UM. Autumn 2020. Homework to the course «Information theory».
[should be returned by Dec 15 to be counted in *contrôle continu*]

Problem 1. Prove that for any triple of jointly distributed random variables (α, β, γ)

$$I(\alpha : \beta) \leq I(\alpha : \langle \beta, \gamma \rangle).$$

Problem 2. (a) Find an example of a joint distribution (α, β, γ) such that

$$I(\alpha : \beta) < I(\alpha : \beta | \gamma).$$

(b) Find an example of a joint distribution (α, β, γ) such that

$$I(\alpha : \beta) > I(\alpha : \beta | \gamma).$$

Problem 3. (a) Prove that for any triple of jointly distributed random variables (α, β, γ)

$$H(\gamma) \leq H(\gamma|\alpha) + H(\gamma|\beta) + I(\alpha : \beta).$$

(b) Assume that there are (deterministic) functions F and G such that with probability one $\gamma = F(\alpha) = G(\beta)$. Prove that $H(\gamma) \leq I(\alpha : \beta)$.

Problem 4. Let S be a finite set in $\mathbb{Z}^3$. We denote by $\pi_{ij}[S]$ the projection of S onto the coordinates i and j (e.g., π_{13} applied to the point (x, y, z) gives (x, z)). The cardinality of a set is denoted $|\cdot|$.

(a) Prove that $2 \cdot \log |S| \leq \log |\pi_{12}[S]| + \log |\pi_{13}[S]| + \log |\pi_{23}[S]|$.

Hint : Introduce a uniform distribution $(\alpha_1, \alpha_2, \alpha_3)$ on the elements of S and use the inequality

~~$2H(\alpha_1, \alpha_2, \alpha_3) \leq H(\alpha_1, \alpha_2) + H(\alpha_1, \alpha_2) + H(\alpha_2, \alpha_2).$~~ /* a misprint */

$$2H(\alpha_1, \alpha_2, \alpha_3) \leq H(\alpha_1, \alpha_2) + H(\alpha_1, \alpha_3) + H(\alpha_2, \alpha_3).$$

(b) Prove that $|S|^2 \leq |\pi_{12}[S]| \cdot |\pi_{13}[S]| \cdot |\pi_{23}[S]|$.

Problem 5. Let $\mathbf{m} = (m_1 \dots m_n)$ be an arbitrary random variable distributed in $\{0, 1\}^n$ and $\mathbf{k} = (k_1 \dots k_n)$ be a *uniform* distribution on the same domain $\{0, 1\}^n$. Assume that $\mathbf{m}$ and $\mathbf{k}$ are independent.

Denote by $\mathbf{e}$ the bitwise XOR of $\mathbf{m}$ and $\mathbf{k}$ (i.e., $e_i = m_i \oplus k_i$ for $i = 1, \dots, n$). Prove that

$$I(\mathbf{m} : \mathbf{e}) = 0.$$

Hint : First of all, prove that $H(\mathbf{e}) \leq n$. Then draw a diagram with information quantities for the triple $(\mathbf{m}, \mathbf{k}, \mathbf{e})$ and focus on the mutual information between $\mathbf{m}$ and $\mathbf{e}$.