# Fast Minor Testing in Planar Graphs

Isolde Adler[1], Frederic Dorn[2], Fedor V. Fomin[2],
Ignasi Sau[3], and Dimitrios M. Thilikos[4]

[1] Institut für Informatik, Goethe-Universität, Frankfurt, Germany.
`iadler@informatik.uni-frankfurt.de`
[2] Department of Informatics, University of Bergen, Norway.
`{frederic.dorn,fedor.fomin}@ii.uib.no`
[3] Department of Computer Science, Technion, Haifa, Israel.
`ignasi@cs.technion.ac.il`
[4] Department of Mathematics, National and Kapodistrian
University of Athens, Greece.
`sedthilk@math.uoa.gr`

**Abstract.** Minor containment is a fundamental problem in Algorithmic Graph Theory, as numerous graph algorithms use it as a subroutine. A model of a graph $H$ in a graph $G$ is a set of disjoint connected subgraphs of $G$ indexed by the vertices of $H$, such that if $\{u, v\}$ is an edge of $H$, then there is an edge of $G$ between components $C_u$ and $C_v$. Graph $H$ is a minor of $G$ if $G$ contains a model of $H$ as a subgraph. We give an algorithm that, given a planar $n$-vertex graph $G$ and an $h$-vertex graph $H$, either finds in time $2^{\mathcal{O}(h)} \cdot n + \mathcal{O}(n^2 \cdot \log n)$ a model of $H$ in $G$, or correctly concludes that $G$ does not contain $H$ as a minor. Our algorithm is the first *single-exponential* algorithm for this problem and improves all previous minor testing algorithms in planar graphs. Our technique is based on a novel approach called *partially embedded dynamic programming*.

**Keywords:** graph minors, planar graphs, branchwidth, parameterized complexity, dynamic programming.

## 1 Introduction

For two input graphs $G$ and $H$, the MINOR CONTAINMENT problem is to decide whether $H$ is a minor of $G$. This is a classical NP-complete problem [15], and remains NP-complete even when both graphs $G$ and $H$ are planar, as it is a generalization of the HAMILTONIAN CYCLE problem. When $H$ is fixed, by the celebrated result of Robertson and Seymour [25], there is an algorithm to decide if $H$ is a minor of an input graph $G$ that runs in time $f(h) \cdot n^3$, where $n$ is the number of vertices of $G$, $h$ is the number of vertices in $H$, and $f$ is some recursive function. One of the significant algorithmic implications of this result is that, combined with the Graph Minor Theorem of Robertson and Seymour [27], it shows the polynomial-time solvability of many graph problems, some of which were previously not even known to be decidable [14]. However, these algorithmic results are highly *non-practical*. This triggered an ongoing quest in the Theory of Algorithms since then –next to on simplifying the 20-papers proof of the Graph Minors Theorem– for making Graph Minors constructive and for making its algorithmic proofs practical for a wide range of applications (e.g., [7, 19]).

Unfortunately, in the minor testing algorithm of Robertson and Seymour [25], the function $f(h)$ has an *immense* exponential growth, which makes the algorithm absolutely impractical even for very simple patterns (see [20] for recent theoretical improvements of this function). There were several attempts to improve the running time of the algorithm of Robertson and Seymour. One direction of such improvements is decreasing the degree of the polynomial in $n$. For example, Reed and Li gave a linear time algorithm solving $K_5$-minor containment [24]. The second direction of improvements is towards reducing the exponential dependency in the function $f(h)$, which is a natural direction of study for Parameterized Complexity [13]. A significant step in this direction was done by Hicks [18], who provided in graphs of branchwidth $k$ and $m$ edges an $\mathcal{O}(3^{k^2} \cdot (h+k-1)! \cdot m)$ time algorithm, following the algorithm sketched by Robertson and Seymour [25]. Recently, this was improved to $\mathcal{O}(2^{(2k+1)\log k} \cdot h^{2k} \cdot 2^{2h^2} \cdot m)$ on general graphs, and in planar, and more generally, in graphs of bounded genus, to $2^{\mathcal{O}(k)} \cdot h^{2k} \cdot 2^{\mathcal{O}(h)} \cdot n$ [1].

In this paper we focus on the case where the input graph $G$ is planar.

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PLANAR $H$-MINOR CONTAINMENT
**Input:** A planar graph $G$.
**Objective:** Either find a model $M$ of $H$ in $G$, or conclude that $G$ does not contain such a model.

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By arguments inspired by Bidimensionality Theory [4], it can be shown that the $2^{\mathcal{O}(k)} \cdot h^{2k} \cdot 2^{\mathcal{O}(h)} \cdot n$ time algorithm from [1], combined with the grid minor Theorem of Robertson, Thomas, and Seymour [26], can be used to solve PLANAR $H$-MINOR CONTAINMENT in time $2^{\mathcal{O}(h \log h)} \cdot n + \mathcal{O}(n^2 \cdot \log n)$. This directly sets up the challenge of designing a *single-exponential* (on the size $h$ of the pattern $H$) algorithm for this problem. Over the last four decades, many different algorithmic techniques in planar graphs were developed for different type of problems and algorithms, including approximation [3,6], exact [11,21], and parameterized algorithms [2,5,12]. However, it seems that none of these approaches can be used to speed up the algorithm for PLANAR $H$-MINOR CONTAINMENT.

**Our results and key ideas.** Our main result is the following theorem.

**Theorem 1.** *Given a planar graph $G$ on $n$ vertices and a graph $H$ on $h$ vertices, we can solve* PLANAR $H$-MINOR CONTAINMENT *in time* $2^{\mathcal{O}(h)} \cdot n + \mathcal{O}(n^2 \cdot \log n)$.

That is, we prove that when $G$ is planar the behavior of the function $f(h)$ can be made *single-exponential*, improving over all previous results for this problem [1, 18,25]. In addition, we can *enumerate* and *count* the number of models within the same time bounds. Let us remark that by our theorem, PLANAR $H$-MINOR CONTAINMENT is solvable in polynomial time when the size of the pattern graph $H$ is $\mathcal{O}(\log n)$, therefore substantially improving the existing algorithms for small patterns [9].

In order to prove Theorem 1, we introduce a novel approach of dynamic programming in planar graphs of bounded branchwidth, namely *partially embedded dynamic programming*. This approach is extremely helpful in computing graph

minors but we believe that this technique can be used in many related problems including PLANAR DISJOINT PATHS. Our technique is inspired by the technique of *embedded dynamic programming* introduced in [10] for solving PLANAR SUB-GRAPH ISOMORPHISM for a pattern of size $h$ and an input graph of size $n$ in time $2^{\mathcal{O}(h)} \cdot n$. There, one controls the partial solutions by the ways the separators of $G$ can be routed through the pattern. The difference (and difficulty) concerning PLANAR $H$-MINOR CONTAINMENT is that we look for a model $M$ of size $\mathcal{O}(n)$ out of $2^{\mathcal{O}(n)}$ possible non-isomorphic models of $H$ in $G$. In partially embedded dynamic programming, we look for potential models of $H$ in $G$ with a magnifying glass only at a given separator $S$ of $G$. That is, we consider a collection $\mathcal{A}$ of graphs $A$ arising from 'inflating' a part of $H$, namely the part interacting with $S$. Thus, each $A$ behaves like a subgraph of $G$ inside the intersection with $S$, and outside that intersection $A$ behaves like a minor of $G$; this is why we call our dynamic programming technique '*partially embedded*'.

After giving some preliminaries in Section 2, we first show in Section 3 how PLANAR $H$-MINOR CONTAINMENT can be solved in polynomial time for input graphs of large branchwidth (in comparison to the pattern size). If the branchwidth is small, we compute the collection $\mathcal{A}$ in Section 3.1 and give the partially embedded dynamic programming approach in Section 3.2.

## 2  Preliminaries

We use standard graph terminology, see for instance [8].

**Graphs and graph minors.** All graphs considered in this article are simple and undirected. Given a graph $G$, we denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. A graph $H$ is a *subgraph* of a graph $G$, $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We define graph operation *contracting* edge $e = \{x, y\} \in G$ by removing $e$, including $x$ and $y$, and making a new vertex $v_e$ adjacent to the former neighbors of $x$ and $y$ (excluding $x$ and $y$).

Graph $H$ is a *minor* of graph $G$ (denoted by $H \preceq G$), if $H$ can be obtained from a subgraph of $G$ by a (possibly empty) sequence of edge contractions. In this case we also say that $G$ is a *major* of $H$. Graph $H$ is a *contraction minor* of graph $G$ (denoted by $H \preceq_c G$), if $H$ can be obtained from $G$ by a (possibly empty) sequence of edge contractions.

A *model* of $H$ in $G$ [25] is a mapping $\phi$, that assigns to every edge $e \in E(H)$ an edge $\phi(e) \in E(G)$, and to every vertex $v \in V(H)$ a non-empty connected subgraph $\phi(v) \subseteq G$, such that

(i) the graphs $\{\phi(v) \mid v \in V(H)\}$ are mutually vertex-disjoint and the edges $\{\phi(e) \mid e \in E(H)\}$ are pairwise distinct; and
(ii) for $e = \{u, v\} \in E(H)$, $\phi(e)$ connects $\phi(u)$ with $\phi(v)$.

Thus, $H$ is isomorphic to a minor of $G$ if and only if there exists a model of $H$ in $G$.

With slight abuse of notation, the subgraph $M \subseteq G$ defined by the union of $\{\phi(v) \mid v \in V(H)\}$ and $\{\phi(e) \mid e \in E(H)\}$ is also called a *model* of $H$ in $G$. The edge set of a model $M$ is partitioned into *c-edges (contraction edges)* and *m-edges*

(*minor edges*), which are the edges of $\{\phi(v) \mid v \in V(H)\}$ and $\{\phi(e) \mid e \in E(H)\}$, respectively. For an illustration, see Fig. 1 [⋆][1].

**Branchwidth.** A *branch decomposition* $(T, \mu)$ of a graph $G$ consists of an unrooted ternary tree $T$ (i.e., all internal vertices are of degree three) and a bijection $\mu : L \rightarrow E(G)$ from the set $L$ of leaves of $T$ to the edge set of $G$. We define for every edge $e$ of $T$ the *middle set* $\mathbf{mid}(e) \subseteq V(G)$ as follows: Let $T_1$ and $T_2$ be the two connected components of $T \setminus \{e\}$. Then let $G_i$ be the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$. The *middle set* is the intersection of the vertex sets of $G_1$ and $G_2$, i.e., $\mathbf{mid}(e) = V(G_1) \cap V(G_2)$. The *width* of $(T, \mu)$ is the maximum order of the middle sets over all edges of $T$, i.e., $width(T, \mu) := \max\{|\mathbf{mid}(e)| : e \in E(T)\}$. The *branchwidth* of $G$ is defined as $\mathbf{bw}(G) := \min\{width(T, \mu) \mid (T, \mu) \text{ branch decomposition of } G\}$. Note that for each $e \in E(T)$, $\mathbf{mid}(e)$ is a separator of $G$, unless $\mathbf{mid}(e) = \emptyset$.

**Remark 1** *For every two edges $e, f \in E(T)$ with $e \cap f \neq \emptyset$, we have $|\mathbf{mid}(e) \cup \mathbf{mid}(f)| \leq 1.5 \cdot width(T, \mu)$.*

Intuitively, a graph $G$ has small branchwidth if $G$ is close to being a tree. The fundamental grid minor Theorem says that, roughly, a graph has either small branchwidth, or it contains a large grid as a minor. We use the variant for planar graphs.

**Proposition 1 ([17, 26])** *Given a planar graph $G$ on $n$ vertices with $\mathbf{bw}(G) \geq k$, a model of the $(\lfloor k/3 \rfloor \times \lfloor k/3 \rfloor)$-grid in $G$ can be found in time $\mathcal{O}(n^2 \cdot \log n)$.*

On the other hand, every planar graph is minor of a large enough grid.

**Proposition 2 ([26])** *If $H$ is a planar graph with $|V(H)| + 2|E(H)| \leq \ell$, then $H$ is isomorphic to a minor of the $(2\ell \times 2\ell)$-grid.*

**Planar graphs and equivalent drawings.** Let $\Sigma$ be the unit sphere. A *planar drawing*, or simply *drawing*, of a graph $G$ with vertex set $V(G)$ and edge set $E(G)$ maps vertices to points in the sphere, and edges to simple curves between their end-vertices, such that edges do not cross, except in common end-vertices. A *plane graph* is a graph G together with a planar drawing. A *planar graph* is a graph that admits a planar drawing. The set of *faces* $F(G)$ of a plane graph $G$ is defined as the union of the connected regions of $\Sigma \setminus G$. A subgraph of a plane graph $G$, induced by the vertices and edges incident to a face $f \in F(G)$, is called a *bound* of $f$ (for further reading, see e.g. [8]). Consider any two drawings $G_1$ and $G_2$ of a planar graph $G$. A *homeomorphism* of $G_1$ onto $G_2$ is a homeomorphism of $\Sigma$ onto itself which maps vertices, edges, and faces of $G_1$ onto vertices, edges, and faces of $G_2$, respectively. We call two planar drawings *equivalent*, if there is a homeomorphism from one onto the other.

**Proposition 3 (e.g. [23])** *Every planar $n$-vertex graph has $2^{\mathcal{O}(n)}$ non-equivalent drawings. A set of all such drawings can be computed in time $2^{\mathcal{O}(n)}$.*

**Proposition 4 ([30])** *The number of non-isomorphic edge-maximal planar graphs on $n$ vertices is $2^{\mathcal{O}(n)}$.*

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[1] The parts marked with [⋆] have been moved to the appendix due to space restrictions.

**Nooses and combinatorial nooses.** A *noose* of a $\Sigma$-plane graph $G$ is a simple closed curve in $\Sigma$ that meets $G$ only in vertices. From the Jordan Curve Theorem (e.g. [22]), it follows that nooses separate $\Sigma$ into two regions.

Let $V(N) = N \cap V(G)$ be the vertices and $F(N)$ be the faces intersected by a noose $N$. The *length* of $N$ is $|V(N)|$, the number of vertices in $V(N)$. The clockwise order in which $N$ meets the vertices of $V(N)$ is a cyclic permutation $\pi$ on the set $V(N)$.

A *combinatorial noose* $N_C = [v_0, f_0, v_1, f_1, \ldots, f_{\ell-1}, v_\ell]$ in a plane graph $G$ is an alternating sequence of vertices and faces of $G$, such that

- $f_i$ is a face incident to both $v_i, v_{i+1}$ for all $i < \ell$;
- $v_0 = v_\ell$ and the vertices $v_1, \ldots, v_\ell$ are mutually distinct; and
- if $f_i = f_j$ for any $i \neq j$ and $i, j = 0, \ldots, \ell - 1$, then the vertices $v_i, v_{i+1}, v_j$, and $v_{j+1}$ do not appear in the order $(v_i, v_j, v_{i+1}, v_{j+1})$ on the bound of face $f_i = f_j$.

The *length* of a combinatorial noose $[v_0, f_0, v_1, f_1, \ldots, f_{\ell-1}, v_\ell]$ is $\ell$.

**Remark 2** *The order in which a noose $N$ intersects the faces $F(N)$ and the vertices $V(N)$ of a plane graph $G$ gives a unique alternating face-vertex sequence of $F(N) \cup V(N)$ which is a combinatorial noose $N_C$. Conversely, for every combinatorial noose $N_C$ there exists a noose $N$ with face-vertex sequence $N_C$.*

We will refer to combinatorial nooses simply as nooses if it is clear from the context.

**Proposition 5 ([10])** *Every plane $n$-vertex graph has $2^{\mathcal{O}(n)}$ combinatorial nooses.*

**Sphere cut decompositions.** For a plane graph $G$, we define a *sphere cut decomposition* (or *sc-decomposition*) $\langle T, \mu, \pi \rangle$ as a branch decomposition, which for every edge $e$ of $T$ has a noose $N_e$ that divides $\Sigma$ into two regions $\Delta_1$ and $\Delta_2$ such that $G_i \subseteq \Delta_i \cup N_e$, where $G_i$ is the graph induced by the edge set $\{\mu(f) : f \in L \cap V(T_i)\}$ for $i \in \{1, 2\}$ and $T_1 \dot\cup T_2 = T \setminus \{e\}$. Thus $N_e$ meets $G$ only in $V(N_e) = \mathbf{mid}(e)$ and its length is $|\mathbf{mid}(e)|$. The vertices of $\mathbf{mid}(e) = V(G_1) \cap V(G_2)$ are ordered according to a cyclic permutation $\pi = (\pi_e)_{e \in E(T)}$ on $\mathbf{mid}(e)$.

**Theorem 2 ( [12, 29]).** *Let $G$ be a planar graph of branchwidth at most $k$ without vertices of degree one embedded on a sphere. Then there exists a sc-decomposition of $G$ of width at most $k$.*

## 3    Minor testing in planar graphs

For solving PLANAR $H$-MINOR CONTAINMENT in single-exponential time $2^{\mathcal{O}(h)} \cdot n + \mathcal{O}(n^2 \cdot \log n)$, we introduce in this section the method of *partially embedded dynamic programming*. We present Algorithm 3.1 as a roadmap on how we proceed in proving our main Theorem 1.

We divide Algorithm 3.1 into three parts, presented in the following sections.

From the next proposition we can find a model of $H$ in $G$ in the case of $G$ having large branchwidth.

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**Algorithm 3.1**: The main routine for PLANAR $H$-MINOR CONTAINMENT.

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   **Input**    : A planar graph $G$.
   **Output** : A model $M$ of $H$ in $G$, if it exists.
   Compute sc-decomposition $\langle T, \mu, \pi \rangle$ of $G$ of width $\mathbf{bw}(G)$.
   **if** $\mathbf{bw}(G) > c \cdot h$ *for some constant $c$* **then** Compute $M$
   **else for** *every plane graph $A \succeq H$ produced by* PRE-PROC$(H)$ **do**
         Run partially embedded dynamic programming on $\langle T, \mu, \pi \rangle$ to try to find a
         model $M$ of $A$ in $G$.

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**Proposition 6** [⋆] *Let $G$ and $H$ be planar graphs with $|V(G)| = n$ and $|V(H)| = h$. There exists a small constant $c$ such that if $\mathbf{bw}(G) > c \cdot h$, then $G$ contains a model of $H$, which can be found in time $\mathcal{O}(n^2 \cdot \log n + h^4)$.*

Otherwise, let us assume that $\mathbf{bw}(G) \leq c \cdot h$. In this case, PRE-PROC$(H)$ (basically) computes a list of all plane majors $A$ of $H$ up to a fixed size linear in $h$. This 'preprocessing step' is presented in Section 3.1. In the sequel, we will only be interested in a graph $A$ of our list, if $A$ is a minor of $G$ obtained from $H$ by 'uncontracting' some part, such that on a given subset $S \subseteq V(G)$, our graph $A$ looks like a subgraph of $G$. Finally, in Section 3.2, we proceed by partially embedded dynamic programming bottom-up along a sphere cut decomposition of $G$. Here we make use of the fact that every middle set $S$ yields a separating noose in an embedding of $G$. If $H$ has a model $M \subseteq G$ that intersects $S$, then the noose comes from a noose in $M$, which in turn is present in some major $A$ of $H$ of our list. We use this fact to restrict the number of candidates we need to consider.

### 3.1   Preprocessing

If the branchwidth of $G$ is at most $c \cdot h$, then we compute a sphere cut decomposition of width $\mathcal{O}(h)$ in time $\mathcal{O}(n^2)$ by using the algorithm of [16], and we continue with dynamic programming.

In the first step we do preprocessing. Namely, we compute for $H$ a list of auxiliary graphs $A$ with $H \preceq A \preceq M$ and $|V(A)| = \mathcal{O}(h)$, such that $A$ is a candidate for a model $M$ in $G$. To be precise, we compute a collection $\mathcal{A}$ of 2-edge-colored plane graphs, each consisting of
- a planar graph $A$ with $|V(A)| \leq h + 1.5 \cdot \mathbf{bw}(G)$, such that $H$ is a contraction minor of $A$;
- a bipartition of the edge set $E(A)$ into m-edges and c-edges such that contracting the c-edges of $A_{m,c}$ creates a graph isomorphic to $H$; and
- a drawing $\varPhi$ of $A_{m,c}$.

We describe the preprocessing in Algorithm C.1 [⋆]: the routine PRE-PROC takes as input $H$ and outputs the collection $\mathcal{A}$ of 2-edge-colored plane graphs $\langle A_{m,c}, \varPhi \rangle$.

When doing dynamic programming in Section 3.2, instead of finding a model $M$ of $H$ in $G$, we restrict ourselves to finding such a collection $\mathcal{A}$ consisting of minors of $M$ which represent both $H$ and $M$ in each dynamic programming step.

**Lemma 1.** [⋆] *For every planar graph $H$ on $h$ vertices and every constant $d$, the cardinality of the collection $\mathcal{A}$ of non-isomorphic 2-edge-colored plane graphs on $d \cdot h$ vertices containing a minor isomorphic to $H$ is $2^{\mathcal{O}(h)}$. Furthermore, we can compute $\mathcal{A}$ in time $2^{\mathcal{O}(h)}$.*

Using Lemma 1, we get the following corollary.

**Corollary 1.** *Algorithm C.1 is correct and runs in time $2^{\mathcal{O}(\mathbf{bw}(G)+h)}$.*

### 3.2 Partially embedded dynamic programming

From now on, we will refer to a 2-edge-colored plane graph $\langle A_{m,c}, \Phi \rangle \in \mathcal{A}$ simply as $A$. In this section we present our technique of partially embedded dynamic programming, using it to solve PLANAR $H$-MINOR CONTAINMENT in the case of the input graph $G$ having bounded branchwidth. Before proceeding to a formal description of the dynamic programming, we provide the basic intuition behind our algorithm. Towards this, let us consider graphs $A \in \mathcal{A}$ satisfying $H \preceq_c A$ and $A \preceq G$.

We define subgraphs PAST, PRESENT, and FUTURE of $A$ with $V(A) = V(\text{PAST}) \cup V(\text{PRESENT}) \cup V(\text{FUTURE})$ and $E(A) = E(\text{PAST}) \dot{\cup} E(\text{PRESENT}) \dot{\cup} E(\text{FUTURE})$, such that

- PRESENT $\subseteq G$, (i.e., we can obtain $A$ as a minor of $G$ with PRESENT being subgraph of $G$); and
- $E(\text{PAST}) \subseteq E(H)$, (i.e., we can obtain $H$ as a contraction minor of $A$ without contracting edges in PAST).[2]

Intuitively speaking, in partially embedded dynamic programming, we look for potential models $M$ of $H$ with a magnifying glass only in the separators of the sc-decomposition of $G$. By decontracting $H$ at the separators, we obtain the part PRESENT, which yields a subgraph of $G$ for which we are enabled to apply embedded dynamic programming. For memorizing the rest of the potential model $M$, we contract all necessary edges to PAST in the processed graph and (almost) all edges to FUTURE in the graph remainder. The picture will be made clearer in the sequel.

Given a sc-decomposition of $G$, we proceed with dynamic programming: Every edge $e$ of the sc-decomposition defines a separator $\mathbf{mid}(e) \subseteq V(G)$ and an associated noose $N_e$, which separates the graph $G_{\text{sub}} \subseteq G$ processed so far from $G \setminus G_{\text{sub}}$. At every edge $e$ of the sc-decomposition, we check for every graph $A$ of $\mathcal{A}$ all the ways in which the graph $G_{\text{sub}}$ can be obtained as a major of $A_{\text{sub}} \subseteq A$ with $A_{\text{sub}} = (V(\text{PAST}) \cup V(\text{PRESENT}), E(\text{PAST}) \cup E(\text{PRESENT}))$, where $\mathbf{mid}(e)$ determines $V(\text{PRESENT})$. The noose $N_e$ comes from a noose in $A$, and this is controlled by the ways in which $N_e$ can be routed through the vertices of $A$. The number of solutions we get—the *valid partial solutions*—is bounded by the number of combinatorial nooses in $A$ onto which we can map $N_e$. When updating the valid partial solutions at two incident edges of the sc-decomposition, we unite

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[2] Here, we slightly abuse notation by assuming that edge sets in different graphs are actually the same, instead of introducing bijective mappings. Note that we make no assumption about the edges in FUTURE.

PRESENT and PAST of two solutions and set the graph remainder to FUTURE. In a post-processing step, we contract part of PRESENT, namely those edges with at most one endpoint in the newly obtained separator of the sc-decomposition; this part becomes PAST. We then decontract some edges of FUTURE for the next updating step. This concludes the informal description of the algorithm.

In the remaining part of this section, we will precisely describe and analyze the dynamic programming routine with which we achieve the following result:

**Lemma 2.** *For a plane graph $G$ with a given sc-decomposition $\langle T, \mu, \pi \rangle$ of $G$ of width $\mathbf{bw}(G)$ and a planar graph $H$ on $h$ vertices, we can decide in time $2^{\mathcal{O}(\mathbf{bw}(G)+h)} \cdot n$ whether $G$ contains a model $M$ of $H$.*

**Dynamic programming.** We root the sc-decomposition $\langle T, \mu, \pi \rangle$ at some node $r \in V(T)$. For each edge $e \in T$, let $L_e$ be the set of leaves of the subtree rooted at $e$. The subgraph $G_e$ of $G$ is induced by the edge set $\{\mu(v) \mid v \in L_e\}$. The vertices of $\mathbf{mid}(e)$ form a combinatorial noose $N$ that separates $G_e$ from the residual graph.

Let $A$ be a given plane graph in $\mathcal{A}$. If $A$ is a minor of $G$, then there exists a plane model $M$ of $A$ in $G$. Furthermore, for above noose $N$ the intersection $M \cap N$ forms a noose in both model $M$ and $A$. One basic point of partially embedded dynamic programming is to check how the vertices of the combinatorial noose $N$ are mapped to faces and vertices of $A$. For a combinatorial noose $N_A$ in $A$, we can map $N$ to $N_A$ bounding (clockwise) a unique subgraph $A_{\mathrm{sub}}$ of $A$.

In each step of the algorithm, we compute the solutions for a sub-problem in $G_e$, where each solution consists of three parts, namely
- a plane 2-edge-colored graph $A \in \mathcal{A}$;
- a combinatorial noose $N_A$ in $A$; and
- a mapping $\gamma$ from combinatorial noose $N$ to $N_A$ (defined below).

$N_A$ has the properties that $a)$ it bounds (clockwise) a subgraph $A_{\mathrm{sub}} \subseteq A$ and $b)$ no vertex in $V(A_{\mathrm{sub}}) \setminus V(N_A)$ is incident to a c-edge. The subgraph $A_{\mathrm{sub}}$ is representing the part of model $M$ already computed, whereas the residual graph of $A$ represents the part of $M$ which still has to be verified. For every middle set, we store this information in an array of triples $\langle A, N_A, \gamma \rangle$.

We define now valid mappings between combinatorial nooses and describe how partial solutions are stored in the dynamic programming. Then, we give the different DP-steps and finally verify the approach.

**Valid partial solutions.** For a middle set $\mathbf{mid}(e)$ of the rooted sc-decomposition $\langle T, \mu, \pi \rangle$ of plane graph $G$, $N = N_e$ is the associated combinatorial noose in $G$ with face-vertex sequence of $F(N) \cup V(N)$ separating $G_e$ from the residual graph. Let $\mathfrak{N}$ denote the set of all combinatorial nooses of $A$ whose length is at most the length of $N$ and which bound (clockwise) a subgraph $A_{\mathrm{sub}} \subseteq A$ such that no vertex in $V(A_{\mathrm{sub}}) \setminus V(N_A)$ is an end-vertex of a c-edge. We now map $N$ to nooses $N_A \in \mathfrak{N}$, preserving the order. More precisely, we map vertices of $N$ to both vertices and faces of $A$. Therefore, we consider partitions of $V(N) = V_1(N) \dot\cup V_2(N)$ where vertices in $V_1(N)$ are mapped to vertices of $V(A)$ and vertices in $V_2(N)$ to faces of $F(A)$.

We define a mapping $\gamma : V(N) \cup F(N) \rightarrow V(A) \cup F(A)$ relating $N$ to the combinatorial nooses in $\mathfrak{N}$. For every $N_A \in \mathfrak{N}$ on faces and vertices of set $F(N_A) \cup V(N_A)$ and for every partition $V_1(N) \dot{\cup} V_2(N)$ of $V(N)$, mapping $\gamma$ *is valid* [$\star$] if

a) $\gamma$ restricted to $V_1(N)$ is a bijection to $V(N_A)$;

b) every $v \in V_2(N)$ and $f \in F(N)$ satisfy $\gamma(v) \in F(N_A)$ and $\gamma(f) \in F(N_A)$;

c) for every $v_i \in V(N)$ and subsequence $[f_{i-1}, v_i, f_i]$ of $N$: if $v_i \in V_2(N)$, then face $\gamma(v_i)$ is equal to both $\gamma(f_{i-1})$ and $\gamma(f_i)$, and if $v_i \in V_1(N)$, then vertex $\gamma(v_i)$ is incident to both $\gamma(f_{i-1})$ and $\gamma(f_i)$; and

d) $A_{\mathrm{sub}}$ is a minor of $G_e$ with respect to $a) - c)$.

We assign an array $\Psi_e$ to each $\mathbf{mid}(e)$ consisting of triples, where each triple $\langle A, N_A, \gamma \rangle$ represents a minor candidate $A$ together with a valid mapping $\gamma$ from a combinatorial noose $N$ corresponding to $\mathbf{mid}(e)$ to a combinatorial noose $N_A \in \mathfrak{N}$. The vertices and faces of $N$ are oriented clockwise around the drawing of $G_e$. Without loss of generality, we assume for every $\langle A, N_A, \gamma \rangle$ the orientation of $N_A$ to be clockwise around the drawing of subgraph $A_{\mathrm{sub}}$ of $A$.

**Step 0: Initializing the leaves.** For every parent edge $e$ of a leaf $v$ of $T$, we initialize for every $A \in \mathcal{A}$ the valid mappings from the combinatorial noose bounding the edge $\mu(v)$ of $G$ to every combinatorial noose of length at most two in $A$ (clockwise bounding at most one edge of $A$).

**Step 1a): Update process.** We update the arrays of the middle sets bottom-up in post-order manner from the leaves of $T$ to root $r$. During this updating process it is guaranteed that the 'local' solutions for each minor associated with a middle set of the sc-decomposition are combined into a 'global' solution for the overall graph $G$.

In each dynamic programming step, we compare the arrays of two middle sets $\mathbf{mid}(e)$ and $\mathbf{mid}(f)$ in order to create a new array assigned to the middle set $\mathbf{mid}(g)$, where $e, f$, and $g$ have a vertex of $T$ in common. From [12] we know that the combinatorial noose $N_g$ is formed by the symmetric difference of the combinatorial nooses $N_e, N_f$ and that $G_g = G_e \cup G_f$. In other words, we are ensured that if two solutions on $G_e$ and $G_f$ bounded by $N_e$ and $N_f$ *fit together*, then they form a new solution on $G_g$ bounded by $N_g$. We now determine when two solutions represented as tuples in the arrays $\Psi_e$ and $\Psi_f$ fit together. We update two triples $\langle A^1, N_A^1, \gamma_1 \rangle \in \Psi_e$ and $\langle A^2, N_A^2, \gamma_2 \rangle \in \Psi_f$ to a new triple in $\Psi_g$ if

- $A^1 = A^2 =: A \in \mathcal{A}$ and every edge of $A$ with no endpoint in $V(N_A^1) \cup V(N_A^2)$ is an m-edge;

- for every $x \in (V(N_e) \cup F(N_e)) \cap (V(N_f) \cup F(N_f))$, we have $\gamma_1(x) = \gamma_2(x)$; and

- for the subgraph $A_{\mathrm{sub}}^1$ of $A$ separated by $N_A^1$ and the subgraph $A_{\mathrm{sub}}^2$ of $A$ separated by $N_A^2$, we have that $E(A_{\mathrm{sub}}^1) \cap E(A_{\mathrm{sub}}^2) = \emptyset$ and $V(A_{\mathrm{sub}}^1) \cap V(A_{\mathrm{sub}}^2) \subseteq \{\gamma(v) \mid v \in V(N_e) \cap V(N_f)\}$.

That is, we only update solutions with the same graph $A$ and with the two nooses $N_A^1$ and $N_A^2$ bounding (clockwise) two edge-disjoint parts of $A$ and intersecting in a consecutive subsequence of both $N_A^1$ and $N_A^2$. If the two solutions on $N_e$

and $N_f$ fit together, we get a valid mapping $\gamma_3 : N_g \to N_A^3$ to a noose $N_A^3$ of $A$ as follows:

- for every $x \in (V(N_e) \cup F(N_e)) \cap (V(N_f) \cup F(N_f)) \cap (V(N_g) \cup F(N_g),)$ we have $\gamma_1(x) = \gamma_2(x) = \gamma_3(x)$;
- for every $y \in (V(N_e) \cup F(N_e)) \setminus (V(N_f) \cup F(N_f))$ we have $\gamma_1(y) = \gamma_3(y)$; and
- for every $z \in (V(N_f) \cup F(N_f)) \setminus (V(N_e) \cup F(N_e))$ we have $\gamma_2(z) = \gamma_3(z)$.

We have that $\gamma_3$ is a valid mapping from $N_g$ to the combinatorial noose $N_A^3$ that bounds subgraph $A_{\mathrm{sub}}^3 = A_{\mathrm{sub}}^1 \cup A_{\mathrm{sub}}^2$.

**Step 1b): Post-processing.** Before adding a triple $\langle A, N_A^3, \gamma_3 \rangle$ to array $\Psi_g$, we need to manipulate $A$ so that $a$) it does not grow too big and $b$) it is suitable for future update operations. In $A$ restricted to subgraph $A_{\mathrm{sub}}^3$, we contract all c-edges with at least one end-vertex not in $N_A^3$ in order to fulfill $a$). Concerning $b$), for every $B \in \mathcal{A}$ we check for all its nooses $N_B$ with $|V(N_B)| = |V(N_A^3)|$ if there is a bijection $\beta$ from $N_A^3$ to $N_B$ such that the following holds. If in a copy of $B$ we contract those c-edges which $i$) are in the subgraph *counter-clockwise* bounded by $N_B$ and $ii$) have at least one end-vertex not in $N_B$, then we obtain a 2-edge-colored graph isomorphic to $A$. We define $\delta = \gamma_3 \circ \beta$ and we replace $\langle A, N_A^3, \gamma_3 \rangle$ in array $\Psi_g$ by those triples $\langle B, N_B, \delta \rangle$ which validate properties $i$) and $ii$).

**Step 2: Termination.** If, at some step, we have a solution where the entire minor $H$ is formed, we terminate the algorithm accepting. That is the case, if for some triple we have that $H \preceq A_{\mathrm{sub}} \preceq A$ and $A_{\mathrm{sub}}$ is bounded by $N_A$. We output model $M$ of $H$ in $G$ represented by this $A$ by reconstructing a solution top-down in $\langle T, \mu, \pi \rangle$. If at root $r$ no $A \in \mathcal{A}$ has been computed, we reject.

**Correctness and running time of the algorithm.** [$\star$]

This completes the proof of Lemma 2.

*Proof of Theorem 1.* We put everything together by verifying Algorithm 3.1. We produce in time $\mathcal{O}(n^2 \cdot \log n)$ a sc-decomposition of input graph $G$ [16]. Next, either we can immediately compute a minor model of $G$ in time $\mathcal{O}(n^2 \cdot \log n + h^4)$ (Proposition 6) or we run our 2-step-algorithm: we produce all majors of the minor pattern (Lemma 1) with Algorithm C.1 in time $2^{\mathcal{O}(h)}$, and run partially embedded dynamic programming in time $2^{\mathcal{O}(h)} \cdot n$ (Lemma 2).  □

## 4   Conclusions and further research

In this paper we showed that PLANAR $H$-MINOR CONTAINMENT is solvable in time $2^{\mathcal{O}(h)} \cdot n + \mathcal{O}(n^2 \cdot \log n)$ for a host graph on $n$ vertices and a pattern $H$ on $h$ vertices. That is, we showed that the problem can be solved in *single-exponential* time in $h$, significantly improving all previously known algorithms. Similar to [10], we can enumerate and count the number of models within the same time bounds.

Let us discuss some interesting avenues for further research concerning minor containment problems. First, it seems possible to solve in single-exponential time other variants of planar minor containment using our approach, like looking for a contraction minor, an induced minor, or a topological minor, as it has been recently done in [1] for general host graphs using completely different techniques. Also, it would be interesting to count the number of non-isomorphic models faster than just by enumerating models and removing isomorphic duplicates.

An important question is if, up to some assumption from complexity theory, the running time of our algorithm is tight. In other words, is there a $2^{o(h)} \cdot n^{\mathcal{O}(1)}$ algorithm (i.e., a *subexponential* algorithm) solving PLANAR $H$-MINOR CONTAINMENT or the existence of such an algorithm would imply the failure of, say, the Exponential Time Hypothesis? A first step could be to study the existence of subexponential algorithms when the pattern is further restricted to be a $k$-outerplanar graph for some constant $k$, or any other subclass of planar graphs.

Conversely, single-exponential algorithms may exist for host graphs more general than planar graphs. The natural candidates are host graphs embeddable in an arbitrary surface. One possible approach could be to use the framework recently introduced in [28] to perform dynamic programming for graphs on surfaces. The main ingredient of this framework is a new type of branch decomposition of graphs on surfaces, called *surface cut decomposition*, which plays the role of sphere cut decompositions for planar graphs.

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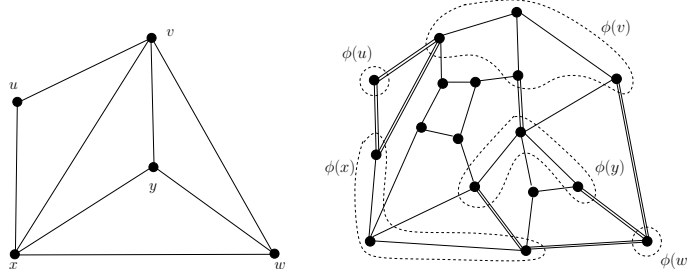
## A   Fig. 1: Minors and models (Section 2)



**Fig. 1.** On the left hand side we illustrate a minor $H$. The figure on the right hand side shows a graph $G$ with a model $M$ of $H$. The subgraphs $\phi$ on $V(H)$ are enclosed by dashed curves and the edges $\{\phi(e) \mid e \in E(H)\}$ are emphasized by double lines.

## B   Proof of Proposition 6

Since any planar graph $H$ with at least three vertices satisfies $|E(H)| \leq 3|V(H)| - 6$, by Proposition 2 any planar graph on $h$ vertices is isomorphic to a minor of the $(14h \times 14h)$-grid.

   With the algorithm of [29], we can find $\mathbf{bw}(G)$ in time $\mathcal{O}(n^2)$. If $\mathbf{bw}(G) > 42h$, then by Proposition 1 we can find in time $\mathcal{O}(n^2 \log n)$ a model of a $(14h \times 14h)$-grid in $G$. From this grid we find a model of $H$ in $G$ using Proposition 2. To conclude, let us discuss how the proof of Proposition 2 provided in [26] can be easily made constructive. For the sake of presentation, we omit many details that can be found in [26]. Indeed, the proof of [26, Fact (1.5)] consists in subdividing $H$ to obtain an auxiliary planar graph $H_1$ with $|V(H_1)| \leq 7h$, which is isomorphic to a minor of a planar Hamiltonian graph $H_2$ with $|V(H_2)| \leq 14h$. It is then proved that such a graph $H_2$ is isomorphic to a minor of a $(|V(H_2)| \times |V(H_2)|)$-grid. Then the proof uses simple operations on vertices, edges, and separating triangles. At one point it uses a Hamiltonian cycle, whose existence is guaranteed by Whitney's Theorem[3], for which there exist constructive versions running in linear time[4]. One can then check that the overall running time is dominated by the inductive proof of [26, Fact (1.4)], in which one must find $\mathcal{O}(h)$ times a separating triangle in a graph on $7h$ vertices. This procedure can be naïvely done in time $\mathcal{O}(h^4)$. Therefore, a rough upper bound for the algorithm that follows from Proposition 2 is $\mathcal{O}(h^4)$.                                    □

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[3] Whitney's Theorem states that each triangulated planar graph without separating triangles is Hamiltonian.

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## C   The preprocessing algorithm: PRE-PROC

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**Algorithm C.1**: The preprocessing algorithm: PRE-PROC.

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**Input**    : Planar graph $H$ on $h$ vertices and $\ell$ edges.
**Output** : Collection $\mathcal{A}$ of 2-edge-colored plane graphs .
Compute all non-isomorphic planar graphs $A$ satisfying
$h \leq |V(A)| \leq 1.5 \cdot \mathbf{bw}(G) + h$ and $|E(A)| \geq \ell$.
**for** *every such graph $A$* **do**
    **for** *every subset $X$ of $E(A)$ of size $\ell$* **do**
        mark the edges in $X$ as m-edges and the edges in $E(A) \setminus X$ as c-edges,
        resulting in a 2-edge-colored graph $A_{m,c}$.

**for** *every graph $A_{m,c}$* **do**
    **if** *the graph resulting from contracting the c-edges of $A_{m,c}$ is isomorphic to*
    *$H$* **then** compute its non-equivalent drawings $\Phi_1, \ldots, \Phi_q$, and
    add the plane graphs $\langle A_{m,c}, \Phi_i \rangle$ for all $1 \leq i \leq q$ to $\mathcal{A}$.

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## D   Proof of Lemma 1

By Proposition 4, the number of non-isomorphic planar graphs $A$ on $d \cdot h$ vertices (for a constant $d$) is $2^{\mathcal{O}(h)}$. We compute this set in time proportional to its size[5]. We partition the edge set of each $A$ into three subsets: the edges that we need to delete, the c-edges, and the m-edges. There are again $2^{\mathcal{O}(h)}$ possible such partitions, and can be computed in the same asymptotic running time. We use the linear time algorithm for planar graph isomorphism[6] to check if after applying the graph operations the resulting graph $A'$ is isomorphic to $H$. If so, we create all non-equivalent drawings of $A$ and add them to $\mathcal{A}$ by using Proposition 3 and Algorithm 4.1 in [10], again in time $2^{\mathcal{O}(h)}$.                              □

## E   Valid partial solutions in partially embedded dynamic programming (Section 3.2)

Items $a$) and $b$) say where to map the faces and vertices of $N$ to. Item $c$) (with $a$)) makes sure that if two vertices $v_h, v_j$ in sequence $N = [\ldots, v_h, \underline{\ldots}, v_j, \ldots]$ are mapped to two vertices $w_i, w_{i+1}$ that appear in sequence $N_A$ as $[\ldots, w_i, f_i, w_{i+1}, \ldots]$,

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[5] using the algorithm of
  Z. Li and S.-I. Nakano. Efficient generation of plane triangulations without repetitions. *In Proc. of the 28th International Colloquium on Automata, Languages and Programming (ICALP)*, volume 2076 of LNCS, pages 433-443, 2001.
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then every face and vertex between $v_h, v_j$ in sequence $N$ (here underlined) is mapped to face $f_i$. Item $d$) simply makes clear that the boundary of $G_e$ is mapped to the boundary $A_{\text{sub}}$ such that contracting some part of $G_e \setminus N$ leads to $A_{\text{sub}}$. For an illustration, see Fig. 2.
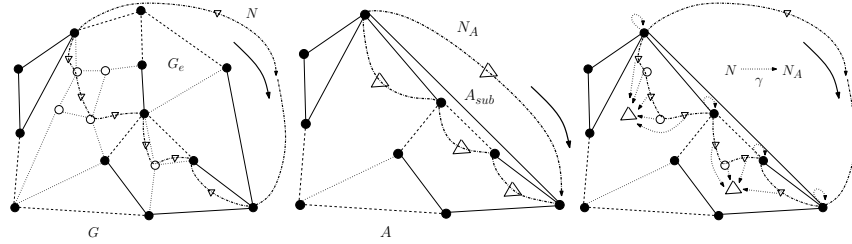


**Fig. 2.** The figure on the left hand side illustrates the graph $G$ of Fig. 1 with an oriented noose $N$ (dashed) enclosing subgraph $G_e$. The figure in the middle shows a graph $A$ of collection $\mathcal{A}$ (that is, $A$ is a major of the graph $H$ of Fig. 1) with a noose $N_A$ enclosing $A_{\text{sub}}$. The figure on the right hand side shows a mapping $\gamma$ from the faces and vertices of $N$ to $N_A$.

## F   Correctness of partially embedded dynamic programming (Section 3.2)

In Corollary 1, we already showed that the preprocessing correctly computes the collection $\mathcal{A}$ of pairwise non-isomorphic 2-edge-colored plane graphs $A$ on on at least $h$ and at most $1.5\mathbf{bw}(G)+h$ vertices containing $H$ as a minor. In the update process on the nooses $N^1, N^2$ of two incident edges of the sc-decomposition, we produce graphs with as many vertices since we have for candidate $A$ of $H$ in $G$ that $A$ intersects $V(N^1) \cup V(N^2)$, and by Remark 1 that $|V(N^1) \cup V(N^2)| \leq 1.5\mathbf{bw}(G)$ and up to $h$ vertices of $A$ might be outside of $N_A^1$ and $N_A^2$.

We have already seen how to map every combinatorial noose of $G$ that identifies a separation of $G$ via a valid mapping $\gamma$ to a combinatorial noose of $A$ determining a separation of $A$. Step 0 ensures that every edge of $A$ is bounded by a combinatorial noose $N_A$ of length two, which is determined by triple $\langle A, N_A, \gamma \rangle$ in an array assigned to a leaf edge of $T$. We need to show that Step $1a$) and $1b$) compute a valid solution for $N_g$ from $N_e$ and $N_f$ for incident edges $e, f, g$. We note that the property that the symmetric difference of the combinatorial nooses $N_e$ and $N_f$ forms a new combinatorial noose $N_g$ is passed on to the combinatorial nooses $N_A^1, N_A^2$, and $N_A^3$ of $A$, too. If the two solutions fit together, then $A_{\text{sub}}^1$ of $A$ separated by $N_A^1$ and subgraph $A_{\text{sub}}^2$ of $A$ separated by $N_A^2$ only intersect in the image of $V(N_e) \cap V(N_f)$. We observe that $N_A^1$ and $N_A^2$ intersect in a consecutive alternating subsequence with order reversed to each other, i.e., $N_A^1 \mid_{N_e \cap N_f} = \overline{N_A^2} \mid_{N_e \cap N_f}$, where $\overline{N_A}$ means the reversed sequence $N_A$. Since every oriented $N_A$ uniquely identifies a separation of $E(A)$, we can easily decide

whether two triples $\langle A, N_A^1, \gamma \rangle \in \Psi_e$ and $\langle A, N_A^2, \gamma \rangle \in \Psi_f$ fit together and bound a new subgraph $A_{\text{sub}}^3$ of $A$. In Step 1$b$), we contract all c-edges in $A_{\text{sub}}^3$ with an end-vertex in $V(A_{\text{sub}}^3) \setminus V(N_A^3)$ such that $A_{\text{sub}}^3$ represents a subgraph of $H$. Next we blow up $\overline{A_{\text{sub}}^3}$, that is the subgraph of $A$ on the other side which $N_A^3$ bounds counter-clockwise. From $\mathcal{A}$ we compute every possible graph partitioned into one subgraph isomorphic to $A_{\text{sub}}^3$ and one subgraph major of $\overline{A_{\text{sub}}^3}$ separated by a noose bijectively mapped from $N_A^3$. If there exists an $A \in \mathcal{A}$ which is a minor of $G$, then at some step we will enter Step 2 and produce the entire model $M$.

## G  Running time of partially embedded dynamic programming (Section 3.2)

By Corollary 1, computing the collection $\mathcal{A}$ in the preprocessing step can be done in time $2^{\mathcal{O}(\mathbf{bw}(G)+h)}$. We now give an upper bound on the size of each array $\Psi$. For each $A \in \mathcal{A}$, the number of combinatorial nooses in $\mathfrak{N}$ we are considering is bounded by the total number of combinatorial nooses in $A$, which is $2^{\mathcal{O}(\mathbf{bw}(G)+h)}$ by Proposition 5. The number of partitions of vertices of any combinatorial noose $N$ is bounded by $2^{|V(N)|}$. Since the order of both $N_A$ and $N$ is given, we only have $2|V(A)|$ possibilities to map vertices of $N$ to $N_A$, once the vertices of $N$ are partitioned. Thus, in an array $\Psi_e$ we may have up to $2^{\mathcal{O}(|V(A)|)} \cdot 2^{|V(N)|} \cdot |V(A|$ triples $\langle A, N_A, \gamma \rangle$. We first create all triples in the arrays assigned to the leaves. Since middle sets of leaves only consist of an edge in $G$, we get arrays of size $\mathcal{O}(|V(A)|^2)$, which we compute in the same asymptotic running time. When updating middle sets $\mathbf{mid}(e), \mathbf{mid}(f)$, we compare every triple of array $\Psi_e$ to every triple in array $\Psi_f$ to check if two triples fit together. We can compute the unique subgraph $A_{\text{sub}}^1$ (resp. $A_{\text{sub}}^2$) described by a triple in $\Psi_e$ (resp. $\Psi_f$), compare two triples in $\Psi_e$ and $\Psi_f$, and create a new triple in time linear in the order of $V(N)$ and $V(H)$. For adding a new triple to $\Psi_g$ in the post-processing, we apply the color coding technique[7] for computing each of the $2^{\mathcal{O}(\mathbf{bw}(G)+h)}$ nooses in $\mathfrak{N}$ in the same asymptotic running time.

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[7] N. Alon, R. Yuster, and U. Zwick. Color-coding. *Journal of the ACM*, 42:844-856, 1995.