Open Problems and Notation

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1 Spectral graph theory

For some background, see [1, 25, 12].

Problem 1.1 *Let $\lambda_1 \geq \lambda_2 \geq \cdots \lambda_n$ the (adjacency matrix) eigenvalues of a graph G. It is known that the size of the largest coclique (independent set of vertices) of G satisfies the bound*

$$\alpha(G) \leq \min\{|i : \lambda_i \geq 0|, |i : \lambda_i \leq 0|\}.$$

Find similar bounds for the k-independence number α_k, $k \geq 1$; that is, the maximum number of vertices which are mutually at distance greater than k (so, $\alpha_1 = \alpha$).

Problem 1.2 *Prove or disprove that, given any graph $G = (V, E)$, we can find a matrix M with entries $M_{uv} = 0$ when $uv \notin E$ such that the above upper bound is sharp.*

Problem 1.3 *For a partition $\mathcal{P}$ of the vertex set V, let $\boldsymbol{B}$ be the weight- (or pseudo-)quotient matrix of $\boldsymbol{A}$. We already know that when the interlacing $\mathrm{ev}\,\boldsymbol{B} \subset \mathrm{ev}\,\boldsymbol{A}$ is tight then $\mathcal{P}$ is weight- (or pseudo-)regular. Look for an "iff"result.*

Problem 1.4 *Some spectral characterizations (such as the spectral excess theorem for diameter three or the upper bound for the diameter) have been obtained by using either of the techniques, eigenvalue interlacing and orthogonal polynomials+orthogonal projections. Thus, we could try to put it all in a common general framework (or, at least, find some interesting relationships between both techniques). Probably, Problems 1.3 and 1.4 are related since someway in both cases we are trying to generalize the conditions (some kind of relationship between $\mathrm{ev}\,\boldsymbol{B}$ and $\mathrm{ev}\,\boldsymbol{A}$?) for having a weight-regular partition.*

Problem 1.5 *Study some possible generalizations of the Hoffman polynomial [26]: A graph $G = (V, E)$ with adjacency matrix $\boldsymbol{A}$ has a (weight-)regular partition iff there exists one (or more) polynomial(s) that, when applied to $\boldsymbol{A}$, gives matrices with blocks being multiple of the all-1 matrix $\boldsymbol{J}$ (with appropriate size), as it happens in the case of connected regular (or general) graphs and biregular graphs.*

2 Distance-regular graphs

Problem 2.1 *(Problem 3 (BCC18.3)) Strongly distance-regular graphs. Proposed by M. A. Fiol. For the definition of a distance-regular graph and related concepts, we refer to Brouwer et al. [3]. A graph G with diameter d is called strongly distance-regular if G is distance-regular and the distance-d graph G_d (in which vertices are adjacent if they have distance d in G) is strongly regular. Examples include [13, 14]:*

1. *Any strongly regular graph.*
2. *Any distance-regular graph with $d = 3$ and third-largest eigenvalue -1;*
3. *Any antipodal distance-regular graph.*

Problem: Prove or disprove that these examples exhaust all possibilities.

Problem 2.2 *Both the intersection parameters p_{ij}^k and the Krein parameters q_{ij}^k are known to be nonnegative. Is this also true for the preintersection parameters ξ_{ij}^k and the preKrein parameters ζ_{ij}^k?*

3 Almost distance-regular graphs

For concepts and notation, see [7, 8].

It would also be interesting to find examples of m-partially distance-regular graphs with m equal (or close) to $d-2$ that are not distance-regular (for all d), is these exist. More specifically, we pose the following problem.

Problem 3.1 *Determine the smallest $m = m_{pdr}(d)$ such that every m-partially distance-regular graph with $d+1$ distinct eigenvalues is distance-regular.*

There are examples of $(D-1)$-walk-regular graphs with diameter D that are not distance-regular, for small D. For larger D, we do not have such examples however, so the question arises if these exist at all.

Problem 3.2 *(a) Determine the smallest $m = m_{wr,D}(D)$ such that every m-walk-regular graph with diameter D is distance-regular.*

(b) Determine the smallest $m = m_{wr,d}(d)$ such that every m-walk-regular graph with $d+1$ distinct eigenvalues is distance-regular.

Problem 3.3 *Define and study the different concepts of almost distance-regularity for "almost distance-biregular graphs". (It is known that if a graph G is distance-regular around each of its vertices, then it is either distance-regular or distance-biregular.)*

Let G be a graph with diameter D and $d+1$ distinct eigenvalues. Let us consider the adjacency algebra

$$\mathcal{A} = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d\} = \operatorname{span}\{\boldsymbol{E}_0, \boldsymbol{E}_1, \boldsymbol{E}_2, \ldots, \boldsymbol{E}_d\}$$

and the distance algebra

$$\mathcal{D} = \operatorname{span}\{\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D\}.$$

Let us consider the following two families of orthogonal polynomials:

- The predistance polynomials $p_0, p_1, \ldots, p_d$ such that

$$\langle p_i, p_j \rangle_\vartriangle = \frac{1}{n}\sum_{l=0}^{d} m_l p_i(\lambda_l) p_j(\lambda_l) = \delta_{ij} n_i = \delta_{ij} p_i(\lambda_0).$$

- The preidempotent polynomials $q_0, q_1, \ldots, q_d$ such that

$$\langle q_i, q_j \rangle_\blacktriangle = \frac{1}{n}\sum_{l=0}^{d} n_l q_i(\lambda_l) q_j(\lambda_l) = \delta_{ij} m_i = \delta_{ij} q_i(\lambda_0).$$

related by the equality

$$\frac{q_j(\lambda_i)}{p_i(\lambda_j)} = \frac{m_j}{n_i} = \frac{q_j(\lambda_0)}{p_i(\lambda_0)}.$$

In this framework, we know that G is i-punctually distance-regular iff any of the following conditions hold

- $\boldsymbol{A}_i \in \mathcal{A}$
- $p_i(\boldsymbol{A}) \in \mathcal{D}$
- $\boldsymbol{A}_i = p_i(\boldsymbol{A})$

Conjecture 3.4 *A graph G is j-punctually eigenspace-distance-regular iff any of the following conditions hold*

- $\boldsymbol{E}_j \in \mathcal{D}$
- $q_j[\boldsymbol{A}] \in \mathcal{A}$
- $\boldsymbol{E}_j = q_j[\boldsymbol{A}]$

where $f[\boldsymbol{A}] = \frac{1}{n}\sum_{i=0}^{d} f(\lambda_i)\boldsymbol{A}_i$ *for any given polynomial* f.

4 Edge-distance-regular graphs

For preliminary studies, see [19, 6].

The *edge-distance-regular* graphs are distance-regular around each of their edges and with the same intersection numbers for each edge. That is, a similar condition that for distance-regular graphs but 'seen' from the edges instead that from the vertices.

Problem 4.1 *Find examples/families of edge-distance-regular graphs not being distance-regular.*

Problem 4.2 *Extend the results/concepts in [7] on almost distance-regular graphs to the case of (almost) edge-distance-regularity. Study new concepts, such as almost edge-distance-regular graphs, edge-walk-regular graphs, etc.*

In general, try to do the same for most of the known results on distance-regular graphs. For instance:

Problem 4.3 *Prove that, for a fixed degree, there are finitely many edge-distance-regular graphs. (In the case of distance-regular graphs, this corresponds to the Itho conjecture that was recently proved (the work is still in revision)).*

Problem 4.4 *Prove or disprove that an edge-distance-regular graph has maximum edge- and/or vertex-connectivity. The same problem for superconnectivity. (In the case of distance-regular graphs, this was a conjecture of Brouwer, now a theorem recently proved).*

Problem 4.5 *Godsil and Shawe-Taylor [24] proved that every distance-regularised graph is either distance-regular or distance-biregular. Prove or disprove that every edge-distance-regularized graphs is edge-distance-regular.*

Problem 4.6 *Find an infinite family of edge-distance-regular graphs not being edge-transitive. In the case of non-vertex-transitive distance-regular graphs, one such a family was* *recently found by Van Dam and Koolen [11]. This is the first known family of non-vertex-transitive distance-regular graphs with unbounded diameter, and it was found by applying the spectral excess theorem for distance-regular graphs [17, 10, 20].*

Problem 4.7 *Find the analogue of association schemes for the case of edge-distance-regular graphs.*

Problem 4.8 *Fins the relationship between edge-distance-regularity in combinatorial sense and in algebraic sense (Terwilliger-like algebras). For the case of distance-regularity, see [5].*

5 Bidirectional digraphs

Given a digraph $G = (V, A)$, with the standard distant function $\mathrm{dist}(\cdot,\cdot) = \mathrm{dist}_G(\cdot,\cdot)$, we can consider the following new concepts:

- The *bidirectional distance* between vertices $u, v \in V$ is defined as
$$\mathrm{dist}^*(u,v) = \min\{\mathrm{dist}(u,v), \mathrm{dist}(v,u)\} = \min\{\mathrm{dist}_G(u,v), \mathrm{dist}_{\overline{G}}(u,v)\}.$$
Notice that, contrarily to dist, the bidirectional distance dist^* is a metric.
- The *bidirectional eccentricity* of a vertex $u \in V$ is
$$\mathrm{ecc}^*(u) = \max\{\mathrm{dist}^*(u,v) : v \in V\}.$$
- The *bidirectional diameter* of G is
$$D^* = \max\{\mathrm{ecc}^*(u) : u \in V\} = \max\{\mathrm{dist}^*(u,v) : u, v \in V\}.$$
- The *bidirectional radius* of G is
$$r^* = \min\{\mathrm{ecc}^*(u) : u \in V\}.$$

Problem 5.1 *For which values of δ and d there exists a δ-regular digraph G with bidirectional diameter $D^* = d$ with exactly a bidirectional d-walk between every pair of vertices. In other words, find solutions of the matrix equation*
$$\boldsymbol{A}^d + (\boldsymbol{A}^\top)^d = \boldsymbol{J},$$
where $\boldsymbol{A}$ is the adjacency matrix of G satisfying $\boldsymbol{A}\delta = \delta\boldsymbol{A}$.

Problem 5.2 *Similar problem for the matrix equation (looking for bidirectional Moore digraphs)*
$$\boldsymbol{I} + \boldsymbol{A} + \boldsymbol{A}^\top + \cdots + \boldsymbol{A}^d + (\boldsymbol{A}^\top)^d = \boldsymbol{J}.$$

6 Notation

Symbol	Definition
a_i	Intersection number $p^i_{1,i}$
α_i	Preintersection number $\xi^i_{1,i}$
$a^{(\ell)}$	Number of crossed walks of length ℓ rooted at any vertex
$a^{(\ell)}_h$	Number of walks of length ℓ between any pair of vertices at distance h
$a^{(\ell)}_u$	Number of crossed walks of length ℓ rooted at vertex u
$a^{(\ell)}_{uv}$	(u,v)-entry of $\boldsymbol{A}^\ell$ or number of ℓ-walks between vertices u and v
$\overline{a}^{(\ell)}_h$	Mean number of ℓ-walks over all pairs of vertices at distance h
$\boldsymbol{A}$	Adjacency matrix of graph G
$\boldsymbol{A}_i$	Adjacency matrix of graph G_i or distance-i matrix of graph G
$\widetilde{\boldsymbol{A}}_i$	Orthogonal projection of $\boldsymbol{A}_i$ onto $\mathcal{A}$
$\boldsymbol{A}_{uv}$	(u,v)-entry of matrix $\boldsymbol{A}$
$\mathcal{A}=\mathcal{A}(G)$	Adjacency or Bose-Mesner algebra of graph G
b_i	Intersection number $p^i_{1,i+1}$
β_i	Preintersection number $\xi^i_{1,i+1}$
$\boldsymbol{B}_i$	i-th graph of an association scheme
$\mathcal{C}$	Set of the numbers $a^{(\ell)}_{uv}$ for $0\le\ell\le d$
c_i	Intersection number $p^i_{1,i-1}$
$d+1$	Number of different eigenvalues of adjacency matrix $\boldsymbol{A}$
$D=D(G)$	Diameter of a graph G
$\mathcal{D}=\mathcal{D}(G)$	Vector space with basis the set of distance matrices of G

Symbol	Definition
$\mathrm{dist}(u,v)$	Distance between vertices u and v
δ	Degree of (regular) graph G
δ_i	Degree of (regular) graph G_i
$\delta(u)$	Degree of vertex u
$\overline{\delta}_i$	Average degree of graph G_i
δ_{ij}	Kronecker delta
$\boldsymbol{e}_u$	Canonical vector of $\mathbb{R}^n$ representing vertex u
$e+1$	Number of graphs is an association scheme
$E=E(G)$	Edge set of a graph G
$\boldsymbol{E}_i$	Idempotent of $\boldsymbol{A}$ corresponding to the orthogonal projection onto $\mathcal{E}_i$
$\mathcal{E}_i$	Eigenspace of eigenvalue λ_i
$\mathrm{ecc}(u)$	eccentricity of vertex u
η_i	Coefficient of x_i in a generic polynomial
$\mathrm{ev}\, G = \mathrm{ev}\, \boldsymbol{A}$	Set of different eigenvalues of graph G
g	Girth of graph G
G	Graph
G_i	Distance-i graph of G
γ_i	Preintersection number $\xi^i_{1,i-1}$
$\Gamma_i(u)$	Set of vertices at distance i from vertex u
h	Parameter for h-punctually distance-polynomial graph
	Parameter for h-punctually distance-regular graph
	Parameter for h-punctually walk-regular graph
	Parameter for h-punctually spectrum-regular graph
h,i,j,k	Distances between vertices ($0\le h,i,j,k\le D$)
	Indexes of distance polynomials
	Parameters of the intersection numbers
H	Hoffman polynomial
i,j,k,l	Indexes of eigenvalues, eigenspaces, idempotents ($0\le i,j,k,l\le d$)
	Indexes of predistance polynomials
	Parameters of preintersection numbers
I	Set of indexes $i\in\{0,1,\ldots,d\}$ for which $q_h(\lambda_i)=0$
$\boldsymbol{I}$	Identity matrix
$\boldsymbol{j}$	All-1 vector
$\boldsymbol{J}$	All-1 matrix
ξ^k_{ij}	Preintersection number for $0\le i,j,k\le d$
$l+1$	Number of eigenvalues λ_i such that $q_h(\lambda_i)=0$
ℓ	Length of a walk
λ	Number of common neighbors of any two vertices of a graph G
$\lambda_i^{m_i}$	Eigenvalue of adjacency matrix $\boldsymbol{A}$ with multiplicity $m_i=m(\lambda_i)$
λ_i^*	Lagrange interpolating polynomial giving the idempotent $\boldsymbol{E}_i=\lambda_i^*(\boldsymbol{A})$
m	Parameter for m-walk-regular graph
	Parameter for m-partially distance-polynomial graph
	Parameter for m-partially distance-regular graph
	Parameter for m-partially walk-regular graph
	Parameter for m-partially spectrum-regular graph

Symbol	Definition
$m_u(\lambda_i)$	u-local multiplicity of λ_i
m_{hi}	Crossed local multiplicity of λ_i for any pair of vertices at distance h
$\overline{m}_{hi}$	Mean crossed local multiplicity of λ_i over all pairs of vertices at distance h
$m_{uv}(\lambda_i)$	Crossed uv-local multiplicity of λ_i
$\boldsymbol{O}$	0-matrix
$\mathbf{0}$	0-vector
p_i	Predistance polynomial with degree i of graph G
	Distance polynomial with degree i of graph G
$\widehat{p_i(\boldsymbol{A})}$	Orthogonal projection of $p_i(\boldsymbol{A})$ onto $\mathcal{D}$
p_{ij}^k	Intersection number for $0 \leq i, j, k \leq D$
ϕ_i	Product of all the terms $\lambda_i - \lambda_j$ for all $j \neq i$
π_i	Product of all the terms $\lvert\lambda_i - \lambda_j\rvert$ for all $j \neq i$
$\mathbb{R}_d[x]$	Vector space of real polynomials with degree at most d
R_i	Relation of an association scheme
$\boldsymbol{S}_k$	Sum of distance matrices up to $\boldsymbol{A}_k$
$\operatorname{sp} G = \operatorname{sp} \boldsymbol{A}$	Spectrum of the adjacency matrix of graph G
sum	Sum of all entries of a matrix
$\mathcal{T}$	Vector space $\mathcal{A} + \mathcal{D}$
$\operatorname{tr} \boldsymbol{A}$	Trace of matrix $\boldsymbol{A}$
$u, v, w, \ldots$	Generic vertices
$V = V(G)$	Vertex set of a graph G
Z	Minimal polynomial of adjacency matrix $\boldsymbol{A}$
$\circ$	Schur or Hadamard component-wise product of matrices
$\sim$	Adjacency between vertices
$\langle p, q\rangle$	Scalar product of polynomials p and q in $\mathbb{R}_d(x)$, defined as $\langle p(\boldsymbol{A}), q(\boldsymbol{A})\rangle$
$\langle \boldsymbol{P}, \boldsymbol{Q}\rangle$	Scalar product of matrices $\boldsymbol{P}, \boldsymbol{Q} \in \mathcal{T}$, defined as $\frac{1}{n}\operatorname{tr}(\boldsymbol{PQ}) = \frac{1}{n}\operatorname{sum}(\boldsymbol{P} \circ Q)$

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