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M2 Recherche - Théorie des langages et pavages V. Poupet, A. Romashchenko, G. Lafitte

Complexité de Kolmogorov (devoir de maison), Dec. 2016

1 Prove that the set of pairs $\{(x, n) \mid \mathrm{K}(x) < n\}$ is recursively enumerable but not decidable.

2 Prove that $2K(x, y, z) \leq K(x, y) + K(x, z) + K(y, z \,|\, x) + O(\log K(x, y, z))$.

3 Prove that the following languages over the alphabet $\{0, 1, \#\}$ are not regular:

(a) the language of words $x\#y$, where x and y are binary words, and x is a factor of y,
(b) the language of words $x\#y$, where x and y are binary words, and y is a factor of x.

4 Denote by $bin(n)$ the binary expansion of an integer n. Prove that for all binary stings x of an even length n that consists of exactly $n/2$ *zeros* and $n/2$ *ones*

(a) $K(x \,|\, bin(n)) \leq n - \frac{1}{2}\log n + O(1)$,
(b) $K(x) \leq n - \frac{1}{2}\log n + O(1)$.

5 Prove that for every Martin-Löf random sequence of bits $\omega_1\omega_2\ldots\omega_n\ldots$

(a) only finitely many (if any) prefixes $\omega_1\omega_2\ldots\omega_n$ of X contain the factor $\underbrace{000\ldots0}_{\sqrt{n}}$,

(b) for all large enough n the prefix $\omega_1\omega_2\ldots\omega_n$ of X contains the factor $\underbrace{111\ldots1}_{\log\log n}$.

6 Prove that for **every** Martin-Löf random sequence of bits $X = \omega_1\omega_2\ldots\omega_n\ldots$

$$\lim_{n\to\infty} \frac{\omega_1 + \omega_2 + \ldots + \omega_n}{n} = \frac{1}{2}.$$

Remark: This statement together with Theorem 3.2 (see the lecture notes) implies *the Law of Large Numbers:* for **almost all** (for all except for a set of measure zero) sequences of bits $X = \omega_1\omega_2\ldots\omega_n\ldots$

$$\lim_{n\to\infty} \frac{\omega_1 + \omega_2 + \ldots + \omega_n}{n} = \frac{1}{2}.$$