QUANTIFICATION IN ORDINARY LANGUAGE from a critic of set-theoretic approaches to a proof-theoretic proposal

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Contents

- Initially (lexical semantics in type theory)
  - *I put all the books in the cellar,* (physical object) *indeed, i already read them all.* (information content)
  - There can be several occurrences of the “same” book.
- Standard quantification (history, linguistic data)
- Models, generalized quantifiers
- Second order and individual concepts
- What is a quantifier (in proof theory)?
  - Generic elements (Hilbert)
  - Cut-elimination
- Conclusion

USUAL QUANTIFICATION

Some, a, there is,…

All, each, any, every,…

Aristotle,
& scholastics (Avicenna, Scott, Ockham)

- *A* and *B* are terms
  (« term » is vague: middle-age distinction bewteen terms, « suppositionnes », eg. Ockham)
  1. All *A* are *B*
  2. Some *A* are *B*
  3. No *A* are *B*
  4. Not all *A* are *B*
  - Rules, syllogisms
  - Remarks:
    - Little about models or truth condition
    - Always a restriction (sorts, kinds,?)
    - « not all » is not lexicalized and some *A* are not *B* has a different focus.

Frege and analytic philosophy

- Attempt of a deductive system
- A single universe where variables « vary »:
  - All A are B
  - $\forall x(A(x) \rightarrow B(x))$
- Deduction, proofs (Hilbert) using a generic element
- Models, truth condition (Tarski)
- Adequation proofs-models:
  completeness theorem (Gödel, Herbrand, ~1930)
  - Whatever is provable is true in any model.
  - What is true in every model is provable.
- Extensions:
  - Logical extensions are possible (intuitionistic, modal,…)
  - No satisfying extension to higher order
  - No proper deductive system for generalized quantifiers

HOW DOES ONE ASSERT, USE OR REFUTE USUAL QUANTIFIED SENTENCES

- « For all » introduction rule
  - (how to prove ∀ as a conclusion)
  - Derive $\forall x P(x)$, from $P(a)$ for an object a without any particular property, i.e. a generic object a.
  - If the domain is known,
    $\forall x P(x)$ can be inferred from a proof of $P(a)$ for each object a of the domain.
    The domain has to be finite to keep proofs finite. The Omega rule of Gentzen is an exception.
- « For all » elimination rule
  - (how to use ∀ as an assumption)
  - From $\forall x P(x)$, one can conclude $P(a)$ for any object a.

HOW DOES ONE ASSERT , USE OR REFUTE USUAL QUANTIFIED SENTENCES

- « Exists » introduction rule
  - (how to prove ∃ as a conclusion) :
  - if for some object *a* *P(a)* is proved,
    then we may infer ∃*x* *P(x)*
- « Exists » elimination rule
  - (how to use ∃ as an assumption):
  - If C holds under the assumption P(a),
    with a only appearing in P(a),
    and if we know that ∃*xP(x)*,
    we may infer C without the assumption P(a).

REFUTATIONS

- $\exists x P(x)$: little can be done apart from proving that all do not have the property.
- $\forall x P(x)$: ***Any dog may bite.***
  this can be refuted in at least two ways:
  - Displaying an object not satisfying P
    Rex would never bite.
  - Asserting that a subset does not satisfy P,
    thus remainig with generic elements:
    Basset hounds do not bite.
- (ideas around Avicenna) a property is always asserted of **a term as part of a class** (distinction homogenous/heterogenous predicate) different sorts rather than a single Fregean universe

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIALS

- Existential are highly common:
  they even are used to structure a discourse as in Discourse Representation Theory.
- Generally with restriction, possibly implicit:
  human beings, things, events, …
  - There's a tramp sittin' on my doorstep
  - Some girls give me money
  - Something happened to me yesterday
- Focus is difficult to account for:
  - Some politicians are crooks.
  - ? Some crooks are politicians.

Usual quantification in ordinary language Universals

- Less common but present.
- With or without restriction:
  - Everyone, everything, anyone, anything,…
  - Every, all, each,…
- Generic (proofs), distributive (models)
  - Whoever, every,…
  - All, each,…
- Sometimes ranges over potentially infinite sets:
  - Each star in the sky is an enormous glowing ball of gas.
  - All groups of stars are held together by gravitational forces.
  - He believes whatever he is told.
  - Maths

Usual quantification in ordinary language universal negative

- With or without restriction:
  - No one, nothing, not any, …
  - No,…
- Generic or distributive:
  - Because no planet's orbit is perfectly circular, the distance of each varies over the course of its year.
  - Porterfield went where no colleague had gone previously this season, realising three figures.
  - I got no expectations.
  - Nothing's gonna change my world.

USUAL QUANTIFICATION IN ORDINARY LANGUAGE EXISTENTIAL NEGATIVE

- Not lexicalised (in every human language?):
  - Not all, not every, …
  - Alternative formulation (different focus): some … are not … / some … do not …
- Harder to grasp (psycholinguistic tests), frequent misunderstandings (→ nothing, no one)
- Rather generic reading:
  - Not Every Picture Tells a Story
  - Everyone is *entitled* to an opinion, but *not every* opinion is *entitled* to student government funding.
- Alternative formulation (different focus):
  - *Some Students Do Not* Participate In Group Experiments Or Projects.

Individual concepts

Alternative view of individuals and quantification

MOTIVATION
FOR INDIVIDUAL CONCEPTS

- Usual semantics with possible worlds:
  It is impossible to believe that
  Tullius≠Cicero
  with rigid designators
- To comme back to the notion of TERM
  - Individuals are particular cases of predicates.
- Quantification is a property of predicates.

First order in second order: proofs

- P is an individual concept whenever IC(P):
  - $\forall x \, \forall \, y(P(x) \wedge P(y) \rightarrow x=y)$
  - Exists x P(x)
- First order quantification
  from second order quantification:
  - $\Pi P \; IC(P) \rightarrow X(P)$
  - $\Sigma P \; IC(P) \; \& \; X(P)$
- As far as proofs are concerned, this is equivalent to first order quantification – if emptyness is allowed implications only (Lacroix & Ciardelli)

MODELS?

- Natural (aka principal models): no completeness
- Henkin models:
  completeness and compactness
  but unnatural,
  e.g. one satisfies all the following formulae:
  - F_0: every injective map is a bijection (Dedekind finite)
  - F_n, $n \geq 1$: there are at least n elements

Generalized quantifiers

Quite common in natural language

Central topic in analytic philosophy (models)

Proofs and refutations?

DEFINITION

- Generalized quantifiers are operators that gives a proposition from two properties (two unary predicates):
  - A restriction
  - A predicate
- Some are definable from usual first order logic:
  - At most two,
  - Exactly three
- And some are not (from compactness):
  - The majority of…
  - Few /a few …
  - Most of… (strong majority + vague)
- Observe that Frege's reduction cannot apply:
  - Most students go out on Thursday evening.
  - For most people, if they are student then they go out on Thursday evening

MODELS / PROOFS

- There are many studies about the models,
  the properties of such quantifiers,
  in particular monotony w.r.t. the restriction or the predicate.
- Formalisation with cardinality are wrong:
  - Most of >>> the majority of
  - Most numbers are not prime.
    Can be found in maths textbooks.
  - Test on “average” people:
    - most number are prime (no)
    - most number are not prime (yes)
  - No cardinality but measure, and what would be the corresponding generic element?
    An object enjoying most of the properties?
- Little is known about the proofs
  (tableaux methods without specific rules, but taking the intended model into account).

« The majority of » Attempt (proof vs. Refutation)

- Two ways of refuting
  the majority of (meaning at least 50%) the A have the property P:
  - Only a minority (less than) of the A has the property P
  - There is another property Q
    which holds for the majority of the A
    with no A satisfying P and Q.
  - What would be a generic majority element?

DEFINE JOINTLY RULES FOR:
1) THE MAJORITY OF
2) A MINORITY OF

- « For all » entails the « majority of »
- If any property Q which is true of the majority of A meets P, then P holds for the majority of the A (impredicative definition, needs further study)
- A minority of A is NOT P
  should be equivalent to
  The majority of A is P
- The majority of does not entail a minority of
- Forall => majority of
- Only a minority => Exists
- *A linguistic remark why do we say « The majority » but « A minority » ?*

What should be the shape of quantifier rules?

Proof-theoretical view: to allow cut-elimination.

IN PROOFS,
FOR ALL IS NOT A LARGE CONJUNCTION

- Existential rule keep the finiteness of proofs: one is enough, from P(b) infer ∃x P(x).
- Universal rule requires either:
  - A known domain D (what is the status of constants)
    - Finite
    - Infinite (loss of the finiteness, recursive descriptions,…)
      → infinite sequents if multiplicative conjunctions
  - Infer ∀x P(x) when P(x) is true of all (each) x in D (Gentzen Omega Rule)
  - A generic element (already in Pythagore)

COMMUNICATION (INTERACTION) BETWEEN PROOFS: CUT RULE

- Cut-rule: two proofs π and ρ may communicate (interact) by means of a formula *A*, i.e. when
  - π ends with a formula *A* and other formulas Γ
  - ρ ends with the negation ~*A* and other formulas Λ
- The communication (interaction) between such a pair of proofs produces a proof which ends with the formulas Γ and the formulas Λ
- Cut-elimination procedure is the development of such a communication (interaction)

A SPECIAL CASE OF COMMUNICATION, LEADING TO QUANTIFIERS RULES.

- A proof π of *A(b)* under assumptions Γ
- A proof ρ of *~A(d)* under assumptions Λ
- These proofs may be composed (cut) when one of the following cases holds:
  - The object b is the same as the object *d* (indeed, replace *b* by *d* in *A(b)*, or replace *d* by *b* in *~A(d)*)
  - The object b is generic in π (i.e. it does not occur in the formulas Γ) (indeed, replace *b* by *d* in *A(b)*
  - The object d is generic in ρ (i.e. it does not occur in the formulas Λ) (indeed, replace *d* by *b* in *~A(d)*)

GENERIC OBJECTS : HILBERT'S APPROACH

- Rules for τx:
  - *when τx A(x) has the property A, every object has.*
  - From *A(b)* with *b* generic, infer *A(τxA(x))* [∀xA(x)]
  - From *~A(d)* , infer *~A(τxA(x))* [~∀xA(x)]
  - So, one reduces to general case of cut rule
  - The development of cut rule is: replace *τxA(x)* by *d*
- Rules for εx:
  - *when an object has the property A, εx A(x) has property A.*
  - From *A(b)* with b generic, infer *A(εx~A(x))* [~∃x~A(x)]
  - From *~A(d)* , infer *~ A(εx~A(x))* [∃x~A(x)]
  - So, one reduces to general case of cut rule
  - The development of cut rule is: replace *εx~A(x)* by *d*
- *A(τxA(x)) ↔A(εx~A(x))* [∀xA(x)]
- *A(τx~A(x))↔A(εxA(x))* [∃xA(x)]

HILBERT FUNCTIONS & USUAL FREGEAN RULES ARE EQUIVALENT

- The following equivalences hold:
  - $\forall x A(x) \leftrightarrow A(\tau x A(x))$
  - $\forall x A(x) \leftrightarrow A(\varepsilon x \sim A(x))$
  - "Universal quantification"
- The following equivalence hold:
  - $\exists x A(x) \leftrightarrow A(\varepsilon x A(x))$
  - $\exists x A(x) \leftrightarrow A(\tau x \sim A(x))$
  - "Existential quantification"

THE TWO DEFINITIONS ARE NOT EQUIVALENT FOR GENERALIZED QUANTIFIERS

- Observe that the Fregean definition of quantifiers with a single universe is not possible with generalized quantifiers. Need of quantifiers operating on two predicates:
  1. Most student go out on Thursday nights.
  2. For most people if they are students then they go out on Thursday nights.
  - 1 → 2
- But still we can ask whether it is possible to introduce other quantifiers, in this proof-theoretical way.

NEW QUANTIFIERS? (IN PROOF-THEORY)

- Introduce a pair of quantifiers,
  a variant $\forall^*$ of $\forall$, and a variant $\exists^*$ of $\exists$.
- Decide one of the following two possibilities:
  - $\forall^* x A(x)$ implies $\forall x A(x)$ and so $\exists x A(x)$ implies $\exists^* x A(x)$
  - $\exists^* x A(x)$ implies $\exists x A(x)$ and so $\forall x A(x)$ implies $\forall^* x A(x)$
  - (the second one is more natural…)
- May we define in this way the quantifiers
  "the majority of x" or "most *x* have the property *A*" …
  in accordance with the "rules" suggested earlier?

CONCLUSION

Of this preliminary work

Rules for (Generalized) quantifiers

- Which properties of quantifier rules guarantee that they behave properly in proofs and interaction?
- Is it possible to define a proof system for some generalized quantifiers?
  - Percentage?
  - Vague quantifiers?
  - …
- What are the corresponding notions of generic elements?

PREDICATION,SORTS AND QUANTIFICATION

- How do we take into account the sorts, what linguists call the restriction of the quantifier (in a typed system, a kind of ontology)?
- To avoid a paradox of the Fregean single sort:
  - Garance is tall
    (for a two year old girl).
  - Garance is not tall
    (as a person, e.g. for opening the fridge).
- One quantifier per type or a general quantifier which specializes? In type theory it would be a single constant of the system F:
  - ForAll/Exists: $\Pi X\ ((X \rightarrow t) \rightarrow t)$

« If **all** roads lead to Rome,
most segments of the transportation system lead to Roma Termini! »

Blog ``Ron in Rome''

THANKS

Any question?