UM. Autumn 2019. Homework 8 to the course «Information theory».
[not counted in *contrôle continu*]

Problem 1. A function $F : \{0,1\}^* \to \mathbb{N}$ is a *lower bound* for Kolmogorov complexity, if for all binary strings x

$$C(x) \geq F(x).$$

Prove that every *computable* lower bound for Kolmogorov complexity is trivial in the following sense : there is a number n such $F(x) \leq n$ for all x.

Problem 2. Prove that there exist infinitely many binary strings x such that

$$C(x) \leq \underbrace{\log\log\log\ldots\log}_{100} |x|.$$

Problem 3. Prove that for all binary strings x, y

(i) $I(x : y) \leq C(x) + O(\log(C(x) + C(y)))$,

(ii) $I(x : x) = C(x) + O(1)$.

Problem 4. Let f be a total computable function. Prove that there exist real numbers λ_1, λ_2 such that all binary strings x, y

$$I(f(x) : y) \leq I(x : y) + \lambda_1 \log(C(x) + C(y)) + \lambda_2.$$

Problem 5. Prove that for all binary strings x, y, z

(i) $2C(x, y, z) \leq C(x, y) + C(x, z) + C(y, z) + O(\log N)$,

(ii) $C(z) + C(x, y, z) \leq C(x, z) + C(y, z) + O(\log N)$,

where $N = C(x) + C(y) + C(z)$.

Problem 6. We say that a binary string x *contains only isolated zeros* if x contains no factors 00. Prove that there exist real numbers $\lambda_1 < 1$ and $\lambda_2 > 0$ such that for all binary strings x with only isolated zeros

$$C(x) \leq \lambda_1 |x| + \lambda_2$$

(this mean, in particular, that all long enough x having this property are compressible, i.e., $C(x) < x$).

Problem 7. Let $n = C(x)$, and let $\text{bin}(n)$ denotes the binary expansion of this number. Prove that

$$C(x, \text{bin}(n)) = n + O(1).$$

Problem 8. Prove that the series $\sum\limits_{x \in \{0,1\}^*} C(x)$ diverges.

Problem 9. Prove that for every function $f : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^m$ its deterministic communication complexity is

(i) at most $2n$;

(ii) at most $n + m$.

Problem 10. Denote $\max_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}^n$ the function defined as

$$\max_n(x, y) = \begin{cases} x, & \text{if } x \geq y \text{ in the lexicographical order} \\ y, & \text{otherwise.} \end{cases}$$

Prove that deterministic communication complexity of $\max_n$

(i) at most $1.5n$;

(ii) at most $1.1n$;

(iii) [optional] at most $n + O(\sqrt{n})$.

Problem 11. Denote $\mathrm{EQ}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ the function defined as

$$\mathrm{EQ}_n(x, y) = \begin{cases} 1, & \text{if } x = y \\ 0, & \text{otherwise.} \end{cases}$$

Prove that for every $\epsilon > 0$ there is a randomized communication protocol computing this function (Alice and Bob may use their private random bits ; for every $x, y \in \{0,1\}^n$ the probability of error is less than ϵ) with communication complexity $O(\log(n/\epsilon))$

Problem 12. Denote $\mathrm{GT}_n : \{0,1\}^n \times \{0,1\}^n \to \{0,1\}$ the function defined as

$$\mathrm{GT}_n(x, y) = \begin{cases} 1, & \text{if } x \geq y \text{ in the lexicographical order} \\ 0, & \text{otherwise.} \end{cases}$$

Prove that for every $\epsilon > 0$ there is a randomized communication protocol computing this function (Alice and Bob may use their private random bits ; for every $x, y \in \{0,1\}^n$ the probability of error is less than ϵ) with communication complexity $O(\log^2(n/\epsilon))$.

Remark : For this problem there is a randomized communication protocol with communication complexity only $O(\log(n/\epsilon))$. But the proof of this fact is more involved.