UM. Autumn 2019. Homework 6 to the course «Information theory».
[should be returned by Dec 10 to be counted in *contrôle continu*]

Problem 1. The *random erasure code* has the two letters alphabet $\{0, 1\}$ on the input and the three letter alphabet $\{0, 1, ?\}$ on the output, with the following conditional probabilities :

$$\begin{array}{l} \text{Prob}[\text{ output} = 0 \mid \text{input} = 0\ ] = 1 - \epsilon, \\ \text{Prob}[\text{ output} = 1 \mid \text{input} = 1\ ] = 1 - \epsilon, \\ \text{Prob}[\text{ output} = \ ? \mid \text{input} = 0\ ] = \epsilon, \\ \text{Prob}[\text{ output} = \ ? \mid \text{input} = 1\ ] = \epsilon. \end{array}$$

Compute Shannon's capacity of this channel.

Problem 2. The *asymmetric random binary code* has the two letters alphabet $\{0, 1\}$ on the input and the three letter alphabet $\{0, 1\}$ on the output, with the following conditional probabilities :

$$\begin{array}{l} \text{Prob}[\text{ output} = 0 \mid \text{input} = 0\ ] = 1, \\ \text{Prob}[\text{ output} = 1 \mid \text{input} = 1\ ] = 1 - \epsilon, \\ \text{Prob}[\text{ output} = 0 \mid \text{input} = 1\ ] = \epsilon. \end{array}$$

Compute Shannon's capacity of this channel for $\epsilon = 1/2$.

Problem 3 (optional)**.** Compute Shannon's capacity of the channel from Exercise 2 for an arbitrary ϵ.

Problem 4. Let us choose at random 10 binary strings of length 1000,

$$\begin{array}{rcl} \bar{x}_1 & = & (x_{1,1} \quad x_{1,2} \quad \dots \quad x_{1,1000}) \\ \bar{x}_2 & = & (x_{2,1} \quad x_{2,2} \quad \dots \quad x_{2,1000}) \\ & \vdots & \\ \bar{x}_{10} & = & (x_{10,1} \quad x_{10,2} \quad \dots \quad x_{10,1000}) \end{array}$$

(each bit $x_{i,j}$ is chosen be equal to 0 or 1 with probabilities 1/2). Prove that with a positive probability (and even with a probability > 0.5) the obtained set of words $\{\bar{x}_1, \dots \bar{x}_{10}\}$ is an error correcting code that corrects at least 50 errors, i.e., the Hamming distance between every two words $\bar{x}_i$ and $\bar{x}_j$ is greater than $50 + 50 = 100$.