise, and the formula $\forall x A$ is focused in the conclusion),
  - the weakening rule (where no formula is focused in the premise, and one formula is focused in the conclusion),
  - the contraction rule (where two occurrences of a formula A are focused in the premise, and one occurrence of the same formula is focused in the conclusion).
- Rules with a disjunction of two premisses: two premisses where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent, one conclusion where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent, if at least one of the premises is provable then the conclusion is provable as well. Usually, these rules are presented as a pair of rules with the same conclusion but different premise. The example is the additive disjunction rule or "plus rule" (in each premise one formula is focused, and the disjunction of these formulas is focused in the conclusion);
- Rules with a conjunction of two premisses: two premisses where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent, one conclusion where some formulas are *focused* by the rule and explicitly listed in the presentation of the sequent, if both the premisses are provable then the conclusion is provable as well. The example are:
  - cut rule (in one premises a formula is focused and in the other premise the negation of this formula is focused, no formula is focused in the conclusion),

– multiplicative conjunction rule or "times-rule"and additive conjunction rule or "with rule" (in each premise one formula is focused, and the conjunction of these formulas is focused in the conclusion).

3. Systems of rules. Reversible (negative) rules and non-reversible (positive) rules

At the basis of the understanding the distinction between positive and negative in logic, there is the fact that each rule may be reversible or not reversible with respect to a given system of rules. Reversible rules may be called negative rules, and non-reversible rules may be called positive.

3.1. **Definitions.** By giving a system of rules, we get a sequent calculus and we may define a concept of *provability* with respect to each given sequent calculus.

The system of rules of figure 1 is a sequent calculus for first-order classical logic.

Let us consider the rules which belong to a given sequent calculus and have exactly one formula focused in the conclusion.

A rule with exactly one formula focused in the conclusion, with respect to a given sequent calculus, is called *negative* or *reversible* iff in the given sequent calculus the premise of the rule (i.e. no premise, the disjunction of two premise, or the conjunction of two premisses) is sufficient and necessary condition of the provability of the conclusion of the rule, i.e. for for every formula A which can be the focus of the rule in the conclusion and for every Γ

- if the rule is 0-ary, $\vdash \Gamma, A$ is provable iff $\vdash \Gamma, A$ is conclusion of the rule with focus on A,
- if the rule is strictly unary, $\vdash \Gamma, A$ is provable iff the premise of the rule (with the focus in A in the conclusion) is provable ;
- if the rule is based on the disjunction of two premisses, $\vdash \Gamma, A$ is provable iff at least one of the premise of the rule (with the focus in Ain the conclusion) is provable ;
- if the rule is based on the conjunction of two premisses, $\vdash \Gamma, A$ is provable iff both the premisses of the rule (with the focus in A in the conclusion) are provable.

A rule with exactly one formula focused in the conclusion, with respect to a given sequent calculus, is called *positive* or *not reversible* iff the rule is not a negative one.

The fact that a rule is positive means that the rule is not simply the description of what is in the premisses, but it introduces something new, it creates something: "positive" means "creative", "negative" means "descriptive". This may be explained in each rule. If you have a system of only negative rules, the search of a proof is very easy: from bottom up, i.e. from the conclusion, you always know what is the premise or the premisses of the rules you are using.

In the presence of positive rules, the search of a proof is not so easy: from bottom up, when you decides to use a positive rule, you have to choice between a lot (sometimes, an infinity) of possibilities for the premise or for the premisses of the rule.

3.2. **Negative rules in first-order classical logic.** Let us consider the system of rules of sequent calculus for first-order classical logic (figure 1).

The following rules are negative:

- the multiplicative rule for "falsehood";
- the multiplicative disjunction rule or "par-rule";
- the additive rule for "truth";
- the additive conjunction rule or "with-rule";
- the universal quantification rule.

In the following, we present the proof that these rules are negative i.e. reversible.

The general way to prove that a rule is negative is the following one: one supposes that $\vdash \Gamma, A$ is provable and A is a possible focus of the conclusion of the rule, and one shows that:

- in the case of a 0-ary rule, $\vdash \Gamma, A$ is the conclusion of the rule;
- in the case of a strictly unary rule, from a cut with a provable sequent containing the negation of A one gets the sequent which is the premise of the rule (when A is focused in the conclusion)
- in the case of a rule based on the disjunction of two premisses, from a cut with a provable sequent containing the negation of A one gets A one gets a sequent which is one of the premisses of the rule (when A is focused in the conclusion)
- in the case of a rule based on the conjunction of two premisses, from a cut with a provable sequent containing the negation of A one gets one of the two premisses of the rule (when A is focused in the conclusion) and from a cut with another provable sequent containing also the negation of A one gets the other premise.

So, this way depends only on the duality and cut-rule.

Now, we give the proof of the reversibility of the above listed rules (we use the symbols for corresponding operators in linear logic).

- The multiplicative rule for "falsehood": if $\vdash \Gamma, \bot$ is provable, since the sequent $\vdash, 1$ is provable, by cut-rule also the sequent $\vdash \Gamma$ is provable.
- The multiplicative disjunction rule or "par-rule": if $\vdash \Gamma, A \parr B$ is provable, since the sequent $\vdash B^{\bot} \otimes A^{\bot}, A, B$ is provable, by cut-rule also the sequent $\vdash \Gamma, A, B$ is provable
- The additive rule for "truth": if $\vdash \Gamma, \top$ is provable, then this sequent is also an axiom.
- The additive conjunction rule or "with-rule": if $\vdash \Gamma, A \& B$ is provable, since the sequents $\vdash B^{\bot} \oplus A^{\bot}, A$ and $\vdash B^{\bot} \oplus A^{\bot}, B$ are provable, by cut rule also the sequents $\vdash \Gamma, A$ and $\vdash \Gamma, B$ are provable.
- The universal quantification rule: if $\vdash \Gamma, \forall x A$ is provable, since the sequent $\vdash \exists x A^{\bot}, A[y/x]$ (where y is a variable not free in Γ) is provable, by cut-rule also the sequent $\Gamma, A[y]$ is provable.

3.3. Positive rules in first-order classical logic. The following rules (i.e. the other rules, among the list of rules presented in figure 1) are positive:

- the multiplicative rule for "truth";
- the multiplicative conjunction rule or "times-rule";
- the additive rule for "falsehood";
- the additive disjunction rule or "plus-rule";
- the existential quantification rule.

Below, we shall give the proof of the fact that these rules are positive, i.e. not reversible.

The general way to prove that a rule is positive is the following one: one takes a formula A which is a possible focus of the conclusion of the rule, and one shows that the sequent formed by A and the negation of A (i.e. A in the context Γ where Γ is the negation of A) is obviously provable (since it is an axiom) but it is not a conclusion of the rule (with focus on A) since the context is not the one required or since the premise or the premisses of the rules are not cut-free provable.

So, this way depends only on the duality and cut-elimination theorem.

Now, we give the proof of the non-reversibility of the above listed rules (in each rule, we represent each logical operator by using the corresponding symbol of linear logic).

- The multiplicative rule for "truth": the sequent $\vdash \bot, 1$ is provable (identity axiom) but it is not conclusion of this rule.
- The multiplicative conjunction rule or "times-rule": when P, Q are atomic non-logical formulas, the sequent $\vdash Q^{\perp} \parr P^{\perp}, P \otimes Q$ is provable (identity axiom) but it is not conclusion of "times-rule" since otherwise the sequent $\vdash P$ or $\vdash Q$ would be cut-free provable (but no such sequent is cut-free provable).
- The additive rule for "falsehood": the sequent $\vdash \top, 0$ is provable (identity axiom) but it is not conclusion of the rule for 0 (indeed, there is no rule for 0).
- the additive disjunction rule or "plus-rule": the sequent $\vdash Q^{\perp} \& P^{\perp}, P \oplus Q$ with P, Q atomic non-logical formulas is provable (identity axiom) but it is not conclusion of "plus-rule" since otherwise the sequent $\vdash Q^{\perp}, P$ or $\vdash P^{\perp}, Q$ would be cut-free provable (but no such sequent is cut-free provable).
- The existential quantification rule: the sequent $\vdash \forall x P^{\perp}(x), \exists x P(x)$ with P atomic non-logical formula is provable (identity axiom) but it is not conclusion of the existential quantifier rules since otherwise the sequent $\vdash P^{\perp}(y), P(x)$ with y different from x would be cut-free provable (but no such sequent is cut-free provable).

4. Positive and negative operators, in classical logic and in linear logic

Let us call *logical operators* the logical constants, the logical connectives, the logical quantifiers.

Given a sequent calculus, a logical operator is called *positive* iff it is defined by a positive rule, and it is called *negative* iff it is defined by a negative rule.

Moreover, a formula A is called *positive* iff the main logical operator in A is positive, and it is called *negative* iff the main logical operator in A is negative.

Remark that, under this natural definition, atomic formulas without logical operators cannot be classifies into positive or negative formulas.

Given the sequent calculus for first-order classical logic,

- the classical logical constants V ("true") is both positive and negative, since it is definable both by a positive rule and a negative rule, due to weakening and contraction rules;
- the classical logical constant F ("false") is both positive and negative, since it is definable both by a positive rule and a negative rule, due to weakening and contraction rules;
- the classical logical connective $\wedge$ ("and“) is both positive and negative, since it is definable both by a positive rule and a negative rule, due to weakening and contraction rules;
- the classical logical connective $\vee$ ("or“) is both positive and negative, since it is definable both by a positive rule and a negative rule, due to weakening and contraction rules;
- the classical logical quantifier $\forall$ ("for all“) is negative, since it is defined by a negative rule;
- the a au classical logical quantifier $\exists$ ("for some“) is positive, since it is defined by a positive rule.

So, the presence of structural rules in classical logic does not allow to distinguish between positive and negative operators, and between positive and negative formulas, except for the quantifiers and formulas beginning with a quantifier.

I.e.: the presence of structural rules produces the collapse of the distinction positive/negative in propositional logic, whereas does not produce a similar effect for quantifiers and for quantified formulas.

Linear logic without exponentials allows to distinguish between positive and negative logical constant, and between positive and negative logical connectives, as well as between positive and negative quantifiers, as follows:

- positive operators: 1 (multiplicative truth), 0 (additive falsehood), $\otimes$ (multiplicative conjunction), $\oplus$ (additive disjunction), $\exists$ (existential quantifier);
- negative operators: $\perp$ (multiplicative falsehood), $\top$ (additive truth), $\invamp$ (multiplicative disjunction), & (additive conjunction), $\forall$ (universal quantifier);
- the dual of a positive operator is a negative operator, and the dual of a negative operator is a positive operator, so that the linear negation $A^{\perp}$ of a formula A is a negative formula if A is a positive formula, and is a positive formula if A is a negative formula.

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(continue...)

Figure 1: the rules of sequent calculus, written on the blackboard during the talk of September 24.

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