Expanders

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Outline

- What is a expander
- Existence of expanders
- Why expanders are useful
- The Eigenvalue connection
- The spectral gap
- Some additional properties
- Explicit constructions

What is a expander

$G = (V, E)$ a graph with $n = |V|$ vertices and $m = |E|$ edges.

- $E(X, Y) = \{xy \in E : x \in X, y \in Y\}$, $e(X, Y) = |E(X, Y)|$.
- Isoperimetric number of G: $i(G) = \min_{|X| \leq n/2} \frac{e(X)}{|X|}$.

- A graph G is a c–expander, $c > 0$, if $i(G) \geq c$.
- $\{G_k, k \in \mathbb{N}\}$ is a (d, c)–expander family if each graph is d–regular and $i(G_k) \geq c$ for all k.

An expander graph is a sparse graph that has strong connectivity properties, quantified using vertex, edge or spectral expansion.

Existence of expanders (a bipartite version)

A random bipartite graph is a (c, d) one sided expander (superconcentrator) with high probability.

- X, Y two sets with $|X| = |Y| = n$.
- Choose d neighbours in Y independently at random for each vertex in X.
- For $S \subset X$, $|S| \leq n/2$ and $T \subset Y$, $|T| \leq c|S|$, the probability that $N(X) \subset Y$ is small.
- Union bound for all (S, T): small probability that the graph is not a (c, d)–expander.

Why expanders are useful

- Complexity of computation of linear transformations in finite fields by a circuit (e.g. Fast Fourier Transform)
  Leslie Valiant (1976): Superconcentrators with linear number of edges.
- Error Correcting Codes
  Construction of (n, k)–linear codes with minimum distance $n/3d$ and rate $1/3d$.
- De–randomization of random algorithms: Design Random Polynomial Algorithms (e.g. primality test, Rabin (1980))
  Probability of failure $1/3d$ (Ajtai, Komlos, Szemeredi, 1987)
- Network design
  high connectivity and small diameter
- Rapidly mixing Markov chains.
- Bounds on treewidth (via separator Lemma)
- ...

The eigenvalue connection

- G connected d–regular graph.
- $L = L(G) = dI - A$ Laplacian matrix of G.
- $0 = \mu_0 < \mu_1 \leq \cdots \leq \mu_{n-1} \leq 2d$ Laplacian spectrum.

$$v^T L v = \sum_{ij \in E} (v_i - v_j)^2.$$

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- $v_S(i) = \begin{cases} |V \setminus S|, & v_i \in S \\ |S|, & v_i \in V \setminus S \end{cases}, \rightarrow \mu_1 \leq \frac{e(S)|V|}{|S||V \setminus S|}.$

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The Cheeger inequality

- v eigenvector of μ_1, $u = v^+ = (max(v_i, 0), 1 \leq i \leq n)$, $supp(v^+) \leq n/2$.
- $\sum_{ij \in E}(u(i)^2 - u(j)^2) \leq \sqrt{2d}|u|\sqrt{u^T L u}$.
- $\sum_{ij \in E}(u(i)^2 - u(j)^2) \geq i(G)|u|^2$.

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- $Lu(i) \leq Lv(i) = \mu_1 v(i) \ \rightarrow \ u^T Lu = \sum_{i\in V} u(i)(Lu(i)) \leq \mu_1|u|$.

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Algebraic definition of expanders

A family $\{G_k : k \in \mathbb{N}\}$ of d–regular graphs is a (d, β)–expanding if $\mu_1(G_k) \geq \beta$ for all k.

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The spectral gap

How large can be μ_1 (i.e., how small can be λ_1 second largest eigenvalue of A)

For a connected d–regular graph G

$$\lambda_1(G) \geq 2\sqrt{d-1} - \frac{2\sqrt{d-1}-1}{b},$$

where $D(G) \geq 2b+2 \geq 4$. (Alon, 1991)

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- Choose $x, y \in V$, $d(x,y) = 2b+2$.
- Define $u(z) = \begin{cases} a_i, & d(z,x) = i \leq b; \\ b_i, & d(z,y) = i \leq b; \\ 0, & \text{otherwsie.} \end{cases}$,

  $a_i = \alpha/(d-1)^{(i-1)/2}$, $b_i = \beta/(d-1)^{(i-1)/2}$ and $u \perp 1$.
- $d - \lambda_1 \overset{Courant-Fisher}{\leq} \frac{1}{|u|} u^T L u \overset{L-formula}{\leq} 1 + (d-1) - 2\sqrt{d-1} + (2\sqrt{d-1}-1)b.$

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A Ramanujan graph is a d–regular graph with

$$\lambda_1 \leq 2\sqrt{d-1}.$$

(Lubotzky, Phillips, Sarnak, 1988)

Some additional properties

G connected nonbipartite d-regular graph with n vertices

Spectral bound on the diameter

$$D \leq \frac{log(n-1)}{log(d/\lambda)}, \ \lambda = \max\{|\lambda_1|, |\lambda_{n-1}|\}$$

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- $A^m > 0 \Rightarrow D \leq m$.
- $A = \sum_{i=0}^{n-1} \lambda_i u_i u_i^T$ spectral decomposition with orthonormal spectral basis.
- $A^m(x,y) = \sum_{i=0}^{n-1} \lambda_i (u_i u_i^T)(x,y) \geq d^m/m - |\sum_{i\geq 1} \lambda_i u_i(x) u_i^T(y)|$.
- $|\sum_{i\geq 1} \lambda_i u_i(x) u_i^T(y)| \leq \lambda^m (\sum_{i\geq 1} u_i(x)^2)^{1/2} (\sum_{i\geq 1} u_i(y)^2)^{1/2} \leq \lambda^m (1-1/n)$.

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Expander families have $D = O(\log n)$

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The Expander Mixing Lemma

For every $S, T \subset V$

$$|e(S,T) - \frac{d|S||T|}{n}| \le \lambda\sqrt{|S||T|}$$

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- $e(S,T) = 1_S A 1_T = \sum_i \lambda_i a_i b_i = d\frac{|S||T|}{n} + \sum_{i\geq 1} \lambda_i a_i b_i$.
- $|e(S,T) - d\frac{|S||T|}{n}| \leq \lambda|a||b| = \lambda\sqrt{|S||T|}$

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Expander families are close to random

Explicit constructions

- Ramanujan graphs with unbounded degree: $K_n (\lambda = 1), Payley(p)\ (\lambda = \sqrt{p}), \ldots$

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- Margulis; Gabber and Galil (1981): 8–regular graphs with m^2 vertices and $\lambda < 8$.

$$V = \mathbb{Z}_m \times \mathbb{Z}_m,\ (x, y) \to \begin{cases} (x \pm y, y), \\ (x \pm (y+1), y), \\ (x, y \pm x), \\ (x, y \pm (x+1)), \end{cases}$$

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- Margulis (1988), Lubotzky, Phillips, Sarnak (1988): Appropriate quotients of infinite d–ary tree.

Fix $p \equiv 1 \pmod 4$ and let $q \equiv 1 \pmod 4$ primes.

$$S = \left\{ \begin{pmatrix} a+ub & c+ud \\ -c+ud & a-ub \end{pmatrix}, \quad \begin{array}{c} a^2+b^2+c^2+d^2 = p, a > 0; \\ u^2 \equiv -1 \pmod q. \end{array} \right\}$$

$Cay(PGL_2(\mathbb{Z}/q\mathbb{Z}), S)$ is a Ramanujan graph with degree $p+1$.

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$$\begin{matrix} G(n, m, \alpha) \\ H(m, d, \beta)0 \end{matrix} \rightarrow G \overset{zz}{\times} H(nm, d^2, \alpha + \beta)$$

$$G_1 = H^2, \qquad H(d^2, d, 1/2)$$

$$G_{n+1} = G_n^2 \overset{zz}{\times} H$$

$$G_n, \ (4^n, d^2, 1/2)$$

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- Friedman (1991): A random d–regular graph has $\lambda \leq 2\sqrt{d-1} + 2\log d + O(1)$.

References

- Fan R.K. Chung. Spectral Graph Theory, CBMS Regional Conference Series in Mathematics, 92, American Mathematical Society (1997)
- S. Hoory, N. Linial, and A. Wigderson. Expander Graphs and their Applications. Bull. AMS, Vol. 43 (4), (2006) 439561.
- M. Ram Murty, Ramanujan Graphs. J. Ramanujan Math. Soc. 18 (2003) 1-20.
- A. Lubotzky, Discrete Graphs, Expander Graphs and Invariant Measures, CRM Series, Birkhauser (1994).