Kernel lower bound for the k-DOMATIC PARTITION problem

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Contents

Kernels, domatic partition

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Kernel

Given a parameterized problem $Q \subseteq \Sigma^* \times \mathbb{N}$, a **kernel** for Q is a **polynomial algorithm** with:

- input: an instance (x, k) of Q
- output: an instance (x', k') of Q

such that:

- $(x, k) \in Q \Leftrightarrow (x', k') \in Q$
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Theorem

$Q \in FPT \Leftrightarrow Q$ has a kernel

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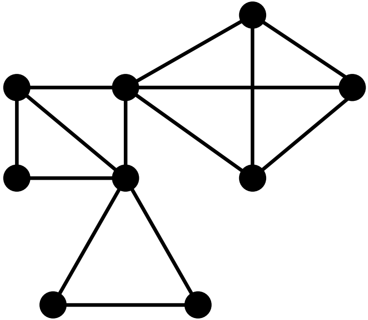

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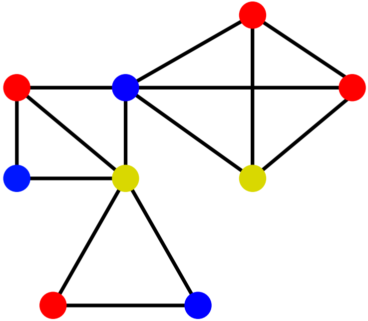

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- FPT when parameterized by *treewidth*(G) (MSO formula)
- 3-DOMATIC PARTITION does **not** admit a polynomial kernel when parameterized by *treewidth*(G) [Bodlaender et al. 2009]
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Sketch of proof:

- **cross-composition** of HYPERGRAPH-2-COLORABILITY to itself
  $\Rightarrow$ no polynomial kernel for HYPERGRAPH-2-COLORABILITY
  (parameterized by the number of vertices)
- **polynomial time and parameter transformation** to k-DOMATIC PARTITION

Contents

Lower bound for HYPERGRAPH-2-COLORABILITY

HYPERGRAPH-2-COLORABILITY

Input : a hypergraph $H = (V, E)$
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Parameter : $n = |V|$

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Theorem [Bodlaender, Jansen, Kratsch, 2011]

If there exists a **cross-composition** from an $\mathcal{NP}$-complete problem A to a parameterized problem Q, then Q does not admit a polynomial kernel unless $coNP \subseteq NP/Poly$

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Equivalence relation:

- computable in polynomial time
- partition a set S into less than $\max_{x \in S} |x|^{O(1)}$ classes

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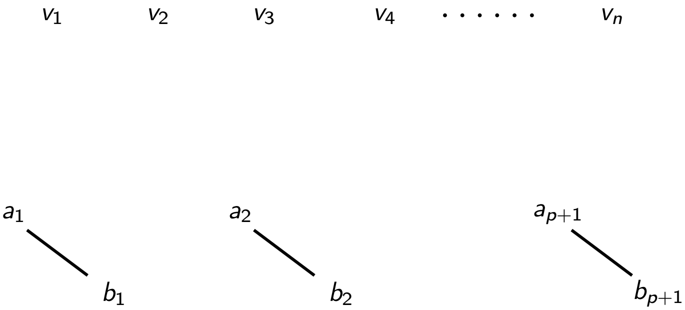

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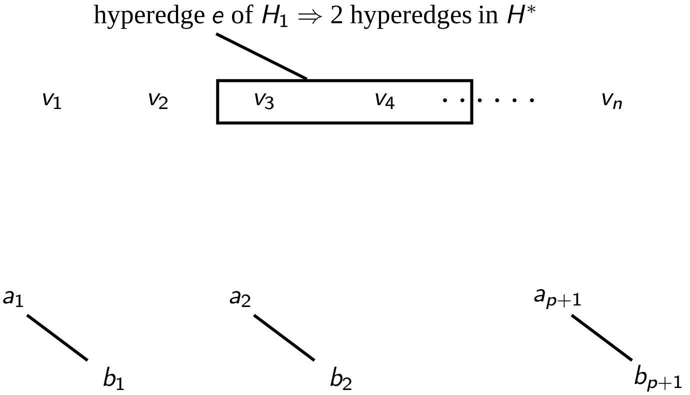

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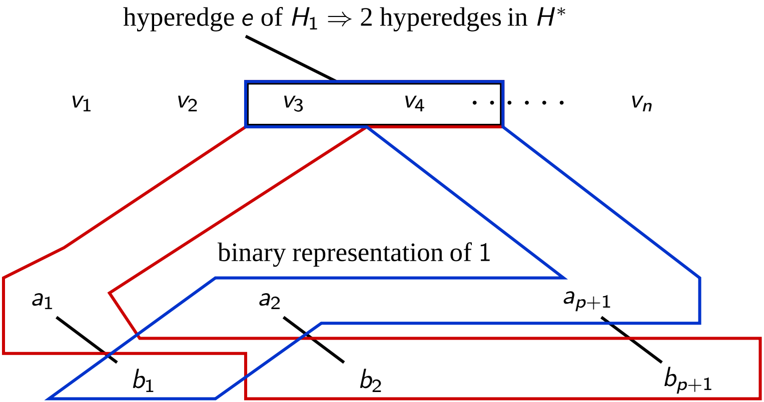

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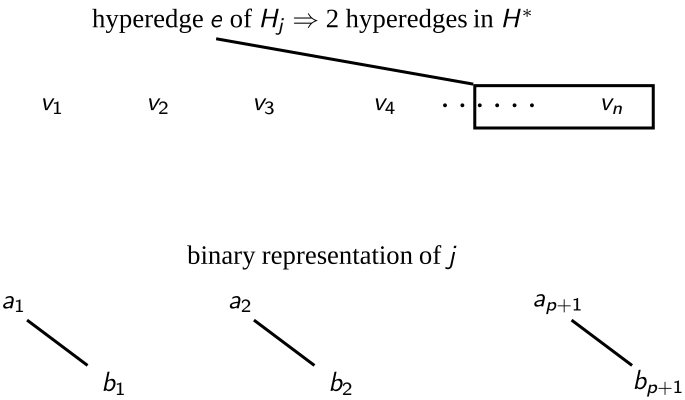

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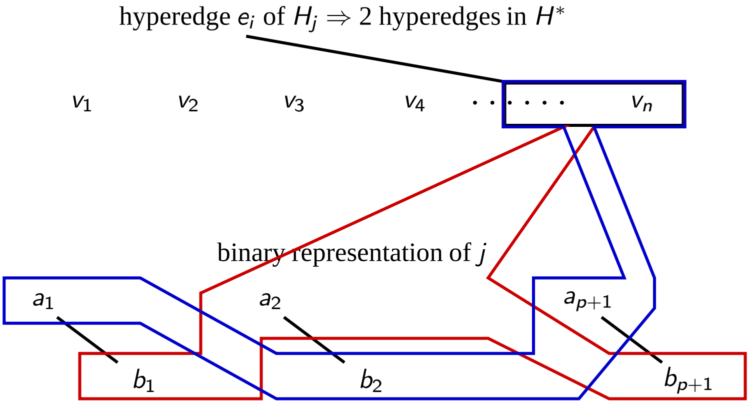

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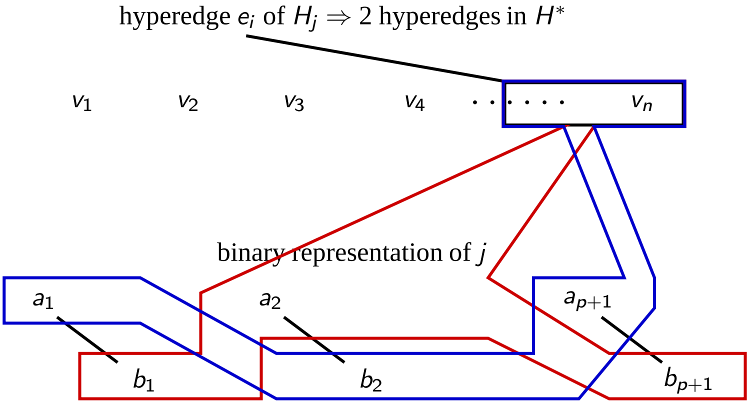

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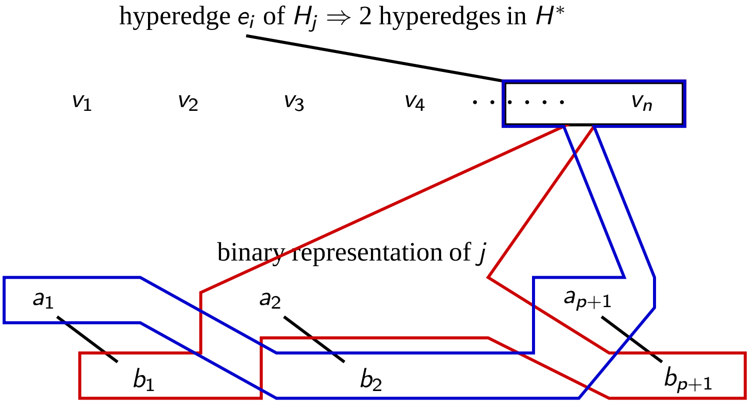

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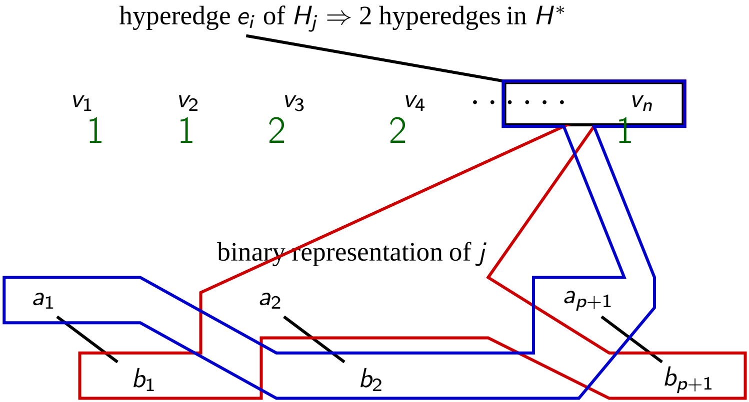

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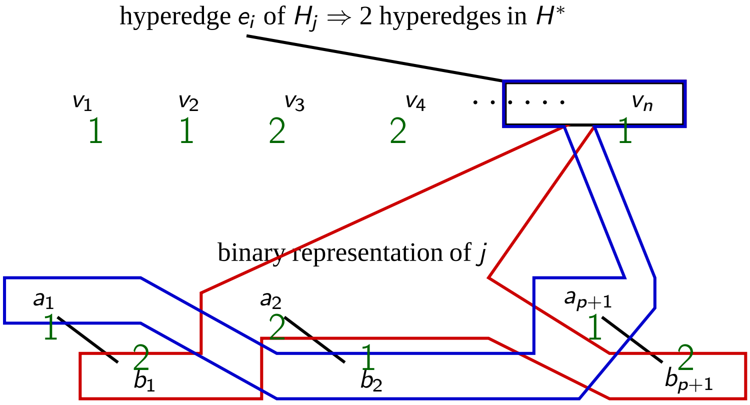

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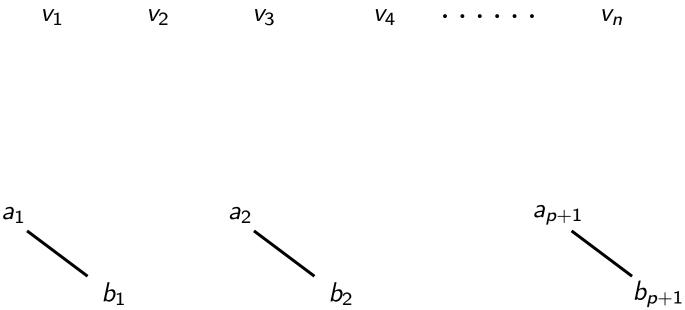

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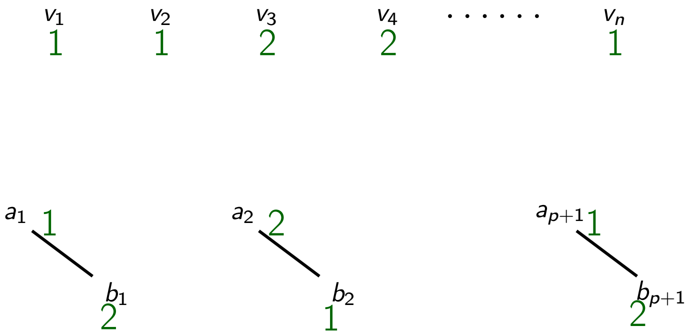

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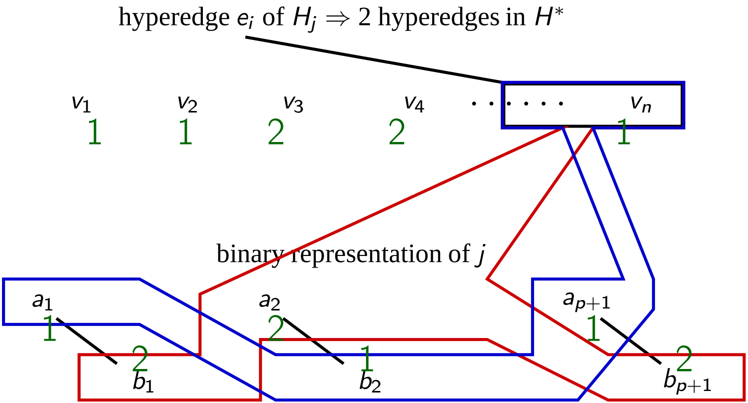

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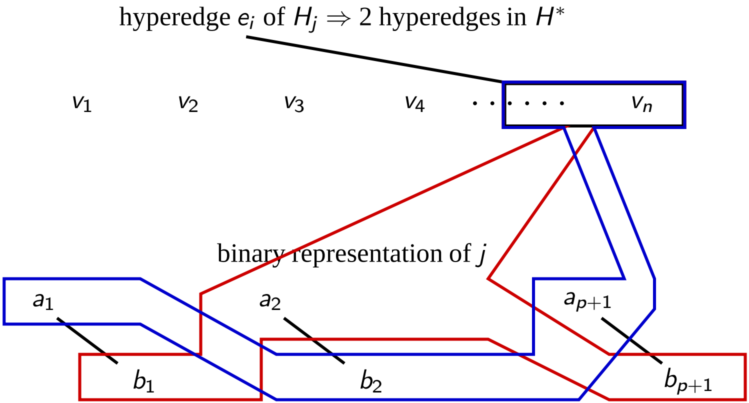

Finally : the number of vertices (parameter) is polynomial in the size of the biggest instance of the sequence + $\log t$

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(proof for $k = 3$, but can be extended for every fixed $k \geq 3$)
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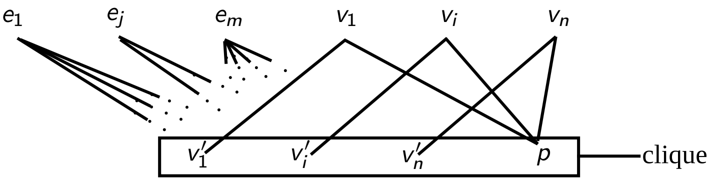

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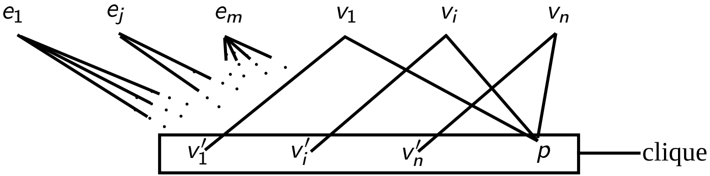
G' has a 3-domatic partition $\Leftrightarrow$ H has a proper 2-coloring.



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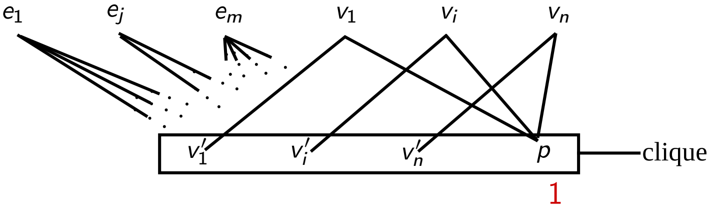

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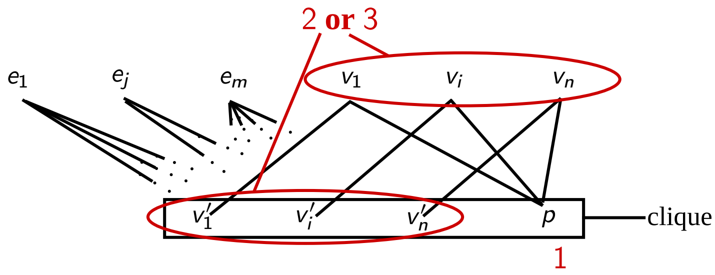

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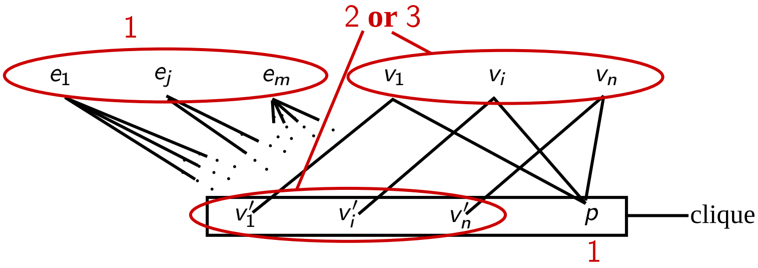

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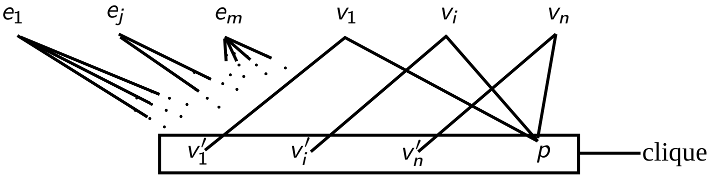
Finally : the clique is a vertex cover (parameter) of size $n+1$



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  - here, distance = set of vertices to remove
    - set of edges to remove
    - set of edges to edit...

Thank you for your attention!