

**QUANTIFICATION
IN ORDINARY LANGUAGE**
from a critic
of set-theoretic approaches
to a proof-theoretic proposal

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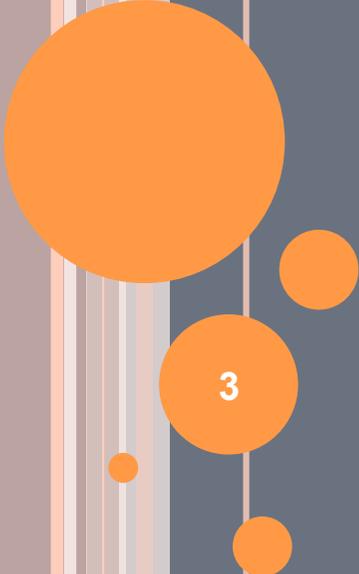
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CONTENTS

- Initially (lexical semantics in type theory)
 - *I put all the books in the cellar*, (physical object)
 - *indeed, i already read them all*. (information content)
 - There can be several occurrences of the “same” book.
- Standard quantification (history, linguistic data)
- Models, generalized quantifiers
- Second order and individual concepts
- What is a quantifier (in proof theory)?
 - Generic elements (Hilbert)
 - Cut-elimination
- Conclusion



USUAL QUANTIFICATION

Some, a, there is,...

All, each, any, every,...

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ARISTOTLE, & SCHOLASTICS (AVICENNA, SCOTT, OCKHAM)

- *A* and *B* are terms
(« term » is vague: middle-age distinction between terms, « suppositionnes », eg. Ockham)
 1. All *A* are *B*
 2. Some *A* are *B*
 3. No *A* are *B*
 4. Not all *A* are *B*
- Rules, syllogisms
- Remarks:
 - Little about models or truth condition
 - Always a restriction (sorts, kinds,?)
 - « not all » is not lexicalized and some *A* are not *B* has a different focus.

FREGE AND ANALYTIC PHILOSOPHY

- Attempt of a deductive system
- A single universe where variables « vary »:
 - All A are B
 - $\forall x(A(x) \rightarrow B(x))$
- Deduction, proofs (Hilbert) using a generic element
- Models, truth condition (Tarski)
- Adequation proofs-models:
completeness theorem (Gödel, Herbrand, ~1930)
 - Whatever is provable is true in any model.
 - What is true in every model is provable.
- Extensions:
 - Logical extensions are possible (intuitionistic, modal,...)
 - No satisfying extension to higher order
 - No proper deductive system for generalized quantifiers

HOW DOES ONE ASSERT, USE OR REFUTE USUAL QUANTIFIED SENTENCES

- « For all » introduction rule
 - (how to prove \forall as a conclusion)
 - Derive $\forall xP(x)$, from $P(a)$ for an object a without any particular property, i.e. a generic object a .
 - If the domain is known, $\forall xP(x)$ can be inferred from a proof of $P(a)$ for each object a of the domain.
The domain has to be finite to keep proofs finite. The Omega rule of Gentzen is an exception.
- « For all » elimination rule
 - (how to use \forall as an assumption)
 - From $\forall xP(x)$, one can conclude $P(a)$ for any object a .

HOW DOES ONE ASSERT , USE OR REFUTE USUAL QUANTIFIED SENTENCES

- « Exists » introduction rule
 - (how to prove \exists as a conclusion):
 - if for some object a $P(a)$ is proved, then we may infer $\exists x P(x)$
- « Exists » elimination rule
 - (how to use \exists as an assumption):
 - If C holds under the assumption $P(a)$, with a only appearing in $P(a)$, and if we know that $\exists x P(x)$, we may infer C without the assumption $P(a)$.

REFUTATIONS

- $\exists xP(x)$: little can be done apart from proving that all do not have the property.
- $\forall xP(x)$: *Any dog may bite.*
this can be refuted in at least two ways:
 - Displaying an object not satisfying P
Rex would never bite.
 - Asserting that a subset does not satisfy P,
thus remaining with generic elements:
Basset hounds do not bite.
- (ideas around Avicenna) a property is always asserted of **a term as part of a class**
(distinction homogenous/heterogenous predicate)
different sorts rather than a single Fregean universe

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIALS

- Existentials are highly common:
they even are used to structure a discourse as in
Discourse Representation Theory.
- Generally with restriction, possibly implicit:
human beings, things, events, ...
 - There's a tramp sittin' on my doorstep
 - Some girls give me money
 - Something happened to me yesterday
- Focus is difficult to account for:
 - Some politicians are crooks.
 - ? Some crooks are politicians.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE UNIVERSALS

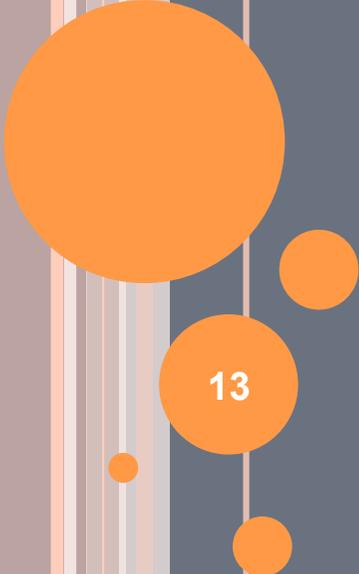
- Less common but present.
- With or without restriction:
 - Everyone, everything, anyone, anything,...
 - Every, all, each,...
- Generic (proofs), distributive (models)
 - Whoever, every,...
 - All, each,...
- Sometimes ranges over potentially infinite sets:
 - Each star in the sky is an enormous glowing ball of gas.
 - All groups of stars are held together by gravitational forces.
 - He believes whatever he is told.
 - Maths

USUAL QUANTIFICATION IN ORDINARY LANGUAGE UNIVERSAL NEGATIVE

- With or without restriction:
 - No one, nothing, not any, ...
 - No,...
- Generic or distributive:
 - Because no planet's orbit is perfectly circular, the distance of each varies over the course of its year.
 - Porterfield went where no colleague had gone previously this season, realising three figures.
 - I got no expectations.
 - Nothing's gonna change my world.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIAL NEGATIVE

- Not lexicalised (in every human language?):
 - Not all, not every, ...
 - Alternative formulation (different focus):
some ... are not ... / some ... do not ...
- Harder to grasp (psycholinguistic tests),
frequent misunderstandings (→ nothing, no one)
- Rather generic reading:
 - Not Every Picture Tells a Story
 - Everyone is *entitled* to an opinion, but *not every* opinion is *entitled* to student government funding.
- Alternative formulation (different focus):
 - *Some Students Do Not Participate In Group Experiments Or Projects.*



INDIVIDUAL CONCEPTS

Alternative view of individuals and quantification

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MOTIVATION FOR INDIVIDUAL CONCEPTS

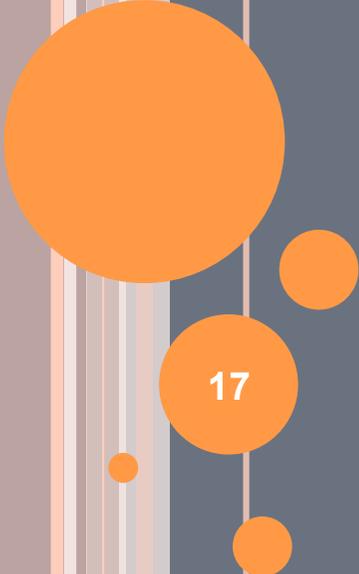
- Usual semantics with possible worlds:
It is impossible to believe that
 Tullius≠Cicero
with rigid designators
- To come back to the notion of TERM
 - Individuals are particular cases of predicates.
- Quantification is a property of predicates.

FIRST ORDER IN SECOND ORDER: PROOFS

- P is an individual concept whenever $IC(P)$:
 - $\forall x \forall y (P(x) \wedge P(y) \rightarrow x=y)$
 - $\exists x P(x)$
- First order quantification from second order quantification:
 - $\Pi P IC(P) \rightarrow X(P)$
 - $\Sigma P IC(P) \& X(P)$
- As far as proofs are concerned, this is equivalent to first order quantification – if emptiness is allowed implications only (Lacroix & Ciardelli)

MODELS?

- Natural (aka principal models): no completeness
- Henkin models:
completeness and compactness
but unnatural,
e.g. one satisfies all the following formulae:
 - F_0 : every injective map is a bijection
(Dedekind finite)
 - $F_n, n \geq 1$: there are at least n elements



GENERALIZED QUANTIFIERS

Quite common in natural language

Central topic in analytic philosophy (models)

Proofs and refutations?

DEFINITION

- Generalized quantifiers are operators that gives a proposition from two properties (two unary predicates):
 - A restriction
 - A predicate
- Some are definable from usual first order logic:
 - At most two,
 - Exactly three
- And some are not (from compactness):
 - The majority of...
 - Few /a few ...
 - Most of... (strong majority + vague)
- Observe that Frege's reduction cannot apply:
 - Most students go out on Thursday evening.
 - For most people, if they are student then they go out on Thursday evening

MODELS / PROOFS

- There are many studies about the models, the properties of such quantifiers, in particular monotony w.r.t. the restriction or the predicate.
- Formalisation with cardinality are wrong:
 - Most of \ggg the majority of
 - Most numbers are not prime.
Can be found in maths textbooks.
 - Test on “average” people:
 - most number are prime (no)
 - most number are not prime (yes)
 - No cardinality but measure, and what would be the corresponding generic element?
An object enjoying most of the properties?
- Little is known about the proofs (tableaux methods without specific rules, but taking the intended model into account).

« THE MAJORITY OF » ATTEMPT (PROOF VS. REFUTATION)

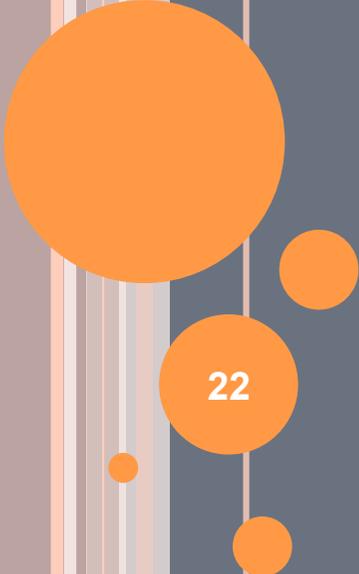
- Two ways of refuting the majority of (meaning at least 50%) the A have the property P:
 - Only a minority (less than) of the A has the property P
 - There is another property Q which holds for the majority of the A with no A satisfying P and Q.
 - What would be a generic majority element?

DEFINE JOINTLY RULES FOR:

1) THE MAJORITY OF

2) A MINORITY OF

- « For all » entails the « majority of »
- If any property Q which is true of the majority of A meets P, then P holds for the majority of the A (impredicative definition, needs further study)
- A minority of A is NOT P should be equivalent to The majority of A is P
- The majority of does not entail a minority of
- Forall => majority of
- Only a minority => Exists
- *A linguistic remark why do we say « The majority » but « A minority » ?*



WHAT SHOULD BE THE SHAPE OF QUANTIFIER RULES?

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Proof-theoretical view: to allow cut-elimination.

IN PROOFS, FOR ALL IS NOT A LARGE CONJUNCTION

- Existential rule keep the finiteness of proofs: one is enough, from $P(b)$ infer $\exists x P(x)$.
- Universal rule requires either:
 - A known domain D (what is the status of constants)
 - Finite
 - Infinite (loss of the finiteness, recursive descriptions,...)
→ infinite sequents if multiplicative conjunctions
 - Infer $\forall x P(x)$ when $P(x)$ is true of all (each) x in D (Gentzen Omega Rule)
 - A generic element (already in Pythagore)

COMMUNICATION (INTERACTION) BETWEEN PROOFS: CUT RULE

- Cut-rule: two proofs π and ρ may communicate (interact) by means of a formula A , i.e. when
 - π ends with a formula A and other formulas Γ
 - ρ ends with the negation $\sim A$ and other formulas Λ
- The communication (interaction) between such a pair of proofs produces a proof which ends with the formulas Γ and the formulas Λ
- Cut-elimination procedure is the development of such a communication (interaction)

A SPECIAL CASE OF COMMUNICATION, LEADING TO QUANTIFIERS RULES.

- A proof π of $A(b)$ under assumptions Γ
- A proof ρ of $\sim A(d)$ under assumptions Λ
- These proofs may be composed (cut) when one of the following cases holds:
 - The object b is the same as the object d (indeed, replace b by d in $A(b)$, or replace d by b in $\sim A(d)$)
 - The object b is generic in π (i.e. it does not occur in the formulas Γ) (indeed, replace b by d in $A(b)$)
 - The object d is generic in ρ (i.e. it does not occur in the formulas Λ) (indeed, replace d by b in $\sim A(d)$)

GENERIC OBJECTS : HILBERT'S APPROACH

- Rules for τx :
 - *when $\tau x A(x)$ has the property A , every object has.*
 - From $A(b)$ with b generic, infer $A(\tau x A(x))$ [$\forall x A(x)$]
 - From $\sim A(d)$, infer $\sim A(\tau x A(x))$ [$\sim \forall x A(x)$]
 - So, one reduces to general case of cut rule
 - The development of cut rule is: replace $\tau x A(x)$ by d
- Rules for ϵx :
 - *when an object has the property A , $\epsilon x A(x)$ has property A .*
 - From $A(b)$ with b generic, infer $A(\epsilon x \sim A(x))$ [$\sim \exists x \sim A(x)$]
 - From $\sim A(d)$, infer $\sim A(\epsilon x \sim A(x))$ [$\exists x \sim A(x)$]
 - So, one reduces to general case of cut rule
 - The development of cut rule is: replace $\epsilon x \sim A(x)$ by d
- $A(\tau x A(x)) \leftrightarrow A(\epsilon x \sim A(x))$ [$\forall x A(x)$]
- $A(\tau x \sim A(x)) \leftrightarrow A(\epsilon x A(x))$ [$\exists x A(x)$]

HILBERT FUNCTIONS & USUAL FREGEAN RULES ARE EQUIVALENT

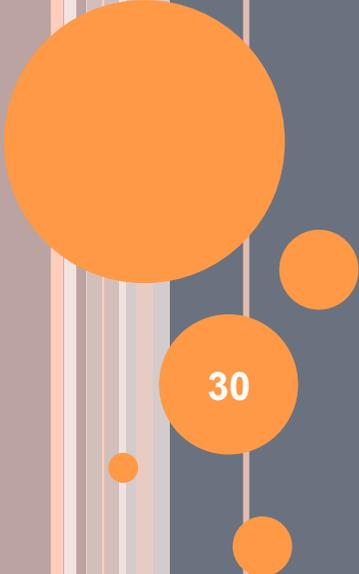
- The following equivalences hold:
 - $\forall xA(x) \leftrightarrow A(\tau xA(x))$
 - $\forall xA(x) \leftrightarrow A(\varepsilon x \sim A(x))$
 - “Universal quantification”
- The following equivalence hold:
 - $\exists xA(x) \leftrightarrow A(\varepsilon xA(x))$
 - $\exists xA(x) \leftrightarrow A(\tau x \sim A(x))$
 - “Existential quantification”

THE TWO DEFINITIONS ARE **NOT** EQUIVALENT FOR GENERALIZED QUANTIFIERS

- Observe that the Fregean definition of quantifiers with a single universe is not possible with generalized quantifiers. Need of quantifiers operating on two predicates:
 1. Most student go out on Thursday nights.
 2. For most people if they are students then they go out on Thursday nights.
 - $1 \rightarrow 2$
- But still we can ask whether it is possible to introduce other quantifiers, in this proof-theoretical way.

NEW QUANTIFIERS? (IN PROOF-THEORY)

- Introduce a pair of quantifiers, a variant \forall^* of \forall , and a variant \exists^* of \exists .
- Decide one of the following two possibilities:
 - $\forall^*x A(x)$ implies $\forall x A(x)$ and so $\exists x A(x)$ implies $\exists^*x A(x)$
 - $\exists^*x A(x)$ implies $\exists x A(x)$ and so $\forall x A(x)$ implies $\forall^*x A(x)$
 - (the second one is more natural...)
- May we define in this way the quantifiers “the majority of x ” or “most x have the property A ” ... in accordance with the “rules” suggested earlier?



CONCLUSION

Of this preliminary work

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RULES FOR (GENERALIZED) QUANTIFIERS

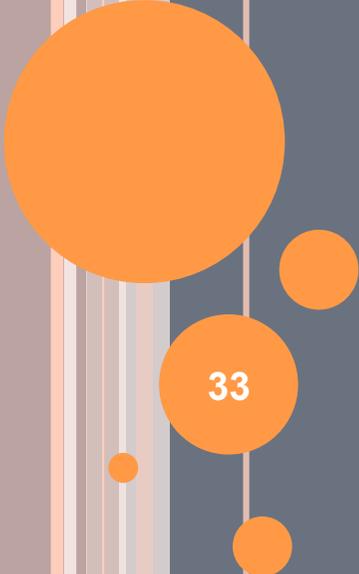
- Which properties of quantifier rules guarantee that they behave properly in proofs and interaction?
- Is it possible to define a proof system for some generalized quantifiers?
 - Percentage?
 - Vague quantifiers?
 - ...
- What are the corresponding notions of generic elements?

PREDICATION, SORTS AND QUANTIFICATION

- How do we take into account the sorts, what linguists call the restriction of the quantifier (in a typed system, a kind of ontology)?
- To avoid a paradox of the Fregean single sort:
 - Garance is tall
(for a two year old girl).
 - Garance is not tall
(as a person, e.g. for opening the fridge).
- One quantifier per type or a general quantifier which specializes? In type theory it would be a single constant of the system F:
 - ForAll/Exists: $P X ((X \rightarrow t) \rightarrow t)$

« If all roads lead to Rome,
most segments of the transportation system
lead to Roma Termini! »

Blog ``Ron in Rome''



THANKS

Any question?