Partition and Measure: a new technique for analyzis of Branch and Bound algorithms

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8 février 2012

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A generic problem Π

Two parameters : p_1, p_0 with $p_0 + p_1 = N$

Three possible situations : A , B, $\bar{A}\wedge \bar{B}$

An hypothesis :

$$p_1 \ge p_0 \Longrightarrow A = \text{TRUE}.$$

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A basic algorithm

Branching : $I \mapsto I', I'$

$$A = \text{TRUE} \implies p_1(l') \le p_1(l) - 4,$$

$$p_0(l') \le p_0(l).$$

$$A = \text{FALSE}, B = \text{TRUE} \implies p_1(l') \le p_1(l) - 2,$$

$$p_0(l') \le p_0(l) + 2.$$

$$A, B = \text{FALSE} \implies p_1(l') \le p_1(l),$$

$$p_0(l') \le p_0(l) - 1.$$

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Standard analyzis is a bit helpless

Since $N = p_0 + p_1$,

 $A = \text{TRUE} \implies T(N) \le 2T(N-4)$ $A = \text{FALSE}, B = \text{TRUE} \implies T(N) \le 2T(N) \quad ???$ $A, B = \text{FALSE} \implies T(N) \le 2T(N-1)$

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Let's do Measure and Conquer instead

Fix
$$k = w_0 p_0 + w_1 p_1$$
,
 $A = \text{TRUE} \implies T(k) \le 2T(k - 4w_1)$
 $A = \text{FALSE}, B = \text{TRUE} \implies T(k) \le 2T(k - 2w_1 + 2w_0)$
 $A, B = \text{FALSE} \implies T(k) \le 2T(k - w_0)$

Optimal is $w_0 = 2w_1/3$, $T(N) \le 2^{3N/2}$

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Can we do something better?



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Another analyzis is possible (1)

- use different measures k_0, k_1 for $p_1 \leq p_0$ and $p_1 \geq p_0$.
- compute a solution T_0 for $p_1 \le p_0$: Optimal is still $w_0 = 2w_1/3$, $T_0(k_0) \le 2^{3k_0/2}$

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Another analyzis is possible (2)

- use T_0 as a lower bound for $p_0 = p_1$.
- compute a solution T_1 for $p_1 \ge p_0$: In this part, optimal is $w_0 = w_1$, $T_1(k_1) \le 2^{5k_1/4}$

$$\max\left\{\max_{
ho_1\leq
ho_0}(2^{3k_0/2}),\max_{
ho_1\geq
ho_0}(2^{5k_1/4})
ight\}\leq 2^{5N/4}$$

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The quotient space of instances

Given a set of parameters p_1, \ldots, p_s such that the problem is polynomial for $\sum p_i = 0$, Given two instances I, I',

$$I \sim I' \iff \forall j, p_j(I) = p_j(I')$$

We work in \mathcal{G}_n/\sim

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A Branching Rule



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Projection allows to compare



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Fixing the objective hyperplan



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Division of the space (1)



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Division of the space (2)



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Sets of recurrences

For a branching B_t , in the subspace $S_i: v_{i,t,j} = \langle \widetilde{l} \overrightarrow{\widetilde{H}_j}, W_i \rangle$

$$T_i(k) \leq \max_t \left\{ \sum_j T_i(k - v_{i,t,j}) \right\},$$

which leads to :

$$1 \leq \max_t \left\{ \sum_j 2^{-\alpha_i v_{i,t,j}} \right\}.$$

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Border conditions

Let i' > i such that $S_{i'}$ and $S_i \setminus S_{i+1}$ have a border β , Let $\tilde{l} \in \beta$. $T_{W,i'}(\tilde{l}) \geq T_{W,i}(\tilde{l})$

The latest inequations can be rewritten :

$$\alpha_{i'} \sum w_{i',j} p_j \ge \alpha_i \sum w_{i,j} p_j$$

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Transversal division

(J^d)_{1≤d≤Δ} is a sequence of subspaces of *I_n*/ ~ of increasing dimension. More precisely :

$$J^d = \operatorname{Vect}(e_1, \dots, e_{j^d}) = \{e_1, \dots, e_{j^d}\}^n$$

 $j^{d+1} > j^d$

• for each J^d , we define a partition (S_i^d) exactly like before.

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Set Cover

- A ground set $\mathcal{C} = \{c_1, \cdots, c_n\}$
- A set system $\mathcal{S} = \{S_1, \cdots, S_m\}$
- Goal : a minimum size subset $\mathcal{S}' \subseteq \mathcal{S}$ that covers all elements in $\mathcal{C}.$

Basic algorithm

If no easy case occurs, branch on a set of maximal size : either take it or not.

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Measure and conquer analysis

- Weight w_i to sets of size i, weight v_j of sets of frequency j.
- Analysis of the branching. Δ_{IN} and Δ_{OUT} (decreasing of the total weight when taking S or not) depending on :

 → The size d of the set S we branch on
 → The frequencies of the d elements in S.

 Set of recurrences :

 r_j : number of elements of frequency j in S.
 ∀d, ∀r_j s.t. Σ_j r_j = d, 2^{-αΔ_{IV}} + 2<sup>-αΔ_{OUT} ≤ 1.

 Global complexity 2^α(wmax|S|+vmax|C|)
 </sup>

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Set of recurrences :

 r_i : number of elements of frequency j in S.

$$\forall d, \forall r_i \ s.t. \sum_i r_i = d, \ 2^{-\alpha \Delta_{IN}} + 2^{-\alpha \Delta_{OUT}} \leq 1.$$

Global complexity $2^{\alpha(w_{max}|S|+v_{max}|C|)}$

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Measure and conquer analysis

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Set of recurrences :

 $\begin{array}{l} r_j: \text{number of elements of frequency } j \text{ in } \mathcal{S}. \\ \forall d, \ \forall r_j \ s.t. \sum_j r_j = d, \ 2^{-\alpha \Delta_{IN}} + 2^{-\alpha \Delta_{OUT}} \leq 1. \\ \text{Global complexity } 2^{\alpha(w_{max}|\mathcal{S}| + v_{max}|\mathcal{C}|)} \end{array}$

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Idea of Partition and Measure

- In $R_1 : n_d$ is small or m_2 is large \rightarrow same analysis, but better complexity bound.
- ② In R_2 : there exists one set of size *d* with no element of frequency 2 → restricted set of recurrences ($r_2 = 0$).

Generalization with several hyperplans for each *d* ; Computation : on-going work!

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Idea of Partition and Measure

- In $R_1 : n_d$ is small or m_2 is large \rightarrow same analysis, but better complexity bound.
- In R₂ : there exists one set of size d with no element of frequency 2 → restricted set of recurrences (r₂ = 0).

Generalization with several hyperplans for each d; Computation : on-going work!

A wide range of possibilities...

- Problems where branch and bound technique is efficient.
- Graphs where better branching exists when maximal degree is high.
- Typical other example : maximum independent set.

... with some computation cost

- r parameters, s subspaces $\implies p \times s$ variables.
- leading measure and conquer algorithms are often quite tedious to analyze (4.8×10^6 recurrences for maximum independent set).