UM. Autumn 2019. Homework 1 to the course «Information theory».
[should be returned by Sep 17 to be counted in *contrôle continu*]

Problem 1 (Sorting algorithms)**.**
(a) [optional] Prove that an array of n elements can be sorted with $O(n \log n)$ comparisons (in the worst case). *Reminder : In the class we proved that no algorithm can do this faster than in* $\log(n!) = \Omega(n \log n)$ *comparisons.*

Find the number of comparisons needed in the worst case
(b) to sort an array of size 4 ;
(c) to sort an array of size 5.

Problem 2. We are given a heap of n stones, and we can use balance scales to compare weights of any two stones. In the class we discussed an algorithm that permits to find the heaviest and the second heaviest stone in $n + \lceil \log_2 n \rceil - 2$ weighing. Prove that this algorithm is optimal (no algorithm can do better than this in the worst case).
Hint : Apply the "adversarial argument."

Problem 3. We are given n coins, and one of them is fake. All genuine coins have the same weights, the fake one is heavier or lighter. We can use balance scales to compare weights of any two groups of coins. How many weighings do we need to find the fake coin for (a) $n = 13$ and (b) $n = 14$.
Reminder : We discussed in the class that 3 *weighings are needed for* $n = 12$ *and* 4 *weighings are needed for* $n = 15$.

Problem 4 (optional and difficult)**.** In what follows $\mathrm{Inf}(X)$ stands for Hartley's combinatorial information in a set X, i.e.,

$$\mathrm{Inf}(X) := \log_2 |X|.$$

Let A be a finite set in $\mathbb{Z}^3$. We denote by $\pi_{ij}[A]$ the projection of A onto the coordinates i and j (e.g., π_{13} applied to the point (x, y, z) gives (x, z)). Prove that

$$2 \cdot \mathrm{Inf}(A) \leq \mathrm{Inf}(\pi_{12}[A]) + \mathrm{Inf}(\pi_{13}[A]) + \mathrm{Inf}(\pi_{23}[A]).$$